

Ellipsoidal bounds for static response of framed structures against interactive uncertainties

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(Received April 13, 2007, Accepted November 5, 2007)

Abstract. This paper presents an optimization-based method for computing a minimal bounding ellipsoid that contains the set of static responses of an uncertain braced frame. Based on a non-stochastic modeling of uncertainty, we assume that the parameters both of brace stiffnesses and external forces are uncertain but bounded. A brace member represents the sum of the stiffness of the actual brace and the contributions of some non-structural elements, and hence we assume that the axial stiffness of each brace is uncertain. By using the S -lemma, we formulate a semidefinite programming (SDP) problem which provides an outer approximation of the minimal bounding ellipsoid. The minimum bounding ellipsoids are computed for a braced frame under several uncertain circumstances.

Keywords: semidefinite program; data uncertainty; uncertain linear equation; interval analysis; braced frame.

1. Introduction

This paper presents a solution method for computing ellipsoidal bounds for static response of a braced frame under the assumptions that the external forces are imprecisely known and the stiffness of each brace possesses uncertainty. Structural analyses considering the uncertainties have received fast-growing interests, because structures actually built always have various uncertainties caused by manufacturing errors, limitation of knowledge of input disturbances, observation errors, simplification for modeling, damage or deterioration of structural elements, etc.

Probabilistic uncertainty modelings of structures were studied extensively. Non-probabilistic uncertainty models have also been developed. In a non-probabilistic uncertainty model, a mechanical system is assumed to contain some unknown parameters which are assumed to be bounded. Ben-Haim and Elishakoff (1990) developed the well-known *convex model* approach, with which Pantelides and Ganzerli (1989) proposed a robust truss optimization method. The info-gap decision theory has been proposed by Ben-Haim (2006). Based on the info-gap uncertainty model,

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the authors proposed solution methods for robustness analysis of structures (Kanno and Takewaki 2006a, Takewaki and Ben-Haim 2005).

The interval linear algebra was well developed for the so-called uncertain linear equations (Alefeld and Mayer 2000), which has been employed in structural analyses considering various uncertainties (Chen *et al.* 2002, McWilliam 2001, Muhanna and Mullen 2001, Qiu and Elishakoff 1998). In contrast to probabilistic modelings, non-probabilistic uncertainty modelings require only bounds on the uncertain parameters, and hence it is not necessarily to estimate the probabilistic distribution functions of uncertain parameters. Under the assumption of small variations of uncertain parameters, interval analyses of static response have been developed based on the first-order interval perturbation (Chen *et al.* 2002, Qiu and Elishakoff 1998). Further refinements of the linear interval approach were proposed for static structural analyses including uncertainties (McWilliam 2001, Muhanna and Mullen 2001). A comparison between the convex model analysis and interval analysis was given by Qiu (2003).

Calafiore and El Ghaoui (2004) proposed a method for finding the ellipsoidal bounds of the solution set of uncertain linear equations based on the *semidefinite programming* (SDP) problem (Helmberg 2002), in which the uncertainty of data of the linear equations is described in terms of a so-called linear fractional representation. The authors formulated an SDP problem which provides a confidential ellipsoidal bound for static response of a truss including some bounded uncertain parameters (Kanno and Takewaki 2006b).

In this paper, we aim at obtaining an ellipsoidal bound for static response of an uncertain braced frame. Both the external nodal loads and the stiffnesses of braces are assumed to be uncertain, and to be included in a given bounded set. By using quadratic embedding of the uncertain parameters and the S -lemma (Ben-Tal and Nemirovski 2001), we formulate an SDP problem that provides an outer approximation of the minimum bounding ellipsoid. It is well known that SDP problems can be solved efficiently by using the primal-dual interior-point method (Ben-Tal and Nemirovski 2001, Helmberg 2002), where the number of arithmetic operations required by the algorithm is bounded by a polynomial of the problem size. Hence, our method finds an outer approximation of the minimum bounding ellipsoid within a polynomial time, while most of the methods based on the interval algebra have in general exponential complexity [4, section 6.5.3].

This paper is organized as follows. The remainder of this section is devoted to the introduction of notation and SDP. Section 2 introduces the uncertainty model of a braced frame subjected to load and structural uncertainties. We formulate the problem to find the minimum bounding ellipsoid for the distribution of static responses in section 3. In section 4, we propose an SDP problem which provides a confidential approximation of the minimal bounding ellipsoid. Numerical experiments are presented in section 5 for a braced frame. Finally, conclusions are drawn in section 6.

1.1 Notation

In this paper, all vectors are assumed to be column vectors. The $(m+n)$ -dimensional column vector $(\mathbf{u}^T, \mathbf{v}^T)^T$ consisting of $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{v} \in \mathbb{R}^n$ is often written simply as (\mathbf{u}, \mathbf{v}) . For two sets $\mathcal{A} \subseteq \mathbb{R}^m$ and $\mathcal{B} \subseteq \mathbb{R}^n$, their Cartesian product is defined by $\mathcal{A} \times \mathcal{B} = \{(\mathbf{a}^T, \mathbf{b}^T)^T \in \mathbb{R}^{m+n} \mid \mathbf{a} \in \mathcal{A}, \mathbf{b} \in \mathcal{B}\}$.

Particularly, we write $\mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$. For a vector $\mathbf{p} = (p_i) \in \mathbb{R}^n$, $\|\mathbf{p}\|_2$ and $\|\mathbf{p}\|_\infty$, respectively, denote the standard Euclidean norm and l_∞ -norm of \mathbf{p} defined as

$$\|\mathbf{p}\|_2 = (\mathbf{p}^T \mathbf{p})^{1/2}$$

$$\|\mathbf{p}\|_\infty = \max_{i \in \{1, \dots, n\}} |p_i|$$

For $\mathbf{p} = (p_i) \in \mathbb{R}^n$ and $\mathbf{q} = (q_i) \in \mathbb{R}^n$, we write $\mathbf{p} \geq \mathbf{0}$ and $\mathbf{p} \geq \mathbf{q}$, respectively, if $p_i \geq 0$ ($i = 1, \dots, n$) and $\mathbf{p} - \mathbf{q} \geq \mathbf{0}$.

We write $\text{Diag}(\mathbf{p})$ for the diagonal matrix with a vector $\mathbf{p} \in \mathbb{R}^n$ on its diagonal. For $\mathbf{p}_l \in \mathbb{R}^{nl}$ ($l = 1, \dots, k$), we simply write $\text{Diag}(\mathbf{p}_1, \dots, \mathbf{p}_k)$ instead of $\text{Diag}((\mathbf{p}_1^\top, \dots, \mathbf{p}_k^\top)^\top)$. For $p \in \mathbb{R}$ and $q \in \mathbb{R}$ satisfying $p \leq q$, we denote by $[p, q]$ the interval defined by

$$[p, q] = \{x \in \mathbb{R} \mid p \leq x \leq q\}$$

For a matrix $P \in \mathbb{R}^{n \times n}$, $\text{tr}(P)$ denotes the trace of P , i.e., the sum of the diagonal elements of P .

1.2 Outline of semidefinite program

Let $\mathcal{S}^n \subset \mathbb{R}^{n \times n}$ denote the set of all $n \times n$ real symmetric matrices. We write $P \succeq O$ if $P \in \mathcal{S}^n$ is positive semidefinite.

Let $A_i \in \mathcal{S}^n$ ($i = 1, \dots, m$), $C \in \mathcal{S}^n$, and $\mathbf{b} \in \mathbb{R}^m$ be constant matrices and a constant vector. The *semidefinite programming* (SDP) problem refers to the optimization problem having the form of

$$\max \left\{ \mathbf{b}^\top \mathbf{y} : C - \sum_{i=1}^m A_i y_i \succeq O \right\} \quad (1)$$

where $\mathbf{y} = (y_i) \in \mathbb{R}^m$ is a variable vector (Helmberg 2002).

Recently, SDP has received increasing attention for its wide fields of application (Ben-Tal and Nemirovski 2001, Kanno and Takewaki 2006a, Ohsaki *et al.* 1999). The primal-dual interior-point method, which has been first developed for LP, has been naturally extended to SDP (Ben-Tal and Nemirovski 2001, Helmberg 2002). It is theoretically guaranteed that the primal-dual interior-point method converges to an optimal solution of the SDP problem (1) within the number of arithmetic operations bounded by a polynomial of m and n .

2 Uncertainty model of braced frames

Consider a linearly elastic, rigidly-jointed frame with some pin-jointed braces in the two- or three-dimensional space. Small rotations and small strains are assumed. Let $\mathbf{u} \in \mathbb{R}^{n^d}$ and $\mathbf{f} \in \mathbb{R}^{n^d}$ denote the vectors of nodal displacements and external forces, respectively, where n^d denotes the number of degrees of freedom of nodal displacements. The equilibrium equation is written as

$$K\mathbf{u} = \mathbf{f} \quad (2)$$

where $K \in \mathcal{S}^{n^d}$ denotes the stiffness matrix. In Eq. (2), we assume that K and \mathbf{f} have uncertainties, which shall be rigorously defined in sections 2.2 and 2.3.

2.1 Motivation

An example of a five-story braced frame is illustrated in Fig. 1. The conventional situation is

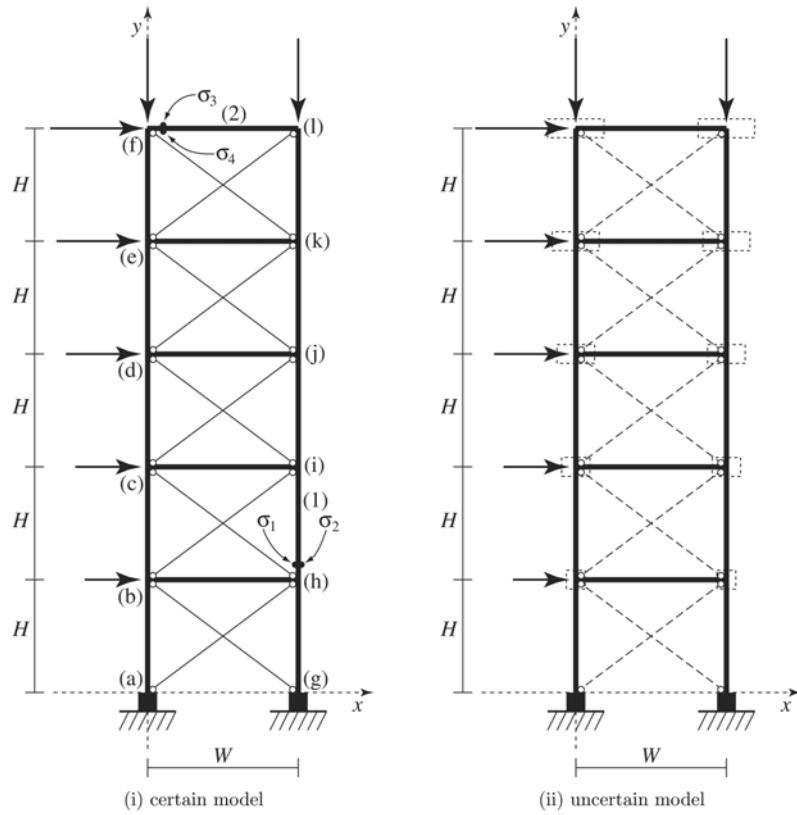


Fig. 1 5-story framed structure

shown in Fig. 1 (i), where nominal static external forces are applied at some nodes.

Usually, the external load is defined as the ‘best’ estimate of the input disturbance, while it is quite difficult to estimate the actual input disturbance precisely. Besides the limitation of knowledge of the external load, a structure designed for only one loading scenario may not be robust sufficiently. Hence, we consider the uncertainty of external loads as illustrated in Fig. 1 (ii), in which we assume that the nodal loads can run through the rectangles depicted with dashed lines.

The frame shown in Fig. 1 have 10 braces. The brace members may play some roles in structural analysis: A brace member represents the sum of the stiffness of the actual brace and the contributions of some non-structural elements. The estimation of additional stiffness caused by the non-structural elements depends on engineers, and hence it may differ drastically. This motivates us to assume that the stiffness of each brace is uncertain in Fig. 1 (ii).

Consequently, we suppose that the braced frame shown in Fig. 1 (ii) is subjected to the uncertain external loads, and the uncertainty of stiffnesses of braces are considered simultaneously. The locations of nodes are assumed to be certain. Our aim is to find a bound on the distribution of static response of the braced frame with the uncertain external load and brace stiffness.

2.2 Uncertainty of nodal load

We assume without loss of generality that \mathbf{f} is decomposed as

$$\mathbf{f} = \begin{pmatrix} \mathbf{f}_u \\ \mathbf{f}_c \end{pmatrix} \quad (3)$$

where $\mathbf{f}_u \in \mathbb{R}^{n^f}$ and $\mathbf{f}_c \in \mathbb{R}^{n^d - n^f}$ denote the vector of *uncertain* elements and that of *certain* elements of \mathbf{f} , respectively. We denote by n^f the number of dimensions of \mathbf{f}_u . In the example of Fig. 1 (ii), \mathbf{f}_u corresponds to the external forces, while \mathbf{f}_c represents the external moments.

Let $\tilde{\mathbf{f}} \in \mathbb{R}^{n^f}$ denote the nominal value of \mathbf{f}_u . Define $\tilde{\mathbf{f}} \in \mathbb{R}^{n^d}$ by $\tilde{\mathbf{f}} = (\tilde{\mathbf{f}}_u^T, \mathbf{f}_c^T)^T$. In order to represent the uncertainty of \mathbf{f}_u , we introduce a parameter vector $\zeta_f = (\zeta_{fj}) \in \mathbb{R}^{n^f}$. We assume that \mathbf{f}_u depends on ζ_f as

$$f_j = \tilde{f}_j + f_j^0 \zeta_{fj}, \quad j = 1, \dots, n^f \quad (4)$$

where $\mathbf{f}^0 = (f_i^0) \in \mathbb{R}^{n^f}$ is a constant vector satisfying $f_j^0 \neq 0$ ($j = 1, \dots, n^f$). Note that f_j^0 represents the magnitude of uncertainty of the j th component f_j of \mathbf{f}_u . To define the model of distribution of the unknown vector ζ_f , we define the set \mathcal{Z}_f by

$$\mathcal{Z}_f = \{ \zeta_f \in \mathbb{R}^{n^f} \mid 1 \geq \| \zeta_f \|_\infty \} \quad (5)$$

The set \mathcal{Z}_f is called the *uncertainty set* of ζ_f , and we assume that ζ_f is bounded as

$$\zeta_f \in \mathcal{Z}_f \quad (6)$$

Thus, the uncertainty of the external force \mathbf{f} is modeled by Eqs. (4) and (6).

2.3 Uncertainty of brace stiffness

Let n^m denote the number of brace members which are modeled as truss elements. The vector of cross-sectional areas of brace members is denoted by $\mathbf{a} = (a_i) \in \mathbb{R}^{n^m}$. Suppose that the axial stiffness of each member has uncertainty. We describe the uncertainties of axial stiffness of the braces via the uncertainties of member cross-sectional areas \mathbf{a} .

Let $\tilde{\mathbf{a}} = (\tilde{a}_i) \in \mathbb{R}^{n^m}$ denote the nominal value of \mathbf{a} . The uncertainty of \mathbf{a} is described by using the parameter vector $\zeta_a = (\zeta_{ai}) \in \mathbb{R}^{n^m}$. Suppose that \mathbf{a} depends on ζ_a affinely as

$$a_i = \tilde{a}_i + a_i^0 \zeta_{ai}, \quad i = 1, \dots, n^m \quad (7)$$

where $\mathbf{a}^0 = (a_i^0) \in \mathbb{R}^{n^m}$ is a constant vector satisfying

$$\tilde{a}_i > a_i^0 > 0, \quad i = 1, \dots, n^m \quad (8)$$

Note that a_i^0 represents the magnitude of uncertainty of a_i . Define the set $\mathcal{Z}_a \subset \mathbb{R}^{n^m}$ by

$$\mathcal{Z}_a = \{ \zeta_a \in \mathbb{R}^{n^m} \mid 1 \geq \| \zeta_a \|_\infty \} \quad (9)$$

which is the uncertainty set of ζ_a . We assume that ζ_a is running through \mathcal{Z}_a as

$$\zeta_a \in \mathcal{Z}_a \quad (10)$$

The assumption (8) guarantees that the condition

$$a_i > 0, \quad i = 1, \dots, n^m \quad (11)$$

is satisfied for any $\zeta_a \in \mathcal{Z}_a$. From the mechanical point of view, it is natural to assume that the condition (11) is satisfied, since a denotes the vector of member cross-sectional areas.

2.4 Solution set of uncertain equilibrium equation

Let $K^F \in \mathcal{S}^{n^d}$ denote the stiffness matrix consisting of all beam-column elements. The member stiffness matrix of the i th brace member is proportional to a_i and is denoted by $a_i K_i \in \mathcal{S}^{n^d}$ ($i = 1, \dots, n^m$). Since the stiffness matrix of the braced frame is defined as the superposition of the stiffness matrices due to the frame and the braces, we can write

$$K = K^F + \sum_{i=1}^{n^m} a_i K_i \quad (12)$$

For a brace member, or a truss element, it is known that the row rank of the member stiffness matrix is equal to one. Hence, for each $i = 1, \dots, n^m$, we can write

$$K_i = \mathbf{b}_i \mathbf{b}_i^T \quad (13)$$

where $\mathbf{b}_i \in \mathbb{R}^{n^d}$ is a constant vector.

Consequently, by using Eqs. (4), (6), (7), and (10), the uncertain equilibrium Eq. (2) can be written as

$$\left(K^F + \sum_{i=1}^{n^m} a_i \mathbf{b}_i \mathbf{b}_i^T \right) \mathbf{u} = \mathbf{f} \quad (14)$$

$$\mathbf{a} = \tilde{\mathbf{a}} + \text{Diag}(\mathbf{a}^0) \zeta_a, \quad \zeta_a \in \mathcal{Z}_a \quad (15)$$

$$\mathbf{f} = \tilde{\mathbf{f}} + \begin{pmatrix} \text{Diag}(\mathbf{f}^0) \zeta_f \\ 0 \end{pmatrix}, \quad \zeta_f \in \mathcal{Z}_f \quad (16)$$

Define the set $\mathcal{U} \subset \mathbb{R}^{n^d}$ by

$$\mathcal{U} = \{ \mathbf{u} \in \mathbb{R}^{n^d} \mid (14), (15), (16) \} \quad (17)$$

i.e., \mathcal{U} is the set of all possible solutions to the uncertain linear Eqs. (14)-(16). It follows from (11) and (13) that K is positive definite for all possible $\zeta_a \in \mathcal{Z}_a$, which guarantees $\mathcal{U} \neq \emptyset$.

3 Minimum bounding ellipsoid

3.1 Bounding ellipsoid

An ellipsoid in the r -dimensional space can be described as

$$\mathcal{E} = \{\mathbf{x} \in \mathbb{R}^r \mid \mathbf{x} = \hat{\mathbf{x}} + D\mathbf{z}, 1 \geq \|\mathbf{z}\|_2, \mathbf{z} \in \mathbb{R}^m\} \quad (18)$$

where $\hat{\mathbf{x}} \in \mathbb{R}^r$ is referred to as the *center* of the ellipsoid. The matrix $D \in \mathbb{R}^{r \times m}$ is called the *shape matrix*, and satisfies $\text{rank}(D) = m \leq r$. Define $P \in \mathcal{S}^r$ by $P = DD^T$. Then Eqs. (18) is alternatively written as

$$\mathcal{E}(P, \hat{\mathbf{x}}) = \left\{ \mathbf{x} \in \mathbb{R}^r \mid \begin{pmatrix} P & (\mathbf{x} - \hat{\mathbf{x}}) \\ (\mathbf{x} - \hat{\mathbf{x}})^T & 1 \end{pmatrix} \succeq O \right\} \quad (19)$$

where $P \succeq O$. Note that $\text{tr}(P)$ corresponds to the sum of squares of the semi-axes lengths. We adopt $\text{tr}(P)$ as the measure of size of the ellipsoid Eq. (19).

For $r = 1$, we see that Eq. (19) is reduced to

$$\mathcal{E}(P, \hat{x}) = [\hat{x} - P^{1/2}, \hat{x} + P^{1/2}]$$

which implies that the ellipsoid $\mathcal{E}(P, \hat{\mathbf{x}})$ coincides with an interval. From this observation, we see that finding a bounding ellipsoid includes finding a confidence interval as a particular case.

In this paper, the vector \mathbf{x} in Eq. (19) is taken as a vector of state variables of a frame that we are interested in. An ellipsoid $\mathcal{E}(P, \hat{\mathbf{x}})$ is called a *bounding ellipsoid* of \mathbf{x} if $\mathcal{E}(P, \hat{\mathbf{x}})$ includes all possible realization of \mathbf{x} . Obviously, the bounding ellipsoid is desired to be as ‘tight’ as possible. In section 3.2, we formulate the problem for finding the minimal bounding ellipsoid in the sense of the measure $\text{tr}(P)$.

3.2 Problem formulation

Suppose that we are interested in predicting the set of static response $G^T \mathbf{u} \in \mathbb{R}^r$ of the braced frame, where $G \in \mathbb{R}^{n^d \times r}$ is a constant matrix. Here, $G^T \mathbf{u}$ is regarded as a vector of appropriately chosen parameters representing the mechanical performance of the frame. We investigate two typical choices of G in Examples 3.1 and 3.3 below. Define $\mathcal{U}_G \subseteq \mathbb{R}^r$ by

$$\mathcal{U}_G = \{G^T \mathbf{u} \mid \mathbf{u} \in \mathcal{U}\} \quad (20)$$

where \mathcal{U} has been introduced in Eq. (17). From the definition, an ellipsoid $\mathcal{E}(P, \hat{\mathbf{x}})$ corresponds to a bounding ellipsoid of $G^T \mathbf{u}$ if it satisfies

$$\mathcal{U}_G \subseteq \mathcal{E}(P, \hat{\mathbf{x}})$$

i.e., if $\mathcal{E}(P, \hat{\mathbf{x}})$ is an outer approximation of the set \mathcal{U}_G . Particularly, we attempt to find the minimum bounding ellipsoid in the sense of the measure $\text{tr}(P)$, which is realized by solving the optimization problem

$$\min \{\text{tr}(P) : \mathcal{U}_G \subseteq \mathcal{E}(P, \hat{\mathbf{u}})\} \quad (21)$$

where $P \in \mathcal{S}^r$ and $\hat{\mathbf{u}} \in \mathbb{R}^r$ are the variables. If K and \mathbf{f} are an interval matrix and interval vector, respectively, then finding an exact interval of u_i is known to be an NP-hard problem (Rohn 1997). Thus, it is very difficult to find the global optimal solution of Eq. (21).

Example 3.1 (bounding ellipsoid of nodal displacement). Consider again the example of a plane frame illustrated in Fig. 1. Suppose that we are interested in the distribution of the nodal displacements of the node (b). We assume without loss of generality that the displacement of the

node (b) is denoted by (u_1, u_2, u_3) , where u_1 and u_2 are the displacements in the x - and y -directions, respectively, and u_3 denotes the rotation. Suppose that we are interested in the distribution of (u_1, u_2) . Put

$$G^T = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \end{pmatrix}$$

in Eq. (20), where $r = 2$. Then we can obtain the minimum bounding ellipsoid in \mathbb{R}^2 which includes all possible realizations of (u_1, u_2) by solving Eq. (21).

Example 3.2 (bounding interval of displacement). In the same situation as (3.1), a simpler problem is to find bounding intervals of u_1 and u_2 . The set \mathcal{U}_G coincides with an interval including all possible u_1 by putting

$$G^T = (1 \ 0 \ \cdots \ 0)$$

in Eq. (20) with $r = 1$. Then the minimum bounding interval including all possible realizations of u_1 by solving Eq. (21). Indeed, letting the pair of $P^* \in \mathbb{R}$ and $\hat{u}^* \in \mathbb{R}$ be an optimal solution of Eq. (21), we see that the interval

$$[\hat{u}^* - \sqrt{P^*}, \hat{u}^* + \sqrt{P^*}]$$

corresponds to the minimal bounding interval of u_1 .

Example 3.3 (bounding interval of member stress). We continue to consider the example of Fig. 1. Let σ_1 and σ_2 denote the extreme-fiber normal stress at the lower end of the member (1). Suppose that we are interested in estimating the distribution of (σ_1, σ_2) . Assuming the bi-axial symmetry of the cross-section, let h_i denote the depth of the cross-section of the i th member. We denote by l_i and $T_i \in \mathbb{R}^{6 \times n^d}$, respectively, the member length and the constant transformation matrix from the global coordinate system of the displacements to the local (member) coordinate system. The elastic modulus is denoted by E . Putting

$$G^T = \frac{E}{l_i^2} \begin{pmatrix} -l_i & 3h_i & 2l_i h_i & l_i & -3h_i & l_i h_i \\ -l_i & -3h_i & -2l_i h_i & l_i & 3h_i & -1l_i h_i \end{pmatrix} T_i$$

in Eq. (20) with $r = 2$, the set of stress vectors $(\sigma_1, \sigma_2)^T \in \mathbb{R}^2$ is represented by \mathcal{U}_G . Accordingly, the minimum bounding ellipsoid including all possible realizations of $(\sigma_1, \sigma_2)^T$ is obtained by solving Eq. (21).

4. Semidefinite programming approximation

As discussed in section 3.2, it is very difficult to find the global optimal solution of Eq. (21). The purpose of this section is to construct an efficiently solvable problem which approximates the problem Eq. (21).

Let $e_j \in \mathbb{R}^{n^d}$ denote the j th column vector of the identity matrix $I \in \mathcal{S}^{n^d}$. Define the constant matrices Ψ and E_0 as

$$\Psi = (\mathbf{b}_1, \dots, \mathbf{b}_{n^m}) \in \mathbb{R}^{n^d \times n^m}$$

$$E_0 = \begin{pmatrix} O \\ I \end{pmatrix} \in \mathbb{R}^{n^d \times (n^d - n^f)}$$

For simplicity, define \tilde{K} by

$$\tilde{K} = K^F + \sum_{i=1}^{n^m} \tilde{a}_i K_i \quad (22)$$

Proposition 4.1. $\mathbf{u} \in \mathcal{U}$ if and only if there exists a vector $\mathbf{p} = (p_i) \in \mathbb{R}^{n^m}$ satisfying

$$E_0^T [\tilde{K} \mathbf{u} + \Psi_p - \tilde{\mathbf{f}}] = \mathbf{0} \quad (23)$$

$$e_j^T [\tilde{K} \mathbf{u} + \Psi_p - \tilde{\mathbf{f}}] = f_j^0 \zeta_{fj}, \quad 1 \leq |\zeta_{fj}|, \quad j = 1, \dots, n^f \quad (24)$$

$$p_i = (a_i^0 \mathbf{b}_i^T \mathbf{u}) \zeta_{ai}, \quad 1 \geq |\zeta_{ai}|, \quad i = 1, \dots, n^m \quad (25)$$

Proof. It follows from Eqs. (12), (13), and the definition of K^F in Eq. (22) that the stiffness matrix K can be written as

$$K = \tilde{K} + \sum_{i=1}^{n^m} a_i^0 \zeta_{ai} \mathbf{b}_i \mathbf{b}_i^T$$

Observe that the relation

$$\sum_{i=1}^{n^m} a_i^0 \zeta_{ai} \mathbf{b}_i \mathbf{b}_i^T \mathbf{u} = (\mathbf{b}_1, \dots, \mathbf{b}_{n^m}) \begin{pmatrix} a_1^0 \zeta_{a1} \mathbf{b}_1^T \mathbf{u} \\ \dots \\ a_{n^m}^0 \zeta_{an^m} \mathbf{b}_{n^m}^T \mathbf{u} \end{pmatrix} = \Psi \begin{pmatrix} a_1^0 \zeta_{a1} \mathbf{b}_1^T \mathbf{u} \\ \dots \\ a_{n^m}^0 \zeta_{an^m} \mathbf{b}_{n^m}^T \mathbf{u} \end{pmatrix}$$

holds. Hence, defining $\mathbf{p} = (p_i) \in \mathbb{R}^{n^m}$ as

$$p_i = a_i^0 \zeta_{ai} \mathbf{b}_i^T \mathbf{u}, \quad i = 1, \dots, n^m \quad (26)$$

we obtain

$$K \mathbf{u} = \tilde{K} \mathbf{u} + \Psi_p$$

Accordingly, Eq. (14) is equivalently rewritten as Eq. (26) and

$$\tilde{K} \mathbf{u} + \Psi_p = \tilde{\mathbf{f}} \quad (27)$$

From the definitions of E_0 and e_j , we see that the condition Eq. (16) is equivalent to

$$E_0^T \mathbf{f} = E_0^T \tilde{\mathbf{f}} \quad (28)$$

$$e_j^T \mathbf{f} = \tilde{f}_j^0 \zeta_{fj}, \quad j = 1, \dots, n^f, \quad \zeta_f \in \mathcal{Z}_f \quad (29)$$

Consequently, Eqs. (14)-(16) are equivalent to Eqs. (26)-(29) and

$$\zeta_a \in \mathcal{Z}_a$$

By substituting Eq. (27) into Eqs. (28) and (29), and by using Eqs. (5) and (9), we see that the conditions (26)-(29) are equivalent to Eqs. (23)-(25).

Proposition 4.1 implies that the system (14)-(16) of uncertain equilibrium equations is equivalent to the system (23)-(25). In comparison with the original system (14)-(16), it is of interest to note that the unknown parameters ζ_f and ζ_a appear only on the right-hand side of Eqs. (23)-(25). We next eliminate these unknown parameters by using the quadratic embedding technique (Calafiore and El Ghaoui 2004, Kanno and Takewaki 2006).

Letting

$$n = n^m + n^d$$

define the vector $\xi \in \mathbb{R}^n$ by

$$\xi = \begin{pmatrix} \mathbf{p} \\ \mathbf{u} \\ 1 \end{pmatrix} \quad (30)$$

Moreover, define the constant matrices $\Omega_0 \in \mathcal{S}^{n+1}$, $\Omega_{\tilde{f}} \in \mathcal{S}^{n+1}$, and $\Omega_{ai} \in \mathcal{S}^{n+1}$ by

$$\Omega_0 = - \begin{pmatrix} \Psi^T \\ \tilde{K} \\ -\tilde{f}^T \end{pmatrix} E_0 E_0^T (\Psi \tilde{K} - \tilde{f}) \quad (31)$$

$$\Omega_{fj} = \text{Diag}(\mathbf{0}, \mathbf{0}, (f_j^0)^2) - \begin{pmatrix} \Psi^T \\ \tilde{K} \\ -\tilde{f}^T \end{pmatrix} e_j e_j^T (\Psi \tilde{K} - \tilde{f}), \quad j = 1, \dots, n^f \quad (32)$$

$$\Omega_{ai} = (a_i^0)^2 \begin{pmatrix} \mathbf{0} \\ \mathbf{b}_i \\ 0 \end{pmatrix} (\mathbf{0}^T \mathbf{b}_i^T 0) - \text{Diag}(e_i, \mathbf{0}, 0), \quad i = 1, \dots, n^m \quad (33)$$

Proposition 4.2. *There exist ζ_f and ζ_a satisfying (23)-(25) if and only if ξ defined by (30) satisfies*

$$\xi^T \Omega_0 \xi \geq 0 \quad (34)$$

$$\xi^T \Omega_{\tilde{f}} \xi \geq 0, \quad j = 1, \dots, n^f \quad (35)$$

$$\xi^T \Omega_{ai} \xi \geq 0, \quad i = 1, \dots, n^m \quad (36)$$

Proof. Observe that the Eq. (23) is equivalently rewritten as the inequality

$$\|E_0^T [\tilde{K} \mathbf{u} + \Psi \mathbf{p} - \tilde{f}] \|_2^2 \leq 0 \quad (37)$$

because the left-hand side of Eq. (37) is nonnegative. From the definition (31) of Ω_0 , we see that (37) is equivalent to Eq. (34). For $s \in \mathbb{R}$, $t \in \mathbb{R}$, $\zeta \in \mathbb{R}$, observe that the implication

$$s = \zeta t, 1 \geq |\zeta| \Leftrightarrow s^2 \leq t^2 \quad (38)$$

holds. By applying Eqs. (38) to (24), we see that there exists a vector ζ_f satisfying Eq. (24) if and only if the inequalities

$$(e_j^T [\tilde{K}\mathbf{u} + \Psi\mathbf{p} - \tilde{\mathbf{f}}])^2 \leq (f_j^0)^2, \quad j = 1, \dots, n^f \quad (39)$$

are satisfied. From the definition (32) of Ω_{f_i} , we see that the condition Eqs. (39) is equivalently rewritten as (35). Moreover, by applying (38) to (25), we see that there exists a vector ζ_a satisfying Eq. (25) if and only if the inequalities

$$p_i^2 \leq (a_i^0 b_i^T \mathbf{u})^2, \quad i = 1, \dots, n^m \quad (40)$$

are satisfied. From the definition Eq. (33) of Ω_{a_i} , we see that the condition (40) is equivalently rewritten as (36).

Proposition 4.2 implies that the uncertain linear Eqs. (23)-(25) are equivalent to a finite number of quadratic inequalities (34)-(36). It should be emphasized that the unknown parameters ζ_f and ζ_a have been eliminated as a result of this quadratic embedding.

We next consider the constraint condition of (21). Let $\mathbf{w}_f = (w_{fj}) \in \mathbb{R}^{n^f}$, $\mathbf{w}_a = (w_{ai}) \in \mathbb{R}^{n^m}$, and $w_0 \in \mathbb{R}$. Define the matrix-valued function $Y: \mathbb{R}^{n^f} \times \mathbb{R}^{n^m} \times \mathbb{R} \rightarrow \mathcal{S}^{r+n+1}$ by

$$Y(\mathbf{w}_f, \mathbf{w}_a, w_0) = \sum_{j=1}^{n^f} w_{fj} \Omega_{fj} + \sum_{i=1}^{n^m} w_{ai} \Omega_{ai} + w_0 \Omega_0 \quad (41)$$

Proposition 4.3. *The condition*

$$\mathcal{U}_G \subseteq \mathcal{E}(\mathbf{P}, \hat{\mathbf{u}})$$

is satisfied if there exist \mathbf{w}_f , \mathbf{w}_a , and w_0 satisfying

$$\begin{pmatrix} P & (O \ G^T \ -\hat{\mathbf{u}}) \\ \begin{pmatrix} O \\ G \\ -\hat{\mathbf{u}}^T \end{pmatrix} \text{Diag}(\mathbf{0}, \mathbf{0}, 1) - Y(\mathbf{w}_f, \mathbf{w}_a, w_0) \end{pmatrix} \succeq O$$

$$\mathbf{w}_f \geq \mathbf{0}, \mathbf{w}_a \geq \mathbf{0}, w_0 \geq 0$$

Proof. It follows from Proposition 4.1 and Proposition 4.2 that the condition $\mathbf{u} \in \mathcal{U}$ is equivalent to (34)-(36). Then this proposition can be shown in a manner similar to Proposition 4.2 in Kanno and Takewaki (2006).

Proposition 4.3 provides a sufficient condition of the constraint condition of the problem (21). It should be emphasized that the matrix Y defined by Eq. (41) depends on the variables $(\mathbf{w}_f, \mathbf{w}_a, w_0)$ linearly. This naturally motivates us to solve the following problem in the variables $P \in \mathcal{S}^r$, $\hat{\mathbf{u}} \in \mathbb{R}^r$, $\mathbf{w}_f \in \mathbb{R}^{n^f}$, $\mathbf{w}_a \in \mathbb{R}^{n^m}$, and $w_0 \in \mathbb{R}$:

Table 1 Definition of the nominal load \tilde{f}

Node	Force (kN)	
	x -direction	y -direction
(b)	300.0	0
(c)	400.0	0
(d)	500.0	0
(e)	600.0	0
(f)	700.0	-2000.0
(l)	0	-2000.0

Table 2 Definition of the uncertainty magnitudes f^0 of the external force

Node	Force (kN)	
	x -direction	y -direction
(b), (h)	20.0	20.0
(c), (i)	20.0	20.0
(d), (j)	60.0	20.0
(e), (k)	80.0	20.0
(f), (l)	100.0	20.0

$$\min \text{tr}(P)$$

$$s.t. \left(\begin{array}{c} P \\ O \\ G \\ -\hat{\mathbf{u}}^T \end{array} \right) \left(\begin{array}{c} O \\ G^T \\ -\hat{\mathbf{u}} \end{array} \right) \geq O \quad \text{Diag}(\mathbf{0}, \mathbf{0}, 1) - Y(\mathbf{w}_f, \mathbf{w}_a, w_0) \right) \geq O \quad (42)$$

$$\mathbf{w}_f \geq \mathbf{0}, \mathbf{w}_a \geq \mathbf{0}, w_0 \geq 0$$

which yields a bounding ellipsoid of \mathcal{U}_G , that is optimal in the sense of the sufficient condition provided by Proposition 4.3. Note that (42) is an SDP problem. Indeed, (42) can be embedded into the standard form (1) of SDP. Hence, we can solve (42) efficiently by using the primal-dual interior-point method (Helmburg 2002).

5. Numerical experiments

The minimum bounding ellipsoids are computed for various static response parameters of the braced frame illustrated in Fig. 1. We solve the SDP problem (42) by using SeDuMi Ver. 1.05 (Sturm 1999), which implements the primal-dual interior-point method for the linear programming problems over symmetric cones. Computation has been carried out on Pentium M (1.5 GHz with 1.0 GB memory) with MATLAB Ver. 6.5.1 (2002).

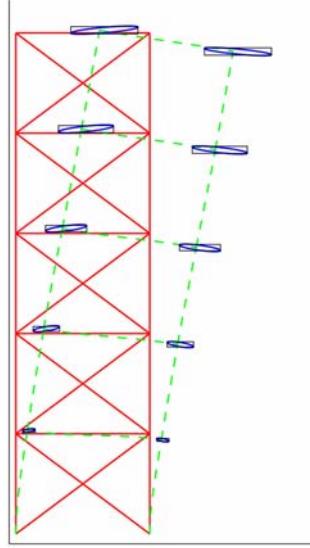


Fig. 2 Bounding ellipsoids and bounding boxes of the nodal displacements for Case 1 and the deformation at the nominal situation (displacements amplified 10 times)

The frame shown in Fig. 1 consists of 10 columns, 5 beams, and 10 braces, i.e., $n^m = 10$. The columns and beams are modeled as the beam-column elements, while the braces are modeled as the truss elements. The nodes (a) and (g) are the fixed-supports, and hence $n^d = 30$. We set $W = 400.0$ cm and $H = 300.0$ cm. The elastic modulus is 200 GPa.

5.1 Bounds for nodal displacements

The cross-sectional area and the second moment of cross section of each beam are 60.0 cm^2 and 500.0 cm^4 , respectively. For each column, the cross-sectional area and the second moment of cross section are set as 40.0 cm^2 and 213.3 cm^4 , respectively. The nominal cross-sectional areas of the braces are

$$\tilde{a}_i = 20.0 \text{ cm}^2, i = 1, \dots, 10$$

As the nominal external force \tilde{f} , the nodal loads are applied to the nodes (b)-(f) and (l) as listed in Table 1. Note that no external moments are applied. The dashed lines in Fig. 2 depict the deformation of the frame with the nominal cross-sectional areas \tilde{a} of the braces subjected to the nominal external load f , where the displacements are amplified 10 times.

As an uncertainty model of f , suppose that uncertain external forces may possibly be applied at all free nodes (b)-(f) and (h)-(l). Note again that no uncertain external moments are considered. Accordingly, we see that $n^f = 20$ in (3). The coefficients f^0 of the uncertainty in (4) are listed in Table 2. Consequently, the external force f is running through the rectangles depicted with the dashed lines in Fig. 1 (ii). As the uncertainty models of a introduced in (7), we consider the following two cases:

$$\text{Case 1: } a_i^0 = 6.0 \text{ cm}^2, i = 1, \dots, 10;$$

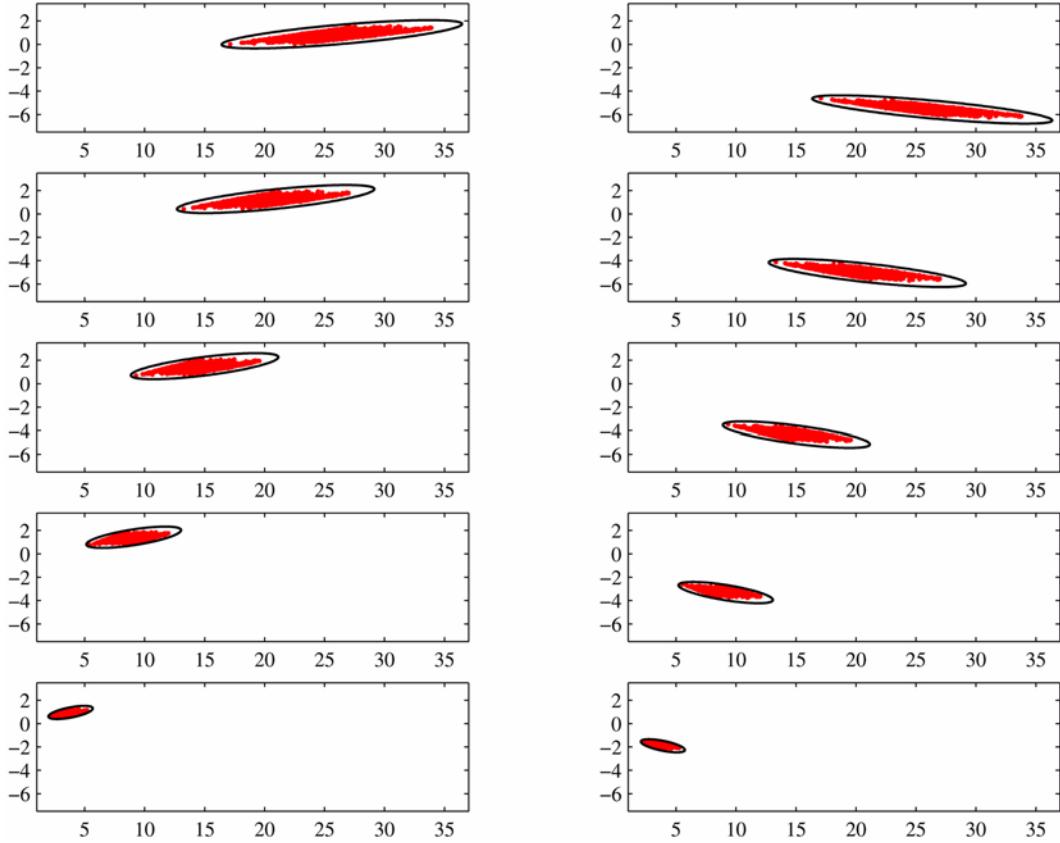


Fig. 3 Bounding ellipsoids and the nodal displacements (in cm) for randomly generated ζ_a and ζ_f in Case 1

$$\text{Case 2: } a_i^0 = 12.0 \text{ cm}^2, i = 1, \dots, 10.$$

We first consider Case 1. Based on the formulation investigated in Example 3.1, we compute the minimal bounding ellipsoid of the nodal displacement of each free node by solving the SDP problem (42). Note that we have solved 10 SDP problems in total. Each SDP problem has 36 variables, 31 linear inequalities, and the constraint that the symmetric matrix in S^{43} should be positive semidefinite. The average and the standard deviation of CPU time, respectively, required for solving one SDP problem are 2.01 sec and 0.32 sec. Similarly, according to Example 3.2, we compute the minimal bounding interval for each component of the displacements by solving the SDP problem (42). Note that we do not compute the bounding intervals for the rotation angles, and hence we have solved 20 SDP problems in total. Each SDP problem has 33 variables, 31 linear inequalities, and the constraint that the symmetric matrix in S^{42} should be positive semidefinite. The average and the standard deviation of CPU time, respectively, required for solving one SDP problem are 2.08 sec and 0.34 sec. The obtained ellipsoids and intervals (boxes) are illustrated in Fig. 2 at the locations of the corresponding nodes.

In order to verify these results, we randomly generate a number of ζ_a and ζ_f , and compute the corresponding displacements. The obtained displacements are shown by many points in Fig. 3 and

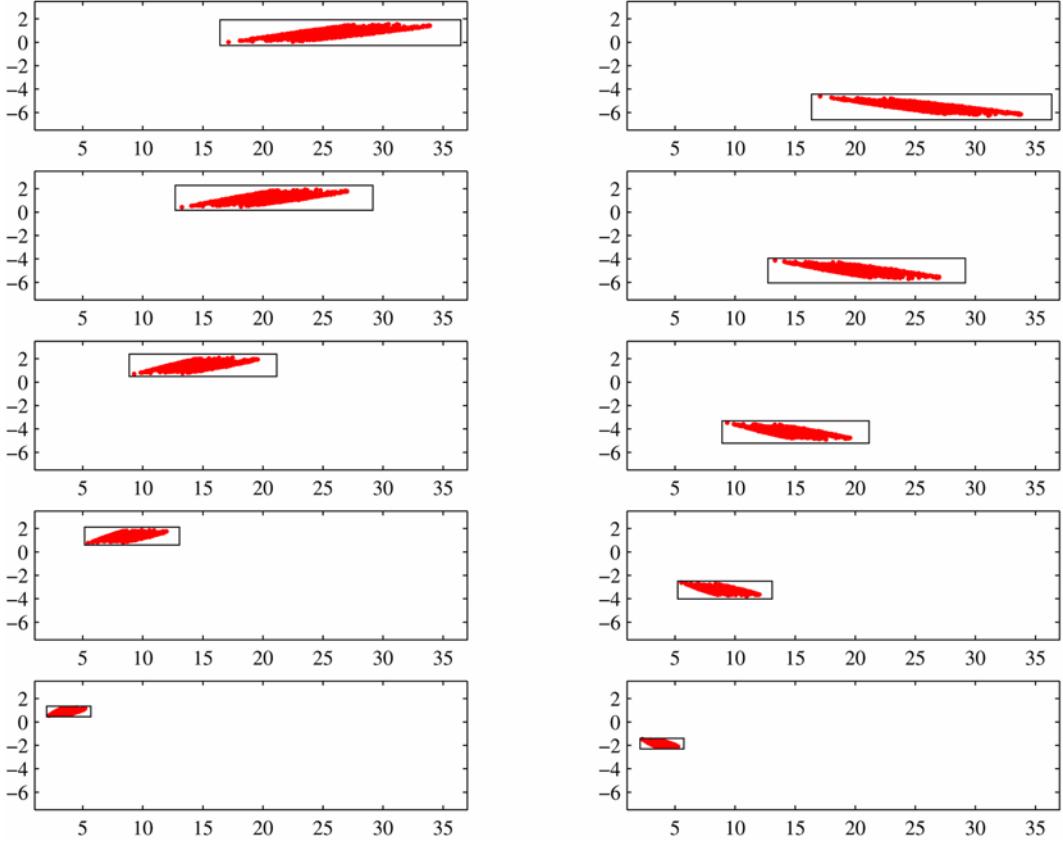


Fig. 4 Bounding boxes and the nodal displacements (in cm) for randomly generated ζ_a and ζ_f in Case 1

Fig. 4 with the bounding ellipsoids and intervals, respectively. It is observed in Fig. 3 and Fig. 4 that all generated displacements are included in the bounds computed, which ensures that the obtained bounding ellipsoids and intervals correspond to outer approximations of the sets of nodal displacements. In addition, the bounding ellipsoids seem to represent the characteristics of distribution of the nodal displacements more clearly and intuitively compared with the bounding intervals. We can also see that these bounds are sufficiently tight.

The bounds of nodal displacements are also computed for Case 2. The minimum ellipsoids and intervals obtained are illustrated in Fig. 5 at the locations of the corresponding nodes. Fig. 6 and Fig. 7 depict the ellipsoids and intervals obtained, respectively, as well as the displacements corresponding to randomly generated ζ_a and ζ_f . It should be emphasized that these bounds are very tight, even though in Case 2 we consider very large magnitudes of perturbations of a .

5.2 Bounds for stresses

We next consider the uncertainties of stresses of beams and columns. The uncertainty modeling of a and f is the same as Case 1 in section 5.1.

Based on the formulation investigated in Example 3.3, we compute the minimal bounding ellipsoid of (σ_1, σ_2) , where σ_1 and σ_2 denote the extreme-fiber normal stress at the lower end of the

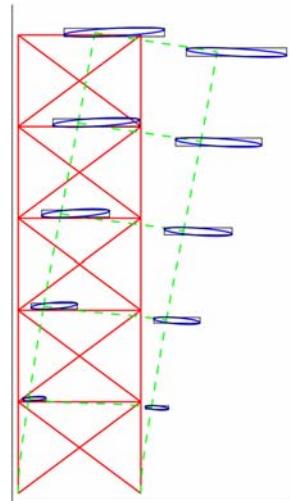


Fig. 5 Bounding ellipsoids and bounding boxes of the nodal displacements for Case 2 and the deformation at the nominal situation (displacements amplified 10 times)

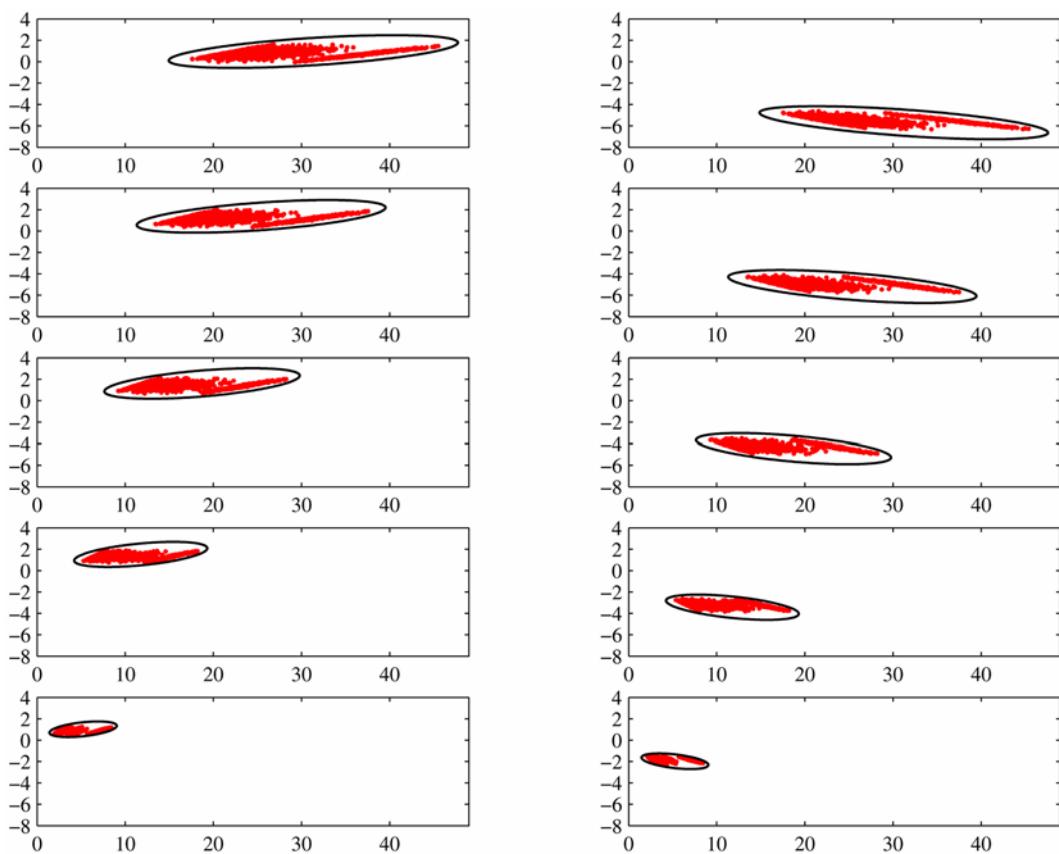


Fig. 6 Bounding ellipsoids and the nodal displacements (in cm) for randomly generated ζ_a and ζ_f in Case 2

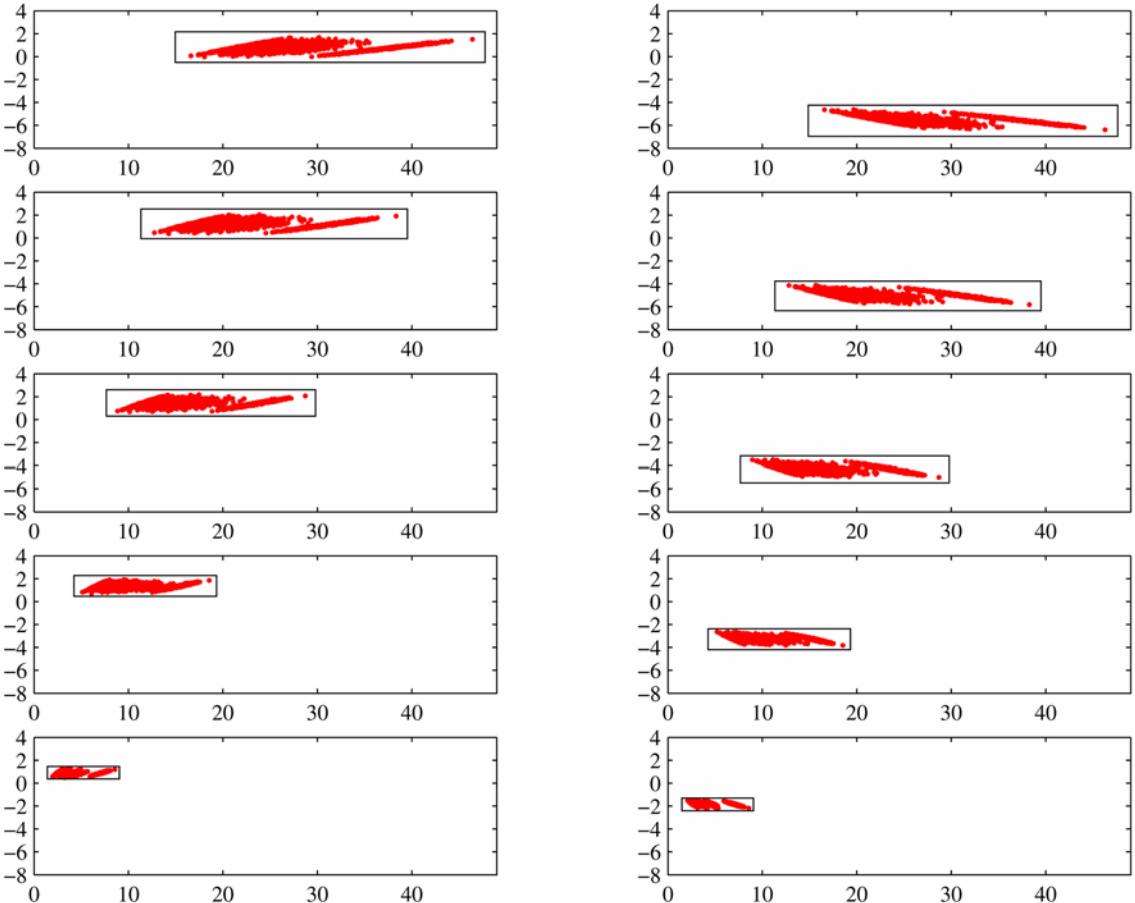


Fig. 7 Bounding boxes and the nodal displacements (in cm) for randomly generated ζ_a and ζ_f in Case 2

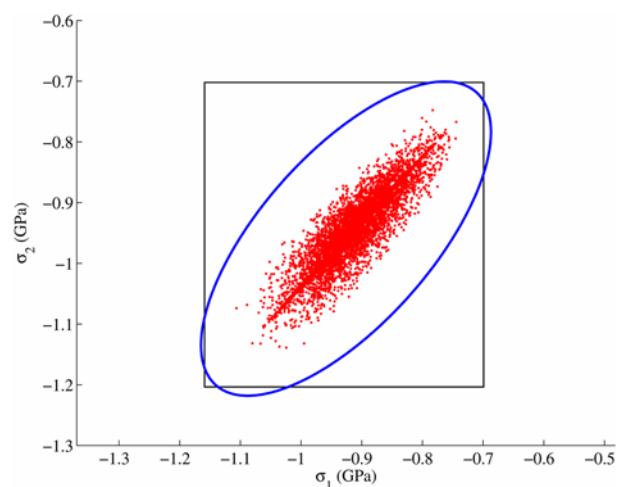


Fig. 8 Bounding ellipsoid and bounding box of the extreme-fiber nominal stress vector (σ_1 , σ_2) at the lower end of the column (1) together with the stresses for randomly generated ζ_a and ζ_f in Case 1

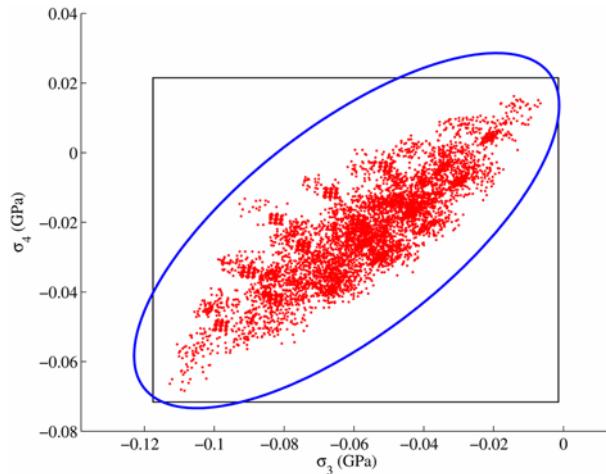


Fig. 9 Bounding ellipsoid and bounding box of the extreme-fiber nominal stress vector (σ_3 , σ_4) at the left end of the beam (2) together with the stresses for randomly generated ζ_a and ζ_f in Case 1

column (1) shown in Fig. 1. The bounding intervals of σ_1 and σ_2 can also be obtained by solving (42) in a manner similar to Example 3.2. The minimum bounding ellipsoid and intervals of (σ_1, σ_2) are illustrated in Fig. 8. We also generate a number of ζ_a and ζ_f randomly, and the corresponding stresses (σ_1, σ_2) are depicted by a number of points in Fig. 8. It is observed in Fig. 8 that all generated stresses are included in the ellipsoidal and interval bounds obtained.

Similarly, consider the extreme-fiber normal stresses σ_3 and σ_4 at the left end of beam (2) shown in Fig. 1. The minimum bounding ellipsoid and box computed are illustrated in Fig. 9, as well as the stresses corresponding to randomly generated ζ_a and ζ_f . It is observed from Fig. 9 that the bounds obtained are sufficiently tight.

6. Conclusions

In this paper, an optimization-based approach has been proposed for computing ellipsoidal deterministic confidence bounds on static response of braced frames including uncertainties. Both of the external loads and the member stiffnesses of braces are assumed to be imprecisely known.

It is shown that an ellipsoidal bound for all realizations of static response of a braced frame can be obtained efficiently by solving a convex optimization problem, which is called a semidefinite programming (SDP) problem. By using the quadratic embedding of uncertainty parameters and the S-lemma, we proposed a sufficient condition that an ellipsoid includes all possible realizations of static response. Based on this sufficient condition, we formulated an SDP problem which approximates the problem for finding the minimal bounding ellipsoid.

It should be emphasized that most of convex model approaches have been developed based on the first-order perturbation, while the proposed method uses a semidefinite relaxation technique. It is known that an SDP problem can be solved within the polynomial time of the problem size by using the primal-dual interior-point method. Hence, our method has polynomial-time complexity of problem size, whereas interval calculus approaches have in general exponential complexity. Compared with confidence intervals, confidence ellipsoids may help intuitive understanding of characteristics

of mechanical behaviors, e.g. distribution of nodal displacements.

In the numerical examples, the SDP problem presented has been solved by using the primal-dual interior-point method. It has been shown that bounding ellipsoids of nodal displacements and stresses of beam-column elements can be obtained effectively. We have also illustrated that the obtained ellipsoidal and interval bounds are sufficiently tight even for large magnitudes of perturbations of external loads and brace member stiffnesses.

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