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A parametric study on the free vibration of a functionally graded material circular plate with non-uniform thickness resting on a variable Pasternak foundation by differential quadrature method

Bassem M. Abdelbaki*, Mohamed E. Sayed Ahmed[a] and Ahmed M. Al Kaisy[b]

Mathematics and Physics Department, Faculty of Engineering, Fayoum University, Fayoum, Egypt

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Abstract. This paper presents a parametric study on the free vibration analysis of a functionally graded material (FGM) circular plate with non-uniform thickness resting on a variable Pasternak elastic foundation. The mechanical properties of the material vary in the transverse direction through the thickness of the plate according to the power-law distribution to represent the constituent components. The equation of motion of the circular plate has been carried out based on the classical plate theory (CPT), and the differential quadrature method (DQM) is employed to solve the governing equations as a semi-analytical method. The grid points are chosen based on Chebyshev-Gauss-Lobatto distribution to achieve acceptable convergence and better accuracy. The influence of geometric parameters, variable elastic foundation, and functionally graded variation for clamped and simply supported boundary conditions on the first three natural frequencies are investigated. Comparisons of results with similar studies in the literature have been presented and two-dimensional mode shapes for particular plates have been plotted to illustrate the effect of variable thickness profile.

Keywords: circular plates; DQM; FGM; free vibration; Pasternak foundation; variable thickness

1. Introduction

In modern industries, circular plates resting on an elastic foundation are among the most widely utilized engineering applications and have been extensively employed for building footings, raft foundation of water tanks, bridge decks, aerospace industries such as air crafts, and other engineering fields, which reflects the importance of circular plates. Furthermore, reducing material consumption is increasingly essential for product development and fabrication. Using 3D printing technology makes it possible to design and produce high-strength lightweight structures. For most components, extra material with regular production methods can be removed at the design stage. During production, material is only used where it is functionally necessary which appears clearly

*Corresponding author, Ph.D. Student, E-mail: bma11@fayoum.edu.eg
[a]Professor, E-mail: mes00@fayoum.edu.eg
[b]Ph.D., E-mail: ama02@fayoum.edu.eg

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in aerospace industry. Recently, researchers have shown a growing interest in a new class of advanced composite material called functionally graded materials (FGMs). This composite material with high mechanical performance can be achieved by continuous variation of the volume fraction of constituent materials (Shen 2016). FGMs were firstly introduced by (Yamanouchi *et al.* 1990). Since then, numerous studies have been conducted to employ and develop high-resistant materials using FGMs.

The dynamic and static analysis of FGM circular plates was investigated by some researchers adopting analytical and numerical solutions (Reddy *et al.* 1999, Ma and Wang 2004, Prakash and Ganapathi 2006, Civalek and Ersoy 2009, Zheng and Zhong 2009). (Ma and Wang 2004) investigated the axisymmetric bending and buckling of FGM circular plates using the theory of third-order shear deformation by studying the effects of gradual change in mechanical properties of the material through the direction of the plate thickness. (Reddy *et al.* 1999) studied the axisymmetric bending analysis of FGM circular and annular plates based on Mindlin-first-order shear deformation theory. The study reveals a relationship between the classical plate theory and the first-order plate theory. (Zheng and Zhong 2009) evaluated the effect of the FGM properties on the axisymmetric bending of circular plates under two types of boundary conditions: elastically supported edge and rigid slipping edge. The exact solution obtained by expanding the transverse displacement function as the Fourier-Bessel series shows a great influence of the FGM properties on the mechanical behavior of the circular plate. (Prakash and Ganapathi 2006) studied the free vibration analysis of FGM circular plates in a thermostatic environment using a finite element approach. A detailed investigation was carried out to study the influences of considered gradient index and temperature on the critical buckling load. (Civalek and Ersoy 2009) employed a discrete singular convolution approach to study the dynamic and bending analysis of circular plates based on Mindlin plate theory. The obtained numerical results of the frequency parameter and deflection of the plate show the accuracy and the efficiency of the adopted approach.

A considerable amount of research has so far explored on the influence of elastic foundation on circular plates. The simplest model employed by the researcher to idealize the behavior of plate-foundation interaction is the well-known Winkler model. In this model, the plate is supported by linear elastic springs, where its reaction is proportional to the transversal displacement of the plate by a proportionality coefficient (k_w) called Winkler's modulus (Birman 2011). (Farhatnia *et al.* 2018) investigated the influence of the Winkler foundation on the thermal-mechanical bending of thick FGMs circular plate under the effect of various mechanical and thermal loading. Gupta *et al.* (Gupta *et al.* 2006) studied the buckling and vibrational behavior of circular plates with variable thickness rested on the Winkler foundation based on classical plate theory. However, other researchers considered Winkler's modulus to be variable, not constant. For example, (Rad and Shariyat 2013) studied the coupling effect of the variation in the distribution of elastic foundation on the bending analysis of FGM annular under the action of normal and in-plane shear tractions. An extension to the Winkler foundation model is extended to include the collaboration between the springs by implementing the Pasternak foundation model, which is a two-parameter model taking into consideration the shear rigidity of the foundation. (Hosseini-Hashemi *et al.* 2010) analyzed buckling and free vibration of FGM circular and annular plates subjected to in-plane compressive forces resting on the Pasternak foundation.(Abdelbaki *et al.* 2021) studied the bending behavior of an FGM circular plate partially resting on a variable Pasternak foundation. The radial stress and deflection were investigated when the plate is not fully resting on elastic foundation besides illustrating the effects of contact area ratio between the plate and the elastic foundation. (Zhou *et al.* 2006) studied the three-dimensional dynamic behavior of circular plates resting on the

Pasternak foundation. The frequency parameter was derived using the Ritz method by considering the effect of the strain energy of the elastic foundation.

A semi-analytical method called the differential quadrature method (DQM) has recently grabbed the attention of many researchers in several engineering domains. Remarkable achievements have been accomplished by many researchers in vibration and bending analysis of plates, shells, and beams. For instance, (Liew *et al*. 1997) presented the vibrational characteristic of uniform thick circular plate based on the linear shear-deformation Mindlin theory using DQM. (Gupta *et al*. 2006) employed DQM for a free vibration analysis of FGM circular plate with variable thickness. In this study, the effect of non-homogeneity and geometric parameters were studied on the natural frequency of the circular plate. (Hamzehkolaei *et al*. 2011) investigated the thermal effect on the flexural analysis of the FGM circular and annular plates. The initial thermal stresses of the circular plate were obtained by DQM and a comparison was performed to confirm the accuracy of the adopted method. (Nie and Zhong 2007) studied the axisymmetric bending of two-functionally graded circular and annular plates using DQM. Numerical results of the deflection of the plate and radial stresses show the accuracy and the computational efficiency of the DQM. (Arshid and Khorshidvand 2018) provided the vibration characteristic of a thin circular plate made up of a porous filled by fluid and merged with piezoelectric actuator patches. Hamilton principle and the classical plate theory were used to obtain the equation of motion of the plate. The DQM was employed to obtain the natural frequency in addition to the radial and circumferential stresses.

In the presented article, a semi-analytical solution for free vibration analysis of FGM circular plate with variable thickness resting on a variable Pasternak foundation is carried out using DQM. The present work attempts to incorporate the effects of the variation in the thickness profile of the plate, FGM power index, and elastic foundation modulus on the natural frequencies and the mode shapes of the circular plate for clamped and simply supported edges. Furthermore, lack of a semi-analytical solution to the problem of circular plates resting on an elastic foundation with a variable-modulus subgrade has existed. Therefore, a more general approach must be investigated to deal with the problem. The numerical results are verified upon comparison with the cited literature to demonstrate the accuracy and computational efficiency of the adopted method. It is worth mentioning that no detailed investigation has been adopted until now to analyze the free vibration of non-uniform functionally graded circular plate supported on two-parameter elastic foundation by the Differential Quadrature method (DQM).

2. Mathematical formulation

2.1 Geometric and foundation parameters

Consider an FGM circular plate with variable thickness resting on a two-parameter elastic foundation. The first parameter is the variable Winkler foundation coefficient ($k_w (r)$) which varies in the radial direction and is represented by a set of linear elastic springs. In addition, the constant Pasternak shear coefficient (k_p) in the transverse direction is represented by a shear layer under the plate. The plate configuration is represented using cylindrical polar coordinates system (r, θ, z) as shown in Fig. 1: where R is the radius of the plate, h_0 is the initial thickness at the center of the plate, $h(r)$ is the thickness at any distance r, and z=0 at the middle of the plate plane. The top and the bottom surfaces are at $z=h(r)/2$ and $z=-h(r)/2$, respectively.

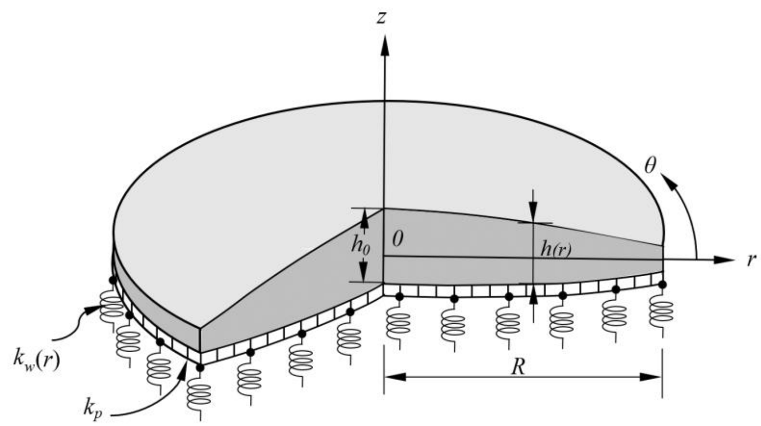


Fig. 1 An FGM circular plate with variable thickness resting on a Pasternak foundation

2.2 Material properties, thickness profile, and foundation parameters

The material properties of the plate gradually change based on the power-law distribution through the thickness direction from ceramic to metal. It is assumed that the material properties (Young's modulus of elasticity and mass density) at any arbitrary point of the thickness located at any distance z from the mid-plane of the plate and any radial distance r, may be calculated based on the following equations

$$E(z) = \left(E_c - E_m\right)\left(\frac{z}{h} + \frac{1}{2}\right)^g + E_m \tag{1}$$

$$\rho(z) = \left(\rho_c - \rho_m\right)\left(\frac{z}{h} + \frac{1}{2}\right)^g + \rho_m \tag{2}$$

where E_c, ρ_c and E_m, ρ_m are Young's modulus of elasticity and mass density per unit of ceramic and metal, respectively, and g is the volume fraction exponent which varies from (0 to ∞), representing a plate with material varying from a fully ceramic plate to a fully metal plate, respectively. Fig. 1 illustrates the variation in modulus of elasticity of the plate through the thickness direction from the lower face to the upper face according to the volume fraction index (g). h is the plate thickness varying parabolically along the radial direction according to the relation taken as follows (Farhatnia, Saadat *et al*. 2019)

$$h(r) = h_0\left(1 + \gamma\left(\frac{r}{R}\right)^{\beta=2}\right) \tag{3}$$

where h_0 denotes the reference thickness at the center of the plate, γ is a geometrical parameter controlling the thickness concavity, as the curvature goes upward in case of $\gamma > 0$, goes downward when $\gamma < 0$, and a uniform thickness when $\gamma = 0$.

By substituting the value of the h which is a function of the radial coordinate in Eqs. (1) and (2), the following equations for $E(z)$ and $\rho(z)$ yield

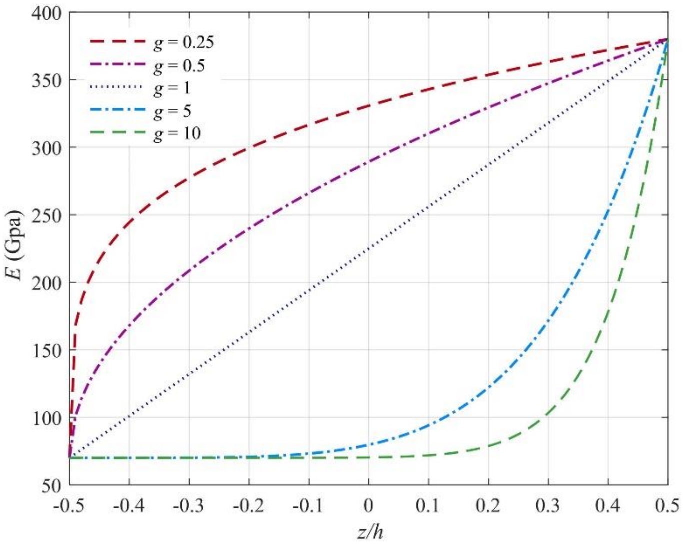


Fig. 2 Variation of material properties through thickness direction.

$$E(z) = \left(E_c - E_m\right)\left(\frac{z}{h_0\left(1+\gamma\left(\frac{r}{R}\right)^{\beta=2}\right)} + \frac{1}{2}\right)^g + E_m \tag{4a}$$

$$\rho(z) = \left(\rho_c - \rho_m\right)\left(\frac{z}{h_0\left(1+\gamma\left(\frac{r}{R}\right)^{\beta=2}\right)} + \frac{1}{2}\right)^g + \rho_m \tag{4b}$$

Assume the plate is resting on a two-parameter elastic foundation, where the elastic modulus in the normal direction is varying parabolically following the adopted relation by (Abdelbaki and Ahmed 2022) that describes the variation in the soil subgrade, and the plate maintains in continuous contact with the foundation. Hence, the $k_w(r)$ modulus can be expressed as

$$k_w(r) = k_0\left(1+\alpha\left(\frac{r}{R}\right)^2\right) \tag{5}$$

where, α is a foundation parameter that manipulates the variation in Winkler foundation modulus.

2.3 Governing equations of free vibration

The differential equation of free vibration of the circular plate resting on the two-parameter elastic foundation can be obtained from the classical plate theory (Leissa 1969) by assuming no external forces and no in-plane stresses, and by adding the terms of Pasternak foundation

(Hosseini-Hashemi *et al*. 2010). So, the equation of motion of the transverse displacement w of an axisymmetric circular plate can be written as (Gupta *et al*. 2006)

$$D\frac{d^4w}{dr^4}+\frac{2}{r}\left(D+r\frac{dD}{r}\right)\frac{d^3w}{dr^3}+\frac{1}{r^2}\left(-D+\left(2+\nu\right)r\frac{dD}{dr}+r^2\frac{d^2D}{dr^2}\right)\frac{d^2w}{dr^2}+ \frac{1}{r^3}\left(D-r\frac{dD}{dr}+r^2\nu\frac{d^2D}{dr^2}\right)\frac{dw}{dr}+k_w(r)w-k_p\nabla^2w=-I\frac{d^2w}{dt^2} \tag{6}$$

where, ∇^2 is the Laplacian operator that can be expressed as follows for axisymmetric problems

$$\nabla^2=\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}\right) \tag{7}$$

D is the elastic flexural rigidity at any radial distance r; w is the deflection of the plate; I is the mass inertia, and ν is Poisson's ratio assumed to be constant (Shen 2016).

The flexural rigidity and mass inertia of the FGM plate can be determined as

$$D(r)=\frac{1}{1-\nu^2}\int_{-h/2}^{h/2}E(z)z^2dz \tag{8}$$

$$I(r)=\int_{-h/2}^{h/2}\rho(z)dz \tag{9}$$

Substituting the value of $\rho(z)$ from Eq. (4a) into Eq. (9), we get the following relation

$$I(r)=\int_{-h/2}^{h/2}\rho(z)dz=\int_{-h/2}^{h/2}\left((\rho_c-\rho_m)\left(\frac{z}{h}+\frac{1}{2}\right)^g+\rho_m\right)dz=\rho_c h_0 A(\beta)B \tag{10}$$

where

$$A(\beta)=\left(1+\gamma\left(\frac{r}{R}\right)^{\beta=2}\right),\qquad B=\left(\frac{\rho_c+\rho_m g}{\rho_c\left(g+1\right)}\right) \tag{11}$$

The harmonic solution of Eq. (6) can be expressed in the form

$$w=w(r)e^{i\omega t} \tag{12}$$

where, ω is the radian frequency of the plate. Therefore, Eq. (6) can be written as

$$D\frac{d^4w}{dr^4}+\frac{2}{r}\left(D+r\frac{dD}{r}\right)\frac{d^3w}{dr^3}+\frac{1}{r^2}\left(-D+\left(2+\nu\right)r\frac{dD}{dr}+r^2\frac{d^2D}{dr^2}\right)\frac{d^2w}{dr^2}+ \frac{1}{r^3}\left(D-r\frac{dD}{dr}+r^2\nu\frac{d^2D}{dr^2}\right)\frac{dw}{dr}-k_p\frac{d^2w}{dr^2}-k_p\frac{1}{r}\frac{dw}{dr}+k_ww=\rho h\omega^2w \tag{13}$$

By introducing the following non-dimensional parameters to simplify the equation of motion

$$\varphi=\frac{r}{R},\quad W=\frac{w}{h},\quad K_w=\frac{\left(k_wR^4\right)}{D_0},\quad K_p=\frac{\left(k_pR^2\right)}{D_0},\quad \Omega=\omega R^2\sqrt{\frac{\rho_c h_0}{D_0}} \tag{14}$$

where, $D_0 = E_0 h_0/12(1-\nu^2)$ is the reference flexural rigidity, Eq. (13) may be re-written as

$$\begin{aligned}&\frac{D}{D_0}\varphi^3\frac{d^4W}{d\varphi^4}+\frac{2\varphi^2}{D_0}\left(D+\varphi\frac{dD}{\varphi}\right)\frac{d^3W}{d\varphi^3}+\frac{\varphi}{D_0}\left(-D+(2+\nu)\varphi\frac{dD}{d\varphi}+\varphi^2\frac{d^2D}{d\varphi^2}\right)\frac{d^2W}{d\varphi^2}+\\&\frac{1}{D_0}\left(D-\varphi\frac{dD}{d\varphi}+\varphi^2\nu\frac{d^2D}{d\varphi^2}\right)\frac{dW}{d\varphi}-K_p\varphi^3\frac{d^2W}{d\varphi^2}-K_p\varphi^2\frac{dW}{d\varphi}+K_w\varphi^3W=\varphi^3\Omega^2A(\beta)BW\end{aligned} \tag{15}$$

2.4 Boundary and regularity conditions

Two boundary conditions at the outer perimeter of the plate are defined in terms of non-dimensional deflection W, as:

- Clamped (C)

$$W(1)=0,\quad \left.\frac{dW}{d\varphi}\right|_{\varphi=1}=0 \tag{16}$$

- Simply-supported (S):

$$W(1)=0,\quad M_\varphi\Big|_{\varphi=1}=-D\left(\frac{d^2W}{d\varphi^2}+\nu\left(\frac{1}{\varphi}\frac{dW}{d\varphi}\right)\right)=0 \tag{17}$$

For the axisymmetric problems, the regularity condition must be implemented to ensure that the slope at the center of the plate is zero (Bert *et al.* 1987). Then, the regularity condition in terms of non-dimensional deflection W can be taken as

$$\left.\frac{dW}{d\varphi}\right|_{\varphi=0}=0 \tag{18}$$

3. Method of solution

3.1 Differential Quadrature method

The concept of the DQM is to approximate the derivative of a function at a grid point as a weighted linear combination of all the functional values at all given grid points in the domain. Therefore, by discretizing the plate into N number of grid points that are applicable in the computational range [0,1], the differential equation can be reduced to a set of algebraic equations. The number of equations depends on the selected number of sample points.

$$\frac{d^nW(r_i)}{dr^n}=\sum_{j=1}^{N}C_{ij}^{(n)}W(x_j),\qquad i=1,2,....,N \tag{19}$$

where $C_{ij}^{(n)}$ denotes for the weighting coefficient, and n indicated the nth order partial derivative of the function $W(r)$. The determination of the weighting coefficients is the basic procedure in the DQ approximation. Eq. (19) takes the form of Vandermonde in the matrix, and in the case of a large number of grid points, the solution can be ill-conditioned. To overcome this obstacle, the

Lagrange interpolated polynomial was introduced in (Shu 2000) to obtain the weighting coefficients as follows

$$l(r) = \frac{M(r)}{M^{(1)}(r_j)(x - x_j)}, \qquad \text{for } j = 1, 2, ..., N \tag{20}$$

where

$$M(r) = \prod_{j=1}^{N}(r - r_j), \quad M^{(1)}(r_i) = \prod_{j=1, i\neq j}^{N}(r_i - r_j), \qquad \text{for } i, j = 1, 2, ..., N \tag{21}$$

Substituting Eq. (21) in Eq. (20), the weighting coefficients for the first and higher-order derivatives can be determined as follows

$$\begin{gathered} C_{ij}^{(1)} = \frac{M^{(1)}(r_i)}{(r_i - r_j)M^{(1)}(r_j)}, \quad C_{ij}^{(n)} = n\left(C_{ii}^{(n-1)}C_{ij}^{(1)} - \frac{C_{ij}^{(n-1)}}{r_i - r_j}\right), \\ C_{ii}^{(n)} = -\sum_{i=1, j\neq i}^{N} C_{ij}^{(n)} \quad \text{where } i \neq j, \qquad i, j = 1, 2, ..., N \;\text{ and }\; n = 2, 3, ..., N-1. \end{gathered} \tag{22}$$

3.2 Choice of sampling points

The accuracy of the results of the DQM can highly be affected by the mesh distribution. Choosing a uniformly distributed grid can result in less accurate results. For this reason, a refined technique is adopted in the present work to achieve better results for vibration problems (Al Kaisy *et al*. 2007), this technique is Chebyshev-Gauss-Lobatto points given by

$$r_i = \frac{1}{2}\left(1 - \cos\left(\frac{i-1}{N-1}\pi\right)\right), \qquad i = 1, 2, ..., N \tag{23}$$

3.3 Application of the DQM into governing equations and boundary conditions

To solve the non-dimensional governing equation by applying the Differential Quadrature discretization, Eq. (22) is substituted in Eq. (15) as follows

$$\begin{gathered} \frac{D_i}{D_0}\varphi_i^3\sum_{j=1}^{N}C_{ij}^{(4)}W_j + \frac{2\varphi_i^2}{D_0}\left(D_i + \varphi_i\frac{dD_i}{\varphi_i}\right)\sum_{j=1}^{N}C_{ij}^{(3)}W_j + \frac{\varphi_i}{D_0}\left(-D_i + (2+\nu)\varphi_i\frac{dD_i}{d\varphi_i} + \varphi_i^2\frac{d^2D_i}{d\varphi_i^2}\right) \\ \sum_{j=1}^{N}C_{ij}^{(2)}W_j + \frac{1}{D_0}\left(D_i - \varphi_i\frac{dD_i}{d\varphi_i} + \varphi_i^2\nu\frac{d^2D_i}{d\varphi_i^2}\right)\sum_{j=1}^{N}C_{ij}^{(1)}W_j - K_p\varphi_i^3\sum_{j=1}^{N}C_{ij}^{(2)}W_j - K_p\varphi_i^2\sum_{j=1}^{N}C_{ij}^{(1)}W_j + \\ K_w\varphi_i^3W_i = \varphi_i^3\Omega^2A(n_1)BW_i, \qquad i = 2, 3, ..., N-2 \end{gathered} \tag{24}$$

The governing equation has two boundary conditions at the same boundary point. As a result, some difficulties arise during applying DQM. To overcome these difficulties (Bert *et al*. 1988) employed an approach called δ-technique, in which the two boundary points are separated by a small distance δ from the boundaries. Then, one of the boundary conditions is written at the boundary point and the other one is implemented on the δ-point. The boundary and regularity conditions can be represented in the DQ discretized form as:

- Clamped (C)

$$W(1)=0,\quad \sum_{j=1}^{N} C_{ij}^{(1)} W_j = 0 \tag{25}$$

- Simply-supported (S)

$$W(1)=0,\quad M_\varphi\Big|_{\varphi=1} = -D_i\left(\sum_{j=1}^{N} C_{(N-1)j}^{(2)} W_j + \nu\left(\frac{1}{\varphi_i}\sum_{j=1}^{N} C_{(N-1)j}^{(1)} W_j\right)\right)=0 \tag{26}$$

- Regularity condition

$$\sum_{j=1}^{N} C_{1j}^{(1)} W_j = 0 \tag{27}$$

To obtain the natural frequency, the governing equation represented in the DQ approach in Eq. (24) can be represented in the following matrix form (Shu 2000)

$$[A_{db}]\{W_b\}+[A_{dd}]\{W_d\}=\Omega^2\{W_d\} \tag{28}$$

where the domain and boundary points are denoted as d and b, respectively.

Moreover, the boundary conditions can be represented in the following matrix form

$$[A_{bb}]\{W_b\}+[A_{bd}]\{W_d\}=\{0\} \tag{29}$$

The governing equation and the boundary and the regularity conditions should be solved simultaneously. In order to achieve that, the value of $\{W_b\}$ should be obtained as follows

$$\{W_b\}=-[A_{bb}]^{-1}[A_{bd}]\{W_d\} \tag{30}$$

By substituting Eq. (30) into Eq. (28), we obtain eigenvalue equation system as follows

$$\left[-A_{db}A_{bb}^{-1}A_{bd}+A_{dd}\right]\{W_d\}=\Omega^2\{W_d\} \tag{31}$$

Solving Eq. (31) results in a set of equations. A MATLAB code was carried out and the natural frequencies were calculated. The results and the parametric study are introduced in the following section.

4. Results and discussion

4.1 Validation of results

In the present work, the results of the first three non-dimensional natural frequencies for free vibration analysis of circular plate are given for two types of boundary conditions, i.e., the clamped and simply supported. The material properties of the plate used in this paper are taken from Ref. (Shariyat and Alipour 2011) as follows:

To validate the results obtained in the present work, a comparative study is provided in Table 2 for a homogeneous plate with uniform thickness as the simplest case. It was found that 20 grid points were sufficient to achieve the convergence and to provide satisfying accuracy for both clamped and simply supported cases with δ=0.00001. The first three natural frequencies were

Table 1 Material properties of the two-directional FGM circular plate

Parameter	Symbol	Value	Dimension
Ceramic Young's modulus	E_c	380	Gpa
Metal Young's modulus	E_m	70	Gpa
Ceramic mass density	ρ_c	3,800	kg/cm^3
Metal mass density	ρ_m	2,702	kg/cm^3
Poisson's ratio	ν	0.3	-

Table 2 Comparison of natural frequency for homogenous circular plate ($g = 0$) and uniform thickness ($\gamma, K_w, K_p = 0$)

Method	Clamped			Simply supported		
	Ω1	Ω2	Ω3	Ω1	Ω2	Ω3
Present (DQM)	10.2158	39.7711	89.1041	4.9351	29.72	74.156
Ref. (Leissa 1969) (analytical)	10.2158	39.771	89.104	4.977	29.76	74.20
Ref. (Wu *et al*. 2002) (DQM)	10.216	39.771	89.104	4.935	29.72	74.156
Ref. (Shariyat and Alipour 2010) (DTM)	10.2163	39.7732	89.1086	4.9351	29.72	74.1561
Ref. (Senjanovic *et al*. 2014) (FEM)	10.201	39.685	88.784	4.931	29.678	73.953

Table 3 Comparison of natural frequency for a homogenous ($g = 0$) circular plate of linear variation in the thickness ($\beta = 1, K_w, K_p = 0$)

γ	Method	Clamped			Simply supported		
		Ω1	Ω2	Ω3	Ω1	Ω2	Ω3
-0.5	Present	6.1504	27.3002	63.0611	3.5498	21.2385	53.4404
	Ref. (Gupta *et al*. 2006)	6.1504	27.3002	63.0611	3.5498	21.2386	53.4404
-0.3	Present	7.7783	32.4610	73.9467	4.1158	24.7265	62.0704
	Ref. (Gupta *et al*. 2006)	7.7783	32.4610	73.9467	4.1158	24.7265	62.0704
0.3	Present	12.6630	46.7812	103.4123	5.7483	34.5624	85.6205
	Ref. (Gupta *et al*. 2006)	12.6631	46.7813	103.4123	5.7483	34.5625	85.6205
0.5	Present	14.3021	51.3480	112.6359	6.2927	37.7423	93.0342
	Ref. (Gupta *et al*. 2006)	14.3021	51.3480	112.6360	6.2927	37.7423	92.0342

calculated using 20 grid points for the clamped and simply supported edges, respectively. The results of the DQM show excellent agreement with results obtained from (Leissa 1969) and (Wu *et al*. 2002). However, by comparing the present results with the results obtained from another numerical method called the Differential transformation method (DTM), it is observed that there are some differences in the results of the first three natural frequencies in case of clamped boundary conditions while the results are identically matched for a simply supported case when compared to the results from the analytical solution. This reveals that DQM converges faster for different boundary conditions in contrast to DTM. Another comparison was carried out with the results obtained from the finite element method (FEM) to confirm the accuracy and the efficiency of the adopted approach. The results of FEM are considered less accurate results in comparison with the rest of the mentioned numerical methods. For non-uniform thickness, the results of natural frequencies for linear variation in thickness can be compared to the results mentioned in

Table 4 Validation of the effect of elastic foundation parameters on the first three frequencies

K_w	K_p	Method	Clamped			Simply supported		
			Ω1	Ω2	Ω3	Ω1	Ω2	Ω3
200	0	Present	17.4460	42.2107	90.2194	14.9785	32.9132	75.4925
		Ref. (Sharma *et al*. 2011)	17.4460	42.2107	90.2194	14.9785	32.9132	75.4925
	10	Present	19.3082	45.9974	94.5297	16.8025	37.2563	80.2995
		Ref. (Sharma *et al*. 2011)	19.3082	45.9974	94.5297	16.8025	37.2563	80.2995
	25	Present	21.7514	51.1398	100.6458	19.2145	42.9558	87.0135
		Ref. (Sharma *et al*. 2011)	21.7514	51.1398	100.6458	19.2145	42.9558	87.0135
500	0	Present	24.5838	45.6261	91.8670	22.8988	37.1925	77.4540
		Ref. (Sharma *et al*. 2011)	24.5838	45.6261	91.8670	22.8988	37.1925	77.4540
	10	Present	25.9385	49.1504	96.1034	24.1314	41.0857	82.1463
		Ref. (Sharma *et al*. 2011)	25.9385	49.1504	96.1034	24.1314	41.0857	82.1463
	25	Present	27.8051	53.9933	102.1253	25.8688	46.3163	88.7207
		Ref. (Sharma *et al*. 2011)	27.8051	53.9933	102.1253	25.8689	46.3163	88.7207

(Gupta *et al*. 2006) as illustrated in Table 3.

The effect of elastic foundation parameters (K_w, K_p) on the first three natural frequencies were validated according to the results obtained from (Sharma, Srivastava *et al*. 2011) using DQM. The results in Table 4 show exact agreement using the same approach of the present work.

4.2 Parametric study

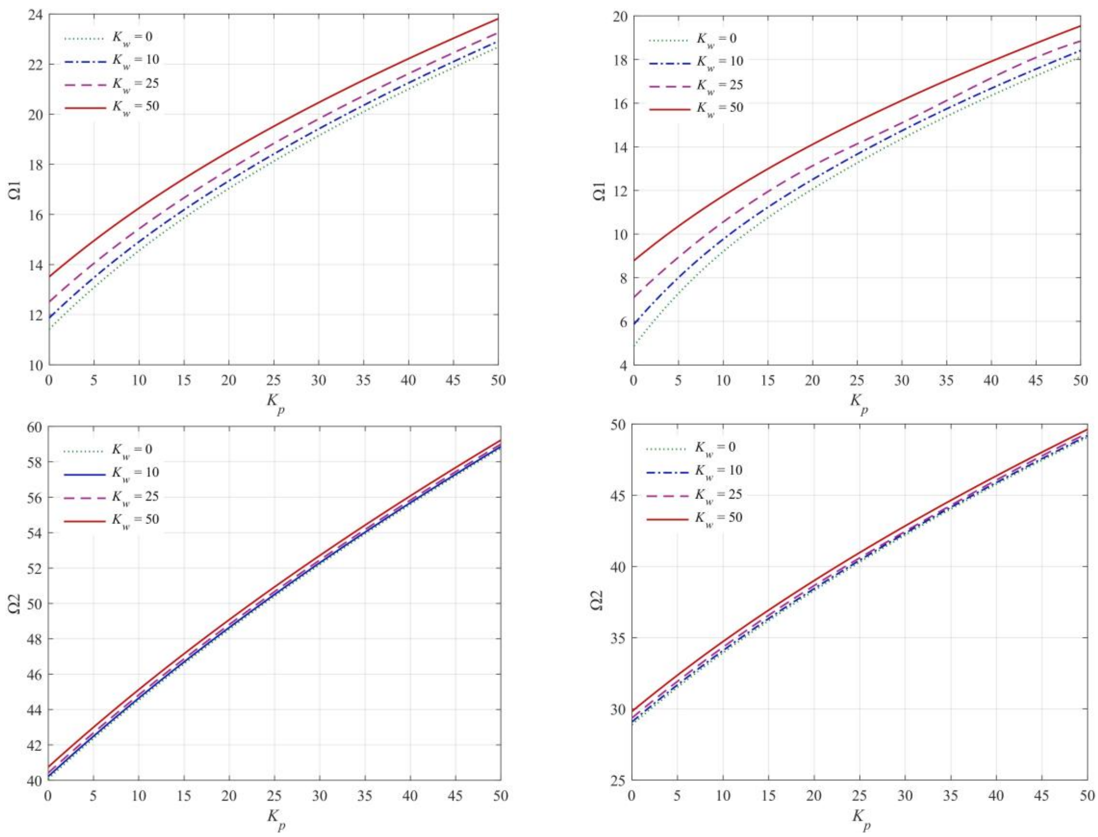
The purpose of the present work is to study the effect of various geometric and elastic foundation parameters on the natural frequency of the circular plate for both clamped and simply supported edges. Firstly, the numerical results presented in Table 5 show the effect of a Pasternak foundation on the first three natural frequencies of a circular plate with variable thickness. According to Table 5, the frequency parameter increases when the curvature direction of the plate changes from downward to upward. However, with the presence of a Pasternak foundation, the effect of the geometric parameter γ decreases significantly in both clamped and simply supported cases. Secondly, Table 6 explains the effect of a Pasternak foundation on a plate with different volume fraction index (g). It can be observed that the natural frequency decreases while the material changes from ceramic to metal. Furthermore, whenever the foundation coefficients are increased the effect of (g) is reduced specially in the simply supported case.

Fig. 3 shows the effects of the Pasternak foundation on the first two natural frequencies. It is noticeable that the fundamental frequency goes up with increasing the foundation parameters. This increase is more evident in case of simply supported edges and negligibly small on the second natural frequency in both cases.

The plots presented in Fig. 4 illustrate the normalized deflection of the circular plate for the first three mode shapes of vibration of a non-uniform circular plate with clamped and simply supported edges $K_w = 50, K_p = 10, \alpha = 0.3$ and $g = 1$. This case aims to study the effect of the geometrical parameter γ on the mode shapes of an FGM circular plate resting on a Pasternak foundation. It was found that the size of nodal circles decreases for thickness profile with geometrical parameter γ with positive value (concavity goes upward), for all mode shapes for both

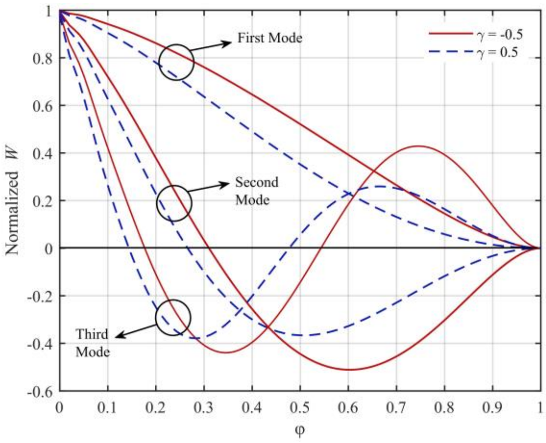
Table 5 Effects of Pasternak foundation on the first three natural frequencies of the circular plate with a parabolic variation in the thickness ($g = 1, \beta = 2, \alpha = 0$)

(K_w, K_p)	γ	Clamped			Simply supported		
		Ω1	Ω2	Ω3	Ω1	Ω2	Ω3
(0,0)	-0.5	5.5173	24.9704	58.1201	3.3603	19.8721	49.8765
	-0.3	6.7186	28.4193	64.9932	3.6633	21.9432	54.9396
	0	8.4988	33.0865	74.1277	4.1057	24.7247	61.6921
	0.3	10.2590	37.3734	82.3746	4.5579	27.2757	67.8079
	0.5	11.4231	40.0848	87.5293	4.8698	28.8918	71.6385
(25,5)	-0.5	9.7786	29.5048	62.9523	8.6708	25.4768	55.6406
	-0.3	10.4879	32.2377	68.9548	8.6615	26.6133	59.5320
	0	11.7174	36.2624	77.3649	8.7027	28.5381	65.3344
	0.3	13.0736	40.1477	85.1848	8.8031	30.5539	70.9096
	0.5	14.0241	42.6630	90.1378	8.9001	31.9110	74.4916
(50,10)	-0.5	12.6693	33.4156	67.4196	11.7703	29.9766	60.7999
	-0.3	13.2063	35.6461	72.6979	11.6803	30.5631	63.7855
	0	14.2015	39.1768	80.4709	11.6014	31.8989	68.7842
	0.3	15.3535	42.7345	87.9028	11.5850	33.5104	73.8796
	0.5	16.1836	45.0849	92.6698	11.6058	34.6628	77.2363

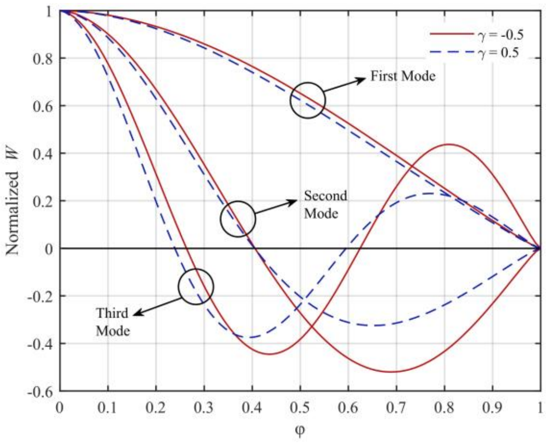


(a) Clamped edge (b) Simply supported edge

Fig. 3 Effects of the Pasternak foundation on the first two natural frequencies, ($\alpha = 0.3, \beta = 2, \gamma = 0.5, g = 1$)



(a) Clamped edge



(b) Simply supported edge

Fig. 4 Effects of geometric parameter γ on the first three normalized mode shapes, for ($K_w = 50, K_p = 10, \alpha = 0.3, \beta = 2, g = 1$)

clamped and simply supported edges except for the fundamental mode in case of simply supported edges, where there is no remarkable change in the size of the nodal circles.

5. Conclusions

The present work discusses the effect of various geometric and foundation parameters on the natural frequency of a circular plate, such as using an FGM for the circular plate changing from ceramic to metal, different functions of variation in the plate thickness, and the presence of variable Winkler parameter with a Pasternak foundation. The Differential Quadrature method is employed to obtain the solution. The next observations can be made:

- The geometry configuration of the plate including the material and thickness profile affects the frequency parameter and the vibrational behavior of the plate.
- A significant reduction in the size of the nodal circles of the mode shapes of the circular plate takes place when the plate becomes thicker at the edge.
- The effects of the Pasternak foundation were included, and it was found that the frequency parameter goes up by increasing the foundation parameters K_w and K_p.
- The frequency parameter can change linearly when the foundation parameter varies linearly. As a result, the behavior of the plate can be predicted according to the type of the soil.
- The results of the adopted approach were compared with the results obtained using different numerical methods from well-known references for special cases to ensure the accuracy of the presented approach.

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