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Mechanical behaviour of advanced composite beams via a simple quasi-3D integral higher-order beam theory

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Abstract. In the present paper, a simple quasi-3D integral higher-order beam theory (HBT) is presented, in which both shear deformation and thickness stretching effects are included for mechanical analysis of advanced composite beams with simply supported boundary conditions, handling mainly bending, buckling, and free vibration problems. The kinematics is based on a novel displacement field which includes the undetermined integral terms and the parabolic function is used in terms of thickness coordinate to represent the effect of transverse shear deformation. The governing equilibrium equations are drawn from the dynamic version of the principle of virtual work; whereas the solution of the problem is obtained by assuming a Navier technique for simply supported advanced composite beams subjected to sinusoidally and uniformly distributed loads. The correctness of the present computational method is checked by comparing the obtained numerical results with quasi-3D solutions found in the literature and with those provided by other shear deformation beam theories. It can be confirmed that the proposed model, which does not involve any shear correction factor, is not only accurate but also simple and useful in solving the static and dynamic response of advanced composite beams.

Keywords: advanced composite beams; kinematics; mechanical analysis; quasi-3D integral HBT

1. Introduction

The advanced composite materials also called functionally graded materials (FGMs) are a new type of lightweight materials that can withstand high surface temperature and challenging environments, and have the potential to support improved structural designs as well as high dimensional stability compared to other conventional composite materials. FGMs were originally developed by a group of scientists in Japan for use in a space plane project as thermal barrier

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materials for aerospace structural applications and fusion reactors (Koizumi 1993, Chan 2001, Schulz *et al.* 2003, Uemura 2003, Watanabe *et al.* 2003, Tarlochan 2012). Nowadays, they have attracted huge attention of researchers from different fields, including, aeronautics, biomedical, electronic, mechanical and civil engineering. Detailed information on the manufacture of the FGMs and its applications may be found in the report by Jha *et al.* (2013). FGM is recognized by a compositional gradient of one material into another, which is entirely unlike conventional composite materials, consisting either of homogeneous mixtures implying a compromise between the properties of the component materials, or two different materials bonded together as in the case of laminated composite materials (Gupta and Talha 2015, Mahamood *et al.* 2017, Bensaid *et al.* 2017, Guerroudj *et al.* 2018, Youcef *et al.* 2020).

Over the last few decades, a considerable amount of research studies have been conducted on the wave propagation, bending, vibration and buckling responses of beam structures made of FGMs, using both analytical and numerical approaches based on various higher-order beam theories. Sallai *et al.* (2009) proposed a new higher-order beam theory by utilizing the principle of virtual works for static bending analysis of sigmoid functionally graded material (S-FGM) beams with simply supported edges and subjected to uniformly distributed transverse loading. Sina *et al.* (2009) presented an analytical solution based on two-dimensional theory of elasticity and Hamilton's principle for free vibration analysis of functionally graded (FG) beams using the first-order beam theory (FBT). Whereas Simsek (2010) studied the fundamental frequencies of FG beams having various boundary conditions using the classical beam theory (CBT), the FBT and different higher-order beam theories (HBTs). In another study, Li *et al.* (2010) presented a general solution based on a third-order beam theory (TBT) for the static and dynamic analysis of FG cantilever beams with power law gradient variation subjected to a uniform pressure. Mahi *et al.* (2010) used a unified HBT for analyzing the temperature-dependent free vibration of symmetric FG beams, in which the material properties change smoothly through the thickness according to a power law distribution (P-FGM), or an exponential law distribution (E-FGM) or a sigmoid law distribution (S-FGM). Thai and Vo (2012) proposed a sinusoidal shear deformation beam theory (SBT) for the bending and free vibration analysis of FG beams. To perform this study, a theoretical kinematics model of the beam was derived assuming a constant transverse displacement and a higher-order variation of axial displacement over the beam thickness. A finite element model based on a refined shear deformation theory has been applied by Vo *et al.* (2014) for the free vibration and buckling analysis of functionally graded sandwich beams with various boundary conditions. In this work, the effects of different parameters on the critical buckling loads and fundamental natural frequencies of the FG sandwich beams are considered. Nguyen *et al.* (2015) proposed a new hyperbolic shear deformation theory for the buckling and free vibration response of isotropic and FG sandwich beams. An analytical solution for buckling and free vibration responses of simply supported sandwich beams made of functionally graded materials is presented by Osofero *et al.* (2016) by using the Hamilton's principle and various quasi-3D theories. Moreover, Ghumare and Sayyad (2017) developed a new fifth-order shear and normal deformation theory for the bending and buckling analysis of FG beams subjected to transverse and axial loadings. In this investigation, the axial and transverse displacements involve polynomial shape functions in order to accommodate the effects of transverse shear and normal deformations. An improved and simpler finite element model having five nodes and ten degrees-of-freedom based on FBT has been presented by Kahya and Turan (2017) for free vibration and buckling analysis of FG beams. Karamanli (2017) used a quasi-3D shear deformation theory and the symmetric smoothed particle hydrodynamics method to investigate the elastostatic behaviour of two-directional FG sandwich beams subjected to a uniformly distributed load for different sets of boundary conditions. An efficient quasi-3D theory was presented by Ait Atmane *et al.* (2017) for bending, buckling and free vibration analysis of FG perfect and imperfect beams resting on elastic foundations, in which the shear deformation has been incorporated with thickness stretching effect by a parabolic variation of all displacements across the beam thickness. Ebrahimi *et al.* (2017) examined the thermo-mechanical vibration characteristics of functionally graded (FG) micro/nanobeams with imperfections in the material composition via a refined hyperbolic beam theory. Mouffoki *et al.* (2017) utilized a new two-unknown trigonometric shear deformation beam theory to investigate the free vibration of nonlocal advanced nanobeams resting on elastic foundation, in which three types of environmental condition are considered. A simple analytical approach was developed by Sayyad and Ghugal (2018a) using a modified exponential beam theory (EBT) to study the static and dynamic behaviour of FG beams with different boundary conditions. Sayyad and Ghugal (2018b) also used the higher-order hyperbolic shear deformation theory developed by Soldatos (1992) to study the bending, buckling and free vibration responses of simply supported FG beams. Ayache *et al.* (2018) have employed a novel four-variable refined beam theory for the free vibration and wave propagation analysis of functionally graded porous beams using a new kinematics model in conjunction with a new function of the porosity factor, while Safa *et al.* (2019) used a refined beam theory for thermo-elastic vibration response of FG beams under thermal loads in which three types of temperature distribution across the beam thickness are considered, uniformity, linearity, and nonlinearity, respectively. Razouki *et al.* (2020) applied a new differential transform method based on a refined higher-order shear deformation theory with only three unknowns to study the static bending of thick FG beams subjected to uniformly distributed loads. According to a unified formulation based on HBT and a Jacobi-Ritz approach, Qin *et al.* (2020) examined the free and forced vibration analyses of functionally grade porous (FGP) beams under uniform temperature rise and various boundary conditions. Recently, Chen and Su (2021) proposed an analytical solution based on the refined zigzag theory (RZT) for vibration in cantilevered functionally graded (FG) sandwich beams. Whereas, Le *et al.* (2021) used a refined third-order shear deformation beam element, in which the transverse displacement is divided into bending and shear components for the free vibration and buckling analysis of FG sandwich beams. The material properties of the beam are evaluated according to two micromechanical models, the Voigt model and Mori-Tanaka scheme.

In the present paper, an analytical solution is presented for the bending, buckling, and free vibration analysis of advanced composite beams using a simplified quasi-3D integral HBT in which both shear deformation and normal transversal deformation effects are included. The displacement field is based on a novel kinematic that introduces undetermined integral terms with a parabolic variation for all displacements, and satisfies the free transverse shear stress conditions on the top and bottom surfaces of the beam without requiring a shear correction factor. The governing equations and its boundary conditions are drawn from the dynamic version of the principle of virtual work. A closed form solution for simply supported FG beams under the uniformly distributed load is obtained by employing a single trigonometric series technique developed by Navier. Numerical results are matched with those computed by other beam theories to exhibit the effects of shear deformation and thickness stretching on displacement, stresses, critical buckling loads and natural frequencies and also to verify the accuracy and effectiveness of the present theory to apply it to the investigation of other types of materials.

2. Theoretical formulation

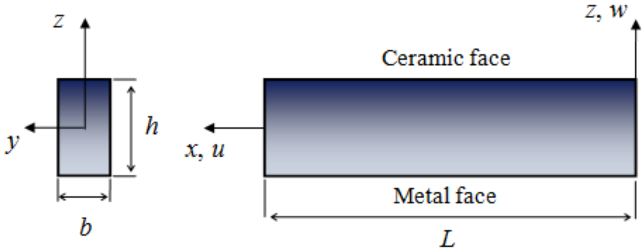

Fig. 1 Geometry and coordinate system of an FG doubly-curved shell

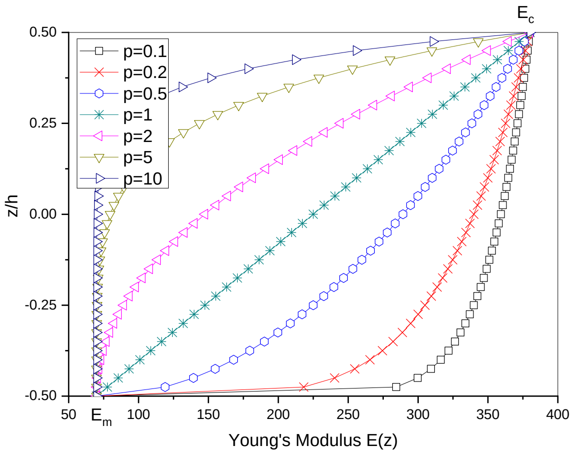

Fig. 2 Variation of the Young's modulus through the FG beam thickness

2.1. FG beams

We consider a simply supported beam consisting of a mixture of both metal and ceramic materials whose mechanical properties vary progressively in the thickness direction, as shown in Fig. 1. The FG beam with length L, width b and a total thickness h is subjected to a transverse mechanical load. For simplicity, Poisson's ratio ν is assumed to be constant, whereas, Young's modulus $E(z)$ and mass density $\rho(z)$ are assumed to vary continuously with a power-law distribution as given by Ghumare and Sayyad (2017)

$$E(z) = (E_c - E_m)\left(\frac{2z+h}{2h}\right)^p + E_m \tag{1a}$$

$$\rho(z) = (\rho_c - \rho_m)\left(\frac{2z+h}{2h}\right)^p + \rho_m \tag{1b}$$

Where subscripts m and c represent the metallic and ceramic constituents, respectively; and p is a non-negative variable parameter referring to the power-law index. Fig. 2 below shows the variation of Young's modulus $E(z)$ through FG beam thickness for various values of this parameter

2.2 Kinematics and constitutive relations

Several types of plate and beam theories are developed for understanding the static and dynamic behaviour of functionally graded structures. In this paper, the displacement field of the proposed integral higher-order shear and normal deformation beam theory can be written in a simpler form as

$$u(x,z) = u_0(x) - z\frac{\partial w_0}{\partial x} + kf(z)\int \theta\,(x)dx,\ w(x,z) = w_0(x) + g(z)\phi(x) \tag{2}$$

Where $u_0(x)$, $w_0(x)$, $\theta(x)$, and $\varphi(x)$ are the four unknown functions of middle surface of the beam. The constant k depends on the geometry. The shape function $f(z)$ is chosen to satisfy the stress-free boundary conditions on the top and bottom surfaces of the FG beam and can be expressed by

$$f(z) = \frac{9z}{8} - \frac{3z^3}{2h^2},\ g(z) = \frac{df(z)}{dz} \tag{3}$$

The infinitesimal strains associated with the displacement field in Eq. (3) are obtained using strain-displacement relationship from linear theory of elasticity

$$\varepsilon_x = \frac{\partial u}{\partial x} = \varepsilon_x^0 + z\varepsilon_x^1 + f(z)\varepsilon_x^2,\ \varepsilon_z = \frac{\partial w}{\partial z} = g'(z)\varepsilon_z^0,\ \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = g(z)\gamma_{xz}^0 \tag{4}$$

where

$$\varepsilon_x^0 = \frac{\partial u_0}{\partial x},\ \varepsilon_x^1 = -\frac{\partial^2 w_0}{\partial x^2},\ \varepsilon_x^2 = k\theta,\ \varepsilon_z^0 = \phi,$$
$$\gamma_{xz}^0 = k\int \theta dx + \frac{\partial \phi}{\partial x} = kA'\frac{\partial \theta}{\partial x} + \frac{\partial \phi}{\partial x},\ A' = -\frac{1}{\alpha^2},\ k = \alpha^2 \tag{5}$$

Using the generalized Hooke's Law, stress-strain relations for FG beam can be written as follows

$$\begin{Bmatrix} \sigma_x \\ \sigma_z \\ \tau_{xz} \end{Bmatrix} = \begin{bmatrix} Q_{11}(z) & Q_{13}(z) & 0 \\ Q_{13}(z) & Q_{33}(z) & 0 \\ 0 & 0 & Q_{55}(z) \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_z \\ \gamma_{xz} \end{Bmatrix} \tag{6}$$

where

$$Q_{11}(z) = Q_{33}(z) = \frac{E(z)}{1-\nu^2},\ Q_{13}(z) = \frac{\nu\, E(z)}{1-\nu^2},\ Q_{55}(z) = \frac{E(z)}{2(1+\nu)} \tag{7}$$

2.3 Governing equations

The dynamic version of the principle of virtual work is used to obtain the governing equations and boundary conditions for the FG beam and can be expressed in analytical form as reported by Sayyad and Ghugal 2016

$$b\int_{-h/2}^{h/2}\int_0^L (\sigma_x\delta\varepsilon_x + \sigma_z\delta\varepsilon_z + \tau_{xz}\delta\gamma_{xz})\ dxdz - \int_0^L \left(q\delta w + N_x^0\frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x}\right)dx +$$
$$b\int_{-h/2}^{h/2}\int_0^L \rho(z)\left(\frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial \delta w}{\partial t}\right)dxdz = 0 \tag{8}$$

By substituting the terms for the virtual displacements and deformations given in Eqs. (2) and (4) into Eq. (8), the principle of virtual work can be restated as follows

$$\int_0^L \left(\begin{array}{c} N_x\delta\varepsilon_x^0 + M_x^b\delta\varepsilon_x^1 + M_x^s\delta\varepsilon_x^2 + R_z\delta\varepsilon_z^0 + S_{xz}^s\gamma_{xz}^0 - q(\delta w_0 + g(z)\delta\varphi) - N_x^0\dfrac{\partial w_0}{\partial x}\dfrac{\partial \delta w_0}{\partial x} \\ +I_0(\dot{u}_0\delta\dot{u}_0 + \dot{w}_0\delta\dot{w}_0) - I_1\left(\dot{u}_0\dfrac{\partial \delta\dot{w}_0}{\partial x} + \dfrac{\partial \dot{w}_0}{\partial x}\delta\dot{u}_0\right) + I_3kA'\left(\dot{u}_0\dfrac{\partial \delta\dot{\theta}}{\partial x} + \dfrac{\partial \dot{\theta}}{\partial x}\delta\dot{u}_0\right) + I_2\dfrac{\partial \dot{w}_0}{\partial x}\dfrac{\partial \delta\dot{w}_0}{\partial x} \\ -I_4kA'\left(\dfrac{\partial \dot{w}_0}{\partial x}\dfrac{\partial \delta\dot{\theta}}{\partial x} + \dfrac{\partial \dot{\theta}}{\partial x}\dfrac{\partial \delta\dot{w}_0}{\partial x}\right) + I_5(kA')^2\dfrac{\partial \dot{\theta}}{\partial x}\dfrac{\partial \delta\dot{\theta}}{\partial x} + I_6(\dot{w}_0\delta\dot{\varphi} + \dot{\varphi}\delta\dot{w}_0) + I_7\dot{\varphi}\delta\dot{\varphi} \end{array} \right) dx = 0 \tag{9}$$

where dot-superscript convention specifies the differentiation with respect to the time variable (t); and I_i $(i = 0, 2, 3, 4, 5, 6, 7)$ are the inertia coefficients. Whereas $(N_x,\ M_x^b,\ M_x^s,\ S_{xz}^s,\ R_z)$ enclosed in Eq. (9) are related to the force and moment resultants, which can be written through the stress components as follows

$$N_x = b\int_{-h/2}^{h/2}\sigma_x dz,\ M_x^b = b\int_{-h/2}^{h/2}\sigma_x z d\,z,\ M_x^s = b\int_{-h/2}^{h/2}\sigma_x f(z) d\,z,$$
$$S_{xz}^s = b\int_{-h/2}^{h/2}\tau_{xz} g(z) d\,z,\ R_z = b\int_{-h/2}^{h/2}\sigma_z g'(z) d\,z \tag{10a}$$

$$(I_0, I_1, I_2, I_3, I_4) = b\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho(z)\big(1, z, z^2, f(z), zf(z)\big)dz,$$
$$(I_5, I_6, I_7) = b\int_{-h/2}^{h/2}\rho(z)(f^2(z), g(z), g^2(z))dz \tag{10b}$$

Substituting expressions for stresses and strains of the present theory into the dynamic version of the principle of virtual work and integrating Eq. (9) by parts according to space and time variables, and collecting the coefficients of δu_0, δw_0, $\delta\theta$ and $\delta\phi$, the governing equations in terms of stress resultants are obtained as follows

$$\begin{aligned}
&\delta u_0\colon \frac{\partial N_x}{\partial x} = I_0\ddot{u}_0 - I_1\frac{\partial \ddot{w}_0}{\partial x} + I_3 kA'\frac{\partial\ddot{\theta}}{\partial x},\\
&\delta w_0\colon \frac{\partial^2 M_x^b}{\partial x^2} + q + N_x^0\frac{\partial^2 w_0}{\partial x^2} = I_0\ddot{w}_0 + I_1\frac{\partial \ddot{u}_0}{\partial x} - I_2\frac{\partial^2 \ddot{w}_0}{\partial x^2} + I_4 kA'\frac{\partial^2\ddot{\theta}}{\partial x^2} + I_6\ddot{\phi},\\
&\delta\theta\colon -kM_x^s + kA'\frac{\partial S_{xz}^s}{\partial x} = -I_3 kA'\frac{\partial \ddot{u}_0}{\partial x} + I_4 kA'\frac{\partial^2 \ddot{w}}{\partial x^2} - I_5 (kA')^2\frac{\partial^2\ddot{\theta}}{\partial x^2},\\
&\delta\phi\colon \frac{\partial S_{xz}^s}{\partial x} - R_z = I_6\ddot{w}_0 + I_7\ddot{\phi}
\end{aligned} \tag{11}$$

Eq. (11) can be expressed in terms of displacement variables by substituting for the stress resultants from Eq. (10). For this case, the governing equations become as follows

$$\begin{aligned}
&\delta u_0\colon A_{11}\frac{\partial^2 u_0}{\partial x^2} - B_{11}\frac{\partial^3 w_0}{\partial x^3} + kE_{11}\frac{\partial\theta}{\partial x} + B_{13}^s\frac{\partial\phi}{\partial x} = I_0\ddot{u}_0 - I_1\frac{\partial\ddot{w}_0}{\partial x} + I_3 kA'\frac{\partial\ddot{\theta}}{\partial x},\\
&\delta w_0\colon B_{11}\frac{\partial^3 u_0}{\partial x^3} - D_{11}\frac{\partial^4 w_0}{\partial x^4} + kF_{11}\frac{\partial^2\theta}{\partial x^2} + D_{13}^s\frac{\partial^2\phi}{\partial x^2} + q + N_x^0\frac{\partial^2 w_0}{\partial x^2}\\
&\qquad = I_0\ddot{w}_0 + I_1\frac{\partial\ddot{u}_0}{\partial x} - I_2\frac{\partial^2\ddot{w}_0}{\partial x^2} + I_4 kA'\frac{\partial^2\ddot{\theta}}{\partial x^2} + I_6\ddot{\phi},\\
&\delta\theta\colon -kE_{11}\frac{\partial u_0}{\partial x} + kF_{11}\frac{\partial^2 w_0}{\partial x^2} + k^2 A'^2 K_{55}^s\frac{\partial^2\theta}{\partial x^2} + kA'K_{55}^s\frac{\partial^2\phi}{\partial x^2} - k^2 H_{11}\theta - kH_{13}^s\phi\\
&\qquad = -I_3 kA'\frac{\partial\ddot{u}_0}{\partial x} + I_4 kA'\frac{\partial^2\ddot{w}}{\partial x^2} - I_5(kA')^2\frac{\partial^2\ddot{\theta}}{\partial x^2},\\
&\delta\phi\colon -B_{13}^s\frac{\partial u_0}{\partial x} + D_{13}^s\frac{\partial^2 w_0}{\partial x^2} + kA'K_{55}^s\frac{\partial^2\theta}{\partial x^2} + K_{55}^s\frac{\partial^2\phi}{\partial x^2} - kH_{13}^s\theta - F_{33}^s\phi = I_6\ddot{w}_0 + I_7\ddot{\phi}
\end{aligned} \tag{12}$$

in which the stiffness coefficients can be defined as follows

$$\begin{aligned}
&(A_{11}, B_{11}, D_{11}) = b\int_{-h/2}^{h/2} Q_{11}(z)(1, z, z^2)dz,\\
&(E_{11}, F_{11}, H_{11}) = b\int_{-h/2}^{h/2} Q_{11}(z)(f(z), z\,f(z), f^2(z))dz,\\
&(B_{13}^s, D_{13}^s, H_{13}^s) = b\int_{-h/2}^{h/2} Q_{13}(z)(g'(z), z\,g'(z), f(z)g'(z))dz,\\
&K_{55}^s = b\int_{-h/2}^{h/2} Q_{55}(z)g^2(z)\,dz,\ F_{33}^s = b\int_{-h/2}^{h/2} Q_{33}(z){g'}^2(z)\,dz
\end{aligned} \tag{13}$$

Table 1 Material properties of FG beam (Vo *et al.* 2015)

Materials	Aluminum (Al)	Alumina (Al_2O_3)
Elasticity modulus (GPa)	70	380
Density (kg/m^3)	2702	3960
Poisson's ratio	0.3	0.3

3. Analytical solutions

To solve the governing equations and to satisfy the boundary conditions of simply supported FG beam based on the proposed Quasi-3D HBT, Navier's technique is used. To this purpose, the trigonometric forms of displacement variables can be taken as

$$\begin{aligned} u_0(x) &= \sum_{m=1}^{\infty} U_m \cos(\alpha x) e^{i\omega t}, \\ w_0(x) &= \sum_{m=1}^{\infty} W_m \sin(\alpha x) e^{i\omega t}, \\ \theta(x) &= \sum_{m=1}^{\infty} \Theta_m \sin(\alpha x) e^{i\omega t}, \\ \phi(x) &= \sum_{m=1}^{\infty} \Phi_m \sin(\alpha x) e^{i\omega t} \end{aligned} \tag{14}$$

where $\alpha = m\pi/L$ and U_m, W_m, Θ_m, Φ_m are the unknown coefficients of the respective Fourier expansions that will be computed for each value of m. Thus, ω is the natural frequency of free vibration of the FG beam. The external transverse load q is also expanded in a single trigonometric series as

$$q(x) = \sum_{m=1}^{\infty} q_m \sin(\alpha x) \tag{15}$$

where

$$\begin{cases} q_m = q_0, \ (m = 1) \text{ For Sinusoidaly distributed load (SDL)} \\ q_m = \frac{4q_0}{m\pi}, \ (m = 1,3,5,...) \text{ For Uniformly distributed load (UDL)} \end{cases} \tag{16}$$

in which q_0 represents the maximum intensity of the distributed load. However, in the buckling case, we assume that the FG beam is subjected to an in-plane compressive load $N_x^0 = -N^0$. By substituting Eqs. (14) and (15) into Eq. (12), the analytical solution can be obtained from the following equations

$$\left\{ \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{12} & K_{22} & K_{23} & K_{24} \\ K_{13} & K_{23} & K_{33} & K_{34} \\ K_{14} & K_{24} & K_{34} & K_{44} \end{bmatrix} - \omega^2 \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{12} & M_{22} & M_{23} & M_{24} \\ M_{13} & M_{23} & M_{33} & M_{34} \\ M_{14} & M_{24} & M_{34} & M_{44} \end{bmatrix} \right\} \begin{Bmatrix} U_m \\ W_m \\ \Theta_m \\ \Phi_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ q_m \\ 0 \\ 0 \end{Bmatrix} \tag{17}$$

where $[K_{ij}]$ and $[M_{ij}]$ correspond to the elements of the stiffness and inertia matrices, respectively, that can be expressed as follows

$$K_{11} = \alpha^2 A_{11},\ K_{12} = -\alpha^3 B_{11},\ K_{13} = -k\alpha E_{11},\ K_{14} = -\alpha B_{13}^s,$$

$$K_{22} = \alpha^2(\alpha^2 D_{11} - N^0),\ K_{23} = k\alpha^2 F_{11},\ K_{24} = \alpha^2 D_{13}^s,$$

$$K_{33} = k^2\left(H_{11} + \alpha^2 A'^2 K_{55}^s\right),\ K_{34} = k(H_{13}^s + \alpha^2 A' K_{55}^s),$$
$$K_{44} = F_{33}^s + \alpha^2 K_{55}^s,\ M_{11} = I_0,\ M_{12} = -I_1\alpha,$$
$$M_{13} = I_3 k A'\alpha,\ M_{14} = 0,\ M_{22} = I_2\alpha^2 + I_0,\ M_{23} = -I_4 k A'\alpha^2, \tag{18}$$
$$M_{24} = I_6,\ M_{33} = I_5(kA')^2\alpha^2,\ M_{34} = 0,\ M_{44} = I_7$$

In the present study, we can note that the solution of the problem given in Eq. (17) allows us to calculate the displacement, stresses, critical buckling load and fundamental natural frequencies responses of FG beams subjected to in-plane and transverse loads.

4. Numerical results and discussion

In this section, the numerical examples are analyzed in order to verify the correctness of the proposed Quasi-3D integral higher-order beam theory and investigate the thickness stretching effect on the mechanical behaviour of advanced composite beams subjected to an in-plane compressive load applied along the *x*-direction and/or to two different types of transverse mechanical loads acting along the *z*-direction with slenderness ratio $L/h = 5$ and $L/h = 20$. As shown in Table 1, FG beams are made from a mixture of Aluminum as metal (Al) and Alumina as ceramic (Al_2O_3).

The following non-dimensional terms have been employed throughout the tables and figures

$$\bar{u} = \frac{100E_m h^3}{qL^4} u(0,z),\ \bar{w} = \frac{100E_m h^3}{qL^4} w\left(\frac{L}{2},z\right),\ \bar{\sigma}_x = \frac{h}{qL}\sigma_x\left(\frac{L}{2},z\right),$$
$$\bar{\tau}_{xz} = \frac{h}{qL}\tau_{xz}(0,z),\ \bar{N}_{cr} = N_{cr}\frac{12L^2}{E_m h^3},\ \bar{\omega} = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}} \tag{19}$$

4.1 Bending analysis

The numerical results of displacements and stresses obtained by the present Quasi-3D integral HBT for simply supported boundary conditions are given in Tables 2 to 5 and compared with those computed using the classical beam theory (CBT) of Euler-Bernoulli (1744), first-order beam theory (FBT) based on Timoshenko's beam theory (Timoshenko 1921), third-order beam theory (TBT) of Reddy (1984) and with those presented by Thai and Vo (2012) based on the refined sinusoidal beam theory (SBT), the analytical solution given by Vo *et al.* (2015) using the Quasi-3D beam theory and the mixed finite element method (MFEM) developed by Madenci (2021) based on sinusoidal shear deformation beam theory. The comparisons of the variations of displacements and stresses through the thickness of the present theory and others shear deformation theories are also plotted in Figs. 3-5.

The comparison of non-dimensional axial and transverse displacements for simply supported FG beams subjected to sinusoidally distributed load (SDL) are illustrated in Tables 2 and 3 for both slenderness ratio $(L/h = 5)$ and 20 and for different values of the power-law index $(p = 0, 1, 2, 5, 10)$. As expected, the present computational method can provide accurate results in comparison with those generated by Vo *et al.* (2015). However, we note that other shear deformation beam models found in the literature which do not include the thickness stretching effect overestimate

Table 2 Comparison of non-dimensional displacements and stresses of simply supported FG beams under SDL (L/h=5)

p	Theory	Model	ε_z	$\bar{u}$	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$
				(-h/2)	(0)	(h/2)	(0)
0	Euler-Bernoulli (1744)	CBT	$= 0$	0.7129	2.2693	3.0396	-
	Timoshenko (1921)	FBT	$= 0$	0.7129	2.5023	3.0396	0.3820
	Reddy (1984)	TBT	$= 0$	0.7251	2.5020	3.0916	0.4769
	Thai and Vo (2012)	SBT	$= 0$	0.7259	2.5016	3.0949	0.4920
	Vo *et al.* (2015)	Quasi-3D	$\neq 0$	0.7155	2.4810	3.0910	0.4769
	Present	Quasi-3D	$\neq 0$	0.7155	2.4810	3.0910	0.4769
1	Euler-Bernoulli (1744)	CBT	$= 0$	1.7588	4.5528	4.6979	-
	Timoshenko (1921)	FBT	$= 0$	1.7588	4.9462	4.6979	0.3820
	Reddy (1984)	TBT	$= 0$	1.7793	4.9458	4.7856	0.4769
	Thai and Vo (2012)	SBT	$= 0$	1.7806	4.9451	4.7913	0.4920
	Vo *et al.* (2015)	Quasi-3D	$\neq 0$	1.7131	4.8455	4.7847	0.4769
	Present	Quasi-3D	$\neq 0$	1.7131	4.8455	4.7847	0.4769
2	Euler-Bernoulli (1744)	CBT	$= 0$	2.3794	5.8346	5.4856	-
	Timoshenko (1921)	FBT	$= 0$	2.3794	6.3452	5.4856	0.3250
	Reddy (1984)	TBT	$= 0$	2.4048	6.3754	5.6004	0.4368
	Thai and Vo (2012)	SBT	$= 0$	2.4063	6.3759	5.6080	0.4527
	Vo *et al.* (2015)	Quasi-3D	$\neq 0$	2.2979	6.2107	5.6013	0.4368
	Present	Quasi-3D	$\neq 0$	2.2979	6.2107	5.6013	0.4368
5	Euler-Bernoulli (1744)	CBT	$= 0$	2.8250	6.8994	6.4382	-
	Timoshenko (1921)	FBT	$= 0$	2.8250	7.6269	6.4382	0.2502
	Reddy (1984)	TBT	$= 0$	2.8644	7.7723	6.6057	0.3856
	Thai and Vo (2012)	SBT	$= 0$	2.8671	7.7793	6.6173	0.4029
	Vo *et al.* (2015)	Quasi-3D	$\neq 0$	2.7418	7.5937	6.6108	0.3856
	Present	Quasi-3D	$\neq 0$	2.7418	7.5937	6.6108	0.3856
10	Euler-Bernoulli (1744)	CBT	$= 0$	2.9488	7.5746	7.7189	-
	Timoshenko (1921)	FBT	$= 0$	2.9488	8.4761	7.7189	0.2735
	Reddy (1984)	TBT	$= 0$	2.9989	8.6530	7.9080	0.4225
	Thai and Vo (2012)	SBT	$= 0$	3.0022	8.6561	7.9196	0.4392
	Vo *et al.* (2015)	Quasi-3D	$\neq 0$	2.8933	8.5088	7.9140	0.4223
	Present	Quasi-3D	$\neq 0$	2.8933	8.5088	7.9140	0.4223

the results because the transverse normal deformation is neglected ($\varepsilon_z = 0$). Nevertheless, both Quasi-3D theories give smaller displacements as compared to other theories, except for the CBT which underestimates the transverse displacement values for all power-law index p due to ignoring the shear deformation effect. Examination of Tables 4 and 5 also reveals that, the present theory gives excellent results with previous research reported by Vo *et al.* (2015) for the non-dimensional axial and transverse shear stresses of thick $L/h = 5$ and thin $L/h = 20$ FG beams subjected to

Table 3 Comparison of non-dimensional displacements and stresses of simply supported FG beams under SDL (L/h=20)

p	Theory	Model	ε_z	$\bar{u}$ (-h/2)	$\bar{w}$ (0)	$\bar{\sigma}_x$ (h/2)	$\bar{\tau}_{xz}$ (0)
0	Euler-Bernoulli (1744)	CBT	= 0	0.1782	2.2693	12.1585	-
	Timoshenko (1921)	FBT	= 0	0.1782	2.2839	12.1585	0.3820
	Reddy (1984)	TBT	= 0	0.1784	2.2839	12.1715	0.4774
	Thai and Vo (2012)	SBT	= 0	0.1784	2.2838	12.1724	0.4927
	Vo *et al.* (2015)	Quasi-3D	≠ 0	0.1783	2.2826	12.1712	0.4774
	Present	Quasi-3D	≠ 0	0.1783	2.2826	12.1712	0.4774
1	Euler-Bernoulli (1744)	CBT	= 0	0.4397	4.5528	18.7918	-
	Timoshenko (1921)	FBT	= 0	0.4397	4.5774	18.7918	0.3820
	Reddy (1984)	TBT	= 0	0.4400	4.5774	18.8137	0.4774
	Thai and Vo (2012)	SBT	= 0	0.4400	4.5774	18.8151	0.4927
	Vo *et al.* (2015)	Quasi-3D	≠ 0	0.4273	4.5105	18.8131	0.4774
	Present	Quasi-3D	≠ 0	0.4273	4.5105	18.8131	0.4774
2	Euler-Bernoulli (1744)	CBT	= 0	0.5948	5.8346	21.9425	-
	Timoshenko (1921)	FBT	= 0	0.5948	5.8665	21.9425	0.3250
	Reddy (1984)	TBT	= 0	0.5952	5.8684	21.9712	0.4374
	Thai and Vo (2012)	SBT	= 0	0.5953	5.8685	21.9731	0.4534
	Vo *et al.* (2015)	Quasi-3D	≠ 0	0.5728	5.7410	21.9711	0.4374
	Present	Quasi-3D	≠ 0	0.5728	5.7410	21.9711	0.4374
5	Euler-Bernoulli (1744)	CBT	= 0	0.7062	6.8993	25.7527	-
	Timoshenko (1921)	FBT	= 0	0.7062	6.9448	25.7527	0.2502
	Reddy (1984)	TBT	= 0	0.7068	6.9540	25.7947	0.3863
	Thai and Vo (2012)	SBT	= 0	0.7069	6.9545	25.7976	0.4038
	Vo *et al.* (2015)	Quasi-3D	≠ 0	0.6814	6.8197	25.7955	0.3863
	Present	Quasi-3D	≠ 0	0.6814	6.8197	25.7955	0.3863
10	Euler-Bernoulli (1744)	CBT	= 0	0.7372	7.5746	30.8757	-
	Timoshenko (1921)	FBT	= 0	0.7372	7.6309	30.8757	0.2735
	Reddy (1984)	TBT	= 0	0.7380	7.6421	30.9230	0.4231
	Thai and Vo (2012)	SBT	= 0	0.7380	7.6423	30.9259	0.4401
	Vo *et al.* (2015)	Quasi-3D	≠ 0	0.7178	7.5508	30.9241	0.4231
	Present	Quasi-3D	≠ 0	0.7178	7.5508	30.9241	0.4231

uniformly distributed load (UDL). It is noteworthy that FBT and CBT yield identical and smaller axial stress $\bar{\sigma}_x$ values as compared to HBTs and Quasi-3D theory. Further, the Quasi-3D theory provides almost identical axial stress values compared to other shear deformation theories. It should be noted that the increase of power-law index leads to an increase of non-dimensional axial stress (i.e., increasing the power-law index decreases the stiffness of FG beams). However, for the case of thick FG beams in which the shear deformation is very substantial, the proposed mathematical model

Table 4 Comparison of non-dimensional displacements and stresses of simply supported FG beams under UDL (L/h=5)

p	Theory	Model	ε_z	$\bar{u}$ (-h/2)	$\bar{w}$ (0)	$\bar{\sigma}_x$ (h/2)	$\bar{\tau}_{xz}$ (0)
0	Euler-Bernoulli (1744)	CBT	= 0	0.9211	2.8783	3.7500	-
	Timoshenko (1921)	FBT	= 0	0.9211	3.1657	3.7500	0.5976
	Reddy (1984)	TBT	= 0	0.9398	3.1654	3.8020	0.7332
	Thai and Vo (2012)	SBT	= 0	0.9409	3.1649	3.8053	0.7549
	Madenci (2021)	MFEM-SBT	= 0	-	3.1651	3.8044	0.7500
	Vo *et al.* (2015)	Quasi-3D	≠ 0	0.9264	3.1397	3.8005	0.7233
	Present	Quasi-3D	≠ 0	0.9264	3.1397	3.8005	0.7233
1	Euler-Bernoulli (1744)	CBT	= 0	2.2722	5.7746	5.7959	-
	Timoshenko (1921)	FBT	= 0	2.2722	6.2599	5.7959	0.5976
	Reddy (1984)	TBT	= 0	2.3038	6.2594	5.8836	0.7332
	Thai and Vo (2012)	SBT	= 0	2.3058	6.2586	5.8892	0.7549
	Madenci (2021)	MFEM-SBT	= 0	-	6.2591	5.8882	0.7500
	Vo *et al.* (2015)	Quasi-3D	≠ 0	2.2167	6.1338	5.8812	0.7233
	Present	Quasi-3D	≠ 0	2.2167	6.1338	5.8812	0.7233
2	Euler-Bernoulli (1744)	CBT	= 0	3.0740	7.4003	6.7676	-
	Timoshenko (1921)	FBT	= 0	3.0740	8.0303	6.7676	0.5085
	Reddy (1984)	TBT	= 0	3.1129	8.0677	6.8826	0.6706
	Thai and Vo (2012)	SBT	= 0	3.1153	8.0683	6.8901	0.6933
	Madenci (2021)	MFEM-SBT	= 0	-	8.0670	6.8875	0.6791
	Vo *et al.* (2015)	Quasi-3D	≠ 0	2.9729	7.8606	6.8818	0.6622
	Present	Quasi-3D	≠ 0	2.9729	7.8606	6.8818	0.6622
5	Euler-Bernoulli (1744)	CBT	= 0	3.6496	8.7508	7.9428	-
	Timoshenko (1921)	FBT	= 0	3.6496	9.6483	7.9428	0.3914
	Reddy (1984)	TBT	= 0	3.7100	9.8281	8.1106	0.5905
	Thai and Vo (2012)	SBT	= 0	3.7140	9.8367	8.1222	0.6155
	Madenci (2021)	MFEM-SBT	= 0	-	9.8269	8.1189	0.5811
	Vo *et al.* (2015)	Quasi-3D	≠ 0	3.5490	9.6037	8.1140	0.5840
	Present	Quasi-3D	≠ 0	3.5490	9.6037	8.1140	0.5840
10	Euler-Bernoulli (1744)	CBT	= 0	3.8097	9.6072	9.5228	-
	Timoshenko (1921)	FBT	= 0	3.8097	10.7194	9.5228	0.4279
	Reddy (1984)	TBT	= 0	3.8863	10.9381	9.7122	0.6467
	Thai and Vo (2012)	SBT	= 0	3.8913	10.9420	9.7238	0.6708
	Madenci (2021)	MFEM-SBT	= 0	-	10.9388	9.7200	0.6440
	Vo *et al.* (2015)	Quasi-3D	≠ 0	3.7468	10.7578	9.7164	0.6396
	Present	Quasi-3D	≠ 0	3.7468	10.7578	9.7164	0.6396

Table 5 Comparison of non-dimensional displacements and stresses of simply supported FG beams under UDL (L/h=20)

p	Theory	Model	ε_z	$\bar{u}$ (-h/2)	$\bar{w}$ (0)	$\bar{\sigma}_x$ (h/2)	$\bar{\tau}_{xz}$ (0)
0	Euler-Bernoulli (1744)	CBT	= 0	0.2303	2.8783	15.0000	-
	Timoshenko (1921)	FBT	= 0	0.2303	2.8962	15.0000	0.5976
	Reddy (1984)	TBT	= 0	0.2306	2.8962	15.0129	0.7451
	Thai and Vo (2012)	SBT	= 0	0.2306	2.8962	15.0138	0.7686
	Madenci (2021)	MFEM-SBT	= 0	-	2.8962	15.0200	0.7500
	Vo *et al.* (2015)	Quasi-3D	≠ 0	0.2303	2.8947	15.0125	0.7432
	Present	Quasi-3D	≠ 0	0.2303	2.8947	15.0125	0.7432
1	Euler-Bernoulli (1744)	CBT	= 0	0.5680	5.7746	23.1834	-
	Timoshenko (1921)	FBT	= 0	0.5680	5.8049	23.1834	0.5976
	Reddy (1984)	TBT	= 0	0.5686	5.8049	23.2053	0.7451
	Thai and Vo (2012)	SBT	= 0	0.5686	5.8049	23.2067	0.7686
	Madenci (2021)	MFEM-SBT	= 0	-	5.8049	23.2198	0.7500
	Vo *et al.* (2015)	Quasi-3D	≠ 0	0.5520	5.7201	23.2046	0.7432
	Present	Quasi-3D	≠ 0	0.5520	5.7201	23.2046	0.7432
2	Euler-Bernoulli (1744)	CBT	= 0	0.7685	7.4003	27.0704	-
	Timoshenko (1921)	FBT	= 0	0.7685	7.4397	27.0704	0.5085
	Reddy (1984)	TBT	= 0	0.7691	7.4421	27.0991	0.6824
	Thai and Vo (2012)	SBT	= 0	0.7692	7.4421	27.1010	0.7069
	Madenci (2021)	MFEM-SBT	= 0	-	7.4416	27.1125	0.6791
	Vo *et al.* (2015)	Quasi-3D	≠ 0	0.7401	7.2805	27.0988	0.6809
	Present	Quasi-3D	≠ 0	0.7401	7.2805	27.0990	0.6809
5	Euler-Bernoulli (1744)	CBT	= 0	0.9124	8.7508	31.7711	-
	Timoshenko (1921)	FBT	= 0	0.9124	8.8069	31.7711	0.3914
	Reddy (1984)	TBT	= 0	0.9134	8.8182	31.8130	0.6023
	Thai and Vo (2012)	SBT	= 0	0.9134	8.8188	31.8159	0.6292
	Madenci (2021)	MFEM-SBT	= 0	-	8.8155	31.8310	0.5811
	Vo *et al.* (2015)	Quasi-3D	≠ 0	0.8804	8.6479	31.8137	0.6010
	Present	Quasi-3D	≠ 0	0.8804	8.6479	31.8137	0.6010
10	Euler-Bernoulli (1744)	CBT	= 0	0.9524	9.6072	38.0913	-
	Timoshenko (1921)	FBT	= 0	0.9524	9.6767	38.0913	0.4279
	Reddy (1984)	TBT	= 0	0.9536	9.6905	38.1385	0.6596
	Thai and Vo (2012)	SBT	= 0	0.9537	9.6908	38.1414	0.6858
	Madenci (2021)	MFEM-SBT	= 0	-	9.6888	38.1602	0.6440
	Vo *et al.* (2015)	Quasi-3D	≠ 0	0.9275	9.5749	38.1395	0.6583
	Present	Quasi-3D	≠ 0	0.9275	9.5749	38.1395	0.6583

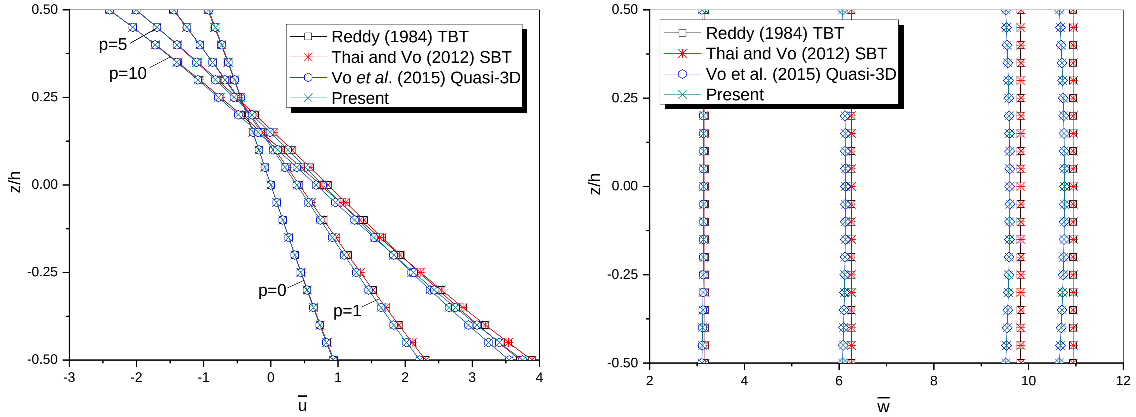

Fig. 3 Variation of non-dimensional displacements through the thickness of simply supported FG beams under UDL (L/h=5)

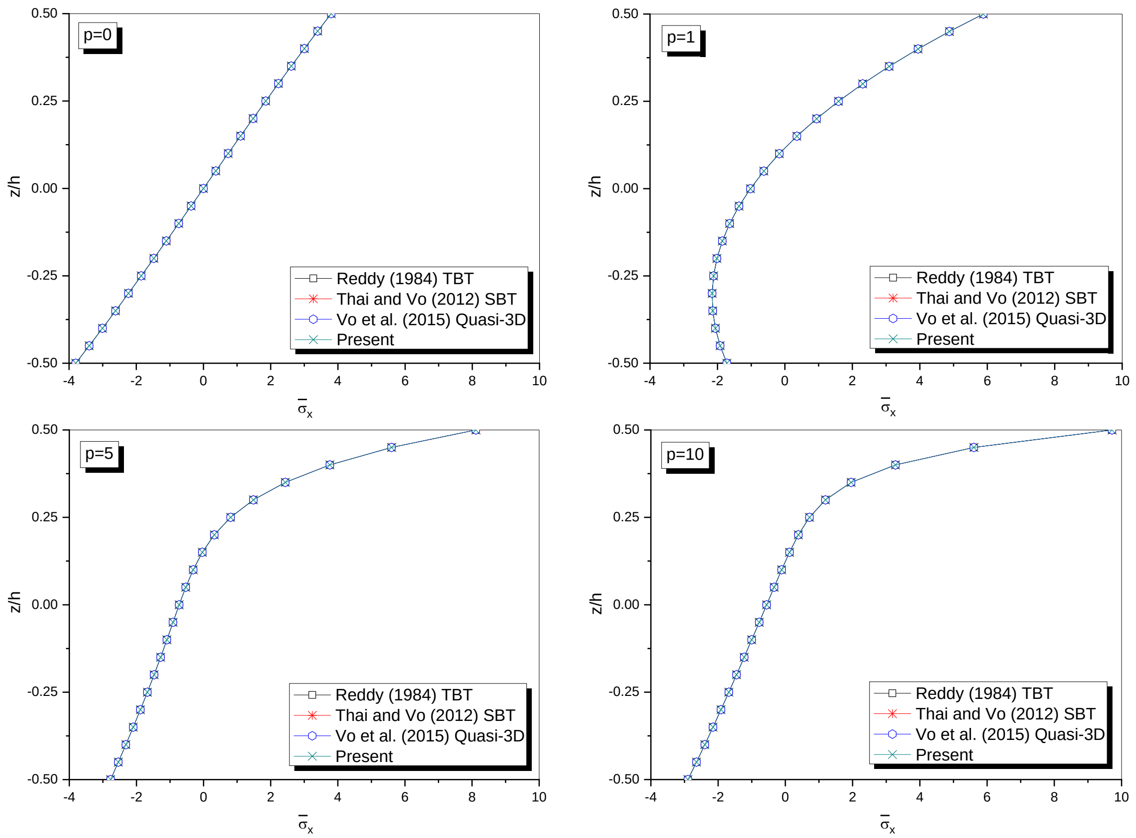

Fig. 4 Variation of non-dimensional axial stresses through the thickness of simply supported FG beams under UDL for different values of the power-law index p (L/h=5)

and Quasi-3D theory of Vo *et al.* (2015) give slightly smaller values of transverse shear stress $\bar{\tau}_{xz}$ than TBT and SBT, because the thickness stretching effect is not taken into account $(\varepsilon_z = 0)$.

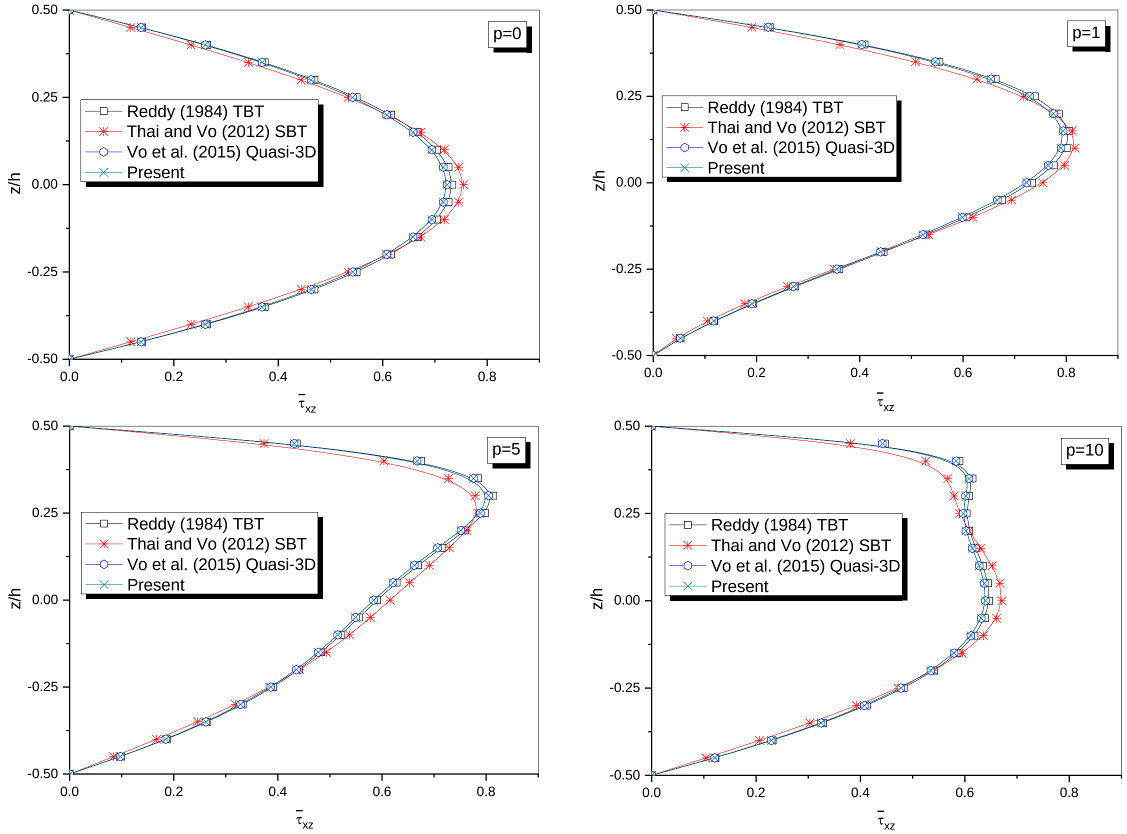

Fig. 5 Variation of non-dimensional transverse shear stresses through the thickness of simply supported FG beams under UDL for different values of the power-law index *p* (*L*/*h*=5)

The graphical results obtained by using the present formulation and other higher-order beam theories, which corresponds to the effect of power-law index on the variations of displacements and stresses through the thickness of thick advanced composite beams under uniform loads are plotted in Figs. 3-5. Also, the results are in excellent agreement with those provided by Vo *et al.* (2015) which confirms the model accuracy. Besides, the Quasi-3D beam theories give a low rate convergence when compared to the other shear deformation theories. For the case of thick homogeneous beam $(p = 0)$ as shown in Fig. 3, the variation of non-dimensional axial stress through the thickness appeared linear.

4.2 Free vibration and buckling analysis

To check the accuracy and effectiveness of the suggested theoretical approach, Tables 6 and 7 present the comparisons of the non-dimensional fundamental frequencies and critical buckling loads of simply supported FG beams subjected to in-plane mechanical loads. The results are estimated for two cases of slenderness ratios $L/h = 5$ and $L/h = 20$ and for various values of the power-law index $(p = 0, 1, 2, 5, 10)$, and compared with those obtained on the basis of the classical beam theory (CBT) of Euler-Bernoulli (1744), first-order shear deformation theory (FBT) of Timoshenko

Table 6 Comparison of non-dimensional fundamental natural frequencies of simply supported FG beams with various power-law indexes

L/h	Theory	Model	ε_z	p				
				0	1	2	5	10
5	Euler-Bernoulli (1744)	CBT	$=0$	5.3953	4.1484	3.7793	3.5949	3.4921
	Timoshenko (1921)	FBT	$=0$	5.1525	3.9902	3.6344	3.4312	3.3135
	Simsek (2010)	TBT	$=0$	5.1527	3.9904	3.6264	3.4012	3.2816
	Thai and Vo (2012)	SBT	$=0$	5.1531	3.9907	3.6263	3.3998	3.2811
	Sayyad and Ghugal (2018b)	HyBT	$=0$	5.1527	3.9904	3.6264	3.4014	3.2816
	Vo *et al.* (2015)	Quasi-3D	$\neq 0$	5.1618	4.0079	3.6442	3.4133	3.2903
	Present	Quasi-3D	$\neq 0$	5.1616	4.0238	3.6689	3.4364	3.3039
20	Euler-Bernoulli (1744)	CBT	$=0$	5.4777	4.2163	3.8472	3.6628	3.5547
	Timoshenko (1921)	FBT	$=0$	5.4603	4.2051	3.8368	3.6509	3.5416
	Simsek (2010)	TBT	$=0$	5.4603	4.2050	3.8361	3.6485	3.5390
	Thai and Vo (2012)	SBT	$=0$	5.4603	4.2051	3.8361	3.6484	3.5389
	Sayyad and Ghugal (2018b)	HyBT	$=0$	5.4603	4.2050	3.8361	3.6485	3.5390
	Vo *et al.* (2015)	Quasi-3D	$\neq 0$	5.4610	4.2347	3.8765	3.6824	3.5590
	Present	Quasi-3D	$\neq 0$	5.4610	4.2357	3.8781	3.6839	3.5599

(1921), and with the corresponding results reported by Simsek (2010) using the third-order beam theory (TBT), higher-order sinusoidal beam theory (SBT) presented by Thai and Vo (2012), higher-order hyperbolic beam theory (HyBT) developed by Sayyad and Ghugal (2018b), as well as the analytical solutions of the Quasi-3D shear deformation model built by Vo *et al.* (2015). For the case of buckling problem, the present results are also compared with the provided results by the existing efficient shear deformation beam theories (Nguyen *et al.* 2013, Vo *et al.* 2014, Sayyad and Ghugal 2018b). For buckling and free vibration problems, the numerical results show that the solutions derived from the present theory are almost identical to those obtained by Vo *et al.* (2015) for different values of the power-law index and slenderness ratio. It can be observed that the results obtained by the Quasi-3D beam model $(\varepsilon_z \neq 0)$ are slightly larger than those of other shear deformation beam theories $(\varepsilon_z = 0)$ found in the literature, which is due to the ignorance of thickness stretching effects. Furthermore, these tables show that the fundamental natural frequencies and critical buckling loads decrease with the increase of the power-law index for both slenderness ratios (This means that as p approaches infinity, the lowest frequency values are obtained). The argument for this is that an increase in the value of the power indices leads to an increase in the percentage of metallic phase, which makes the FG beam more flexible, and thus to a reduction in the values of the fundamental frequency and the critical buckling load. Furthermore, it can also be seen that the increase in slenderness has an effect on the increase in the target parameters calculated in this study.

5. Conclusions

The objective of this paper is to present a simple Quasi-3D integral higher-order shear deformation beam theory for the mechanical analysis of simply supported advanced composite

Table 7 Comparison of non-dimensional critical buckling loads of simply supported FG beams with various power-law indexes

L/h	Theory	Model	ε_z	p				
				0	1	2	5	10
5	Euler-Bernoulli (1744)	CBT	$= 0$	53.5778	26.7054	20.8387	17.6227	16.0517
	Nguyen *et al.* (2013)	FBT	$= 0$	48.8350	24.6870	19.2450	16.0240	14.4270
	Vo *et al.* (2014)	HBT	$= 0$	48.8401	24.6911	19.1605	15.7400	14.1468
	Sayyad and Ghugal (2018b)	HyBT	$= 0$	48.5960	24.5840	19.0710	15.6450	14.0520
	Vo *et al.* (2015)	Quasi-3D	$\neq 0$	49.5901	25.2116	19.6124	16.0842	14.4116
	Present	Quasi-3D	$\neq 0$	49.5905	25.3504	19.7589	16.1607	14.4340
10	Euler-Bernoulli (1744)	CBT	$= 0$	53.5778	26.7054	20.8387	17.6227	16.0517
	Nguyen *et al.* (2013)	FBT	$= 0$	52.3080	26.1710	20.4160	17.1940	15.6120
	Vo *et al.* (2014)	HBT	$= 0$	52.3082	26.1727	20.3936	17.1118	15.5291
	Sayyad and Ghugal (2018b)	HyBT	$= 0$	52.2380	26.1410	20.3660	17.0820	15.5000
	Vo *et al.* (2015)	Quasi-3D	$\neq 0$	52.5361	26.4869	20.7164	17.3580	15.6895
	Present	Quasi-3D	$\neq 0$	52.5361	26.6399	20.8931	17.4803	15.7505
20	Euler-Bernoulli (1744)	CBT	$= 0$	53.5778	26.7054	20.8387	17.6227	16.0517
	Vo *et al.* (2014)	HBT	$= 0$	53.2546	26.5718	20.7275	17.4935	15.9185
	Vo *et al.* (2015)	Quasi-3D	$\neq 0$	53.3075	26.8174	21.0066	17.7048	16.0416
	Present	Quasi-3D	$\neq 0$	53.3144	26.9772	21.1934	17.8416	16.1151

beams under sinusoidally and uniformly distributed loads. The properties of the beam are graded along the thickness direction and are assumed to be varying continuously according to a power-law distribution. The proposed theory is based on a new higher-order displacement model with combining both shear and normal deformation effects by considering undetermined integral terms, thus satisfies the shear stress-free boundary conditions on the upper and lower surfaces of the beam. The governing equations and its boundary conditions are established by means of the dynamic version of the principle of virtual work. Closed form expressions are derived via Navier's type solution to present the bending, buckling and free vibration responses of thick and thin FG beams. The numerical results of displacements, stresses, critical buckling loads and natural frequencies are provided for different power-law index values by using the proposed computational method and compared with various shear deformation theories found in the literature to demonstrate the accuracy and assess the effectiveness of the present theory. Finally, this unified formulation lends ideal conditions to study a number of problems related to the mechanical behaviour of multilayered composite beams and nanobeams under mechanical and thermal loads (Sina *et al.* 2009, Katariya *et al.* 2016, Karamanli 2017, Akbaş *et al.* 2018, Eltaher *et al.* 2018, Selmi and Bisharat 2018, Safa *et al.* 2019, Avcar 2019, Karami and Janghorban 2019, Madenci 2019, Ghumare and Sayyad 2020, Ebrahimi and Hosseini 2021).

Finally, based on this study, we can underline the following points:

1. The proposed theoretical model meets the stress-free boundary conditions on the top and bottom faces of the beam, and do not include a shear correction factor.
2. Both FBT and HBT neglect the effect of thickness stretching, which is perceptible in thick FG

beams.
3. For thick beams made of FG materials, it is generally found that the natural frequency and critical buckling load obtained using the classical Euler-Bernoulli theory are greater than those of other shear deformation models.
4. Numerical results also show significant effects of the power law index and slenderness ratio on the static bending, buckling and dynamic responses of FG beams.
5. The suggested Quasi-3D shear deformation beam theory costs less effort and time of computation, although the thickness stretching effect is considered. Therefore, it is better to use this formulation in the future work to investigate the symmetric and anti-symmetric multilayered composite structures.

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