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A Mathematical model to estimate the wind power using three parameter Weibull distribution

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Abstract. Weibull distribution is a suitable distribution to use in modeling the life time data. It has been found to be a exact fit for the empirical distribution of the wind speed measurement samples. In brief this paper consist of important properties and characters of Weibull distribution. Also we discuss the application of Weibull distribution to wind speed measurements and derive an expression for the probability distribution of the power produced by a wind turbine at a fixed location, so that the modeling problem reduces to collecting data to estimate the three parameters of the Weibull distribution using Maximum likelihood Method.

Keywords: three-parameter Weibull distribution; mean; variance; maximum likelihood method; wind power

1. Introduction

Wind energy has started to compete with other energy resources and it is recently being applied in various industries. Wind energy history, wind-power meteorology, the energy–climate relations, wind-turbine technology, wind economy, wind–hybrid applications and the current status of installed wind energy capacity all over the world reviewed critically with further enhancements and new research trend direction suggestions.

Wind energy can be utilized for a variety of functions ranging from windmills to pumping water and sailing boats. With increasing significance of environmental problems, clean energy generation becomes essential in every aspect of energy consumption. Wind energy is very clean but not persistent for long period of time. In potential wind energy generation studies fossil fuels must be supplemented by wind energy. There are many scientific studies in wind energy domain, which have treated the problem with various approaches Cook (2004). General trends towards wind and other renewable energy resources increased after the energy crises of the 20th century.

As a meteorological variable, wind describes fuel of wind energy. In energy production, wind takes the same role as water, and wind variables should be analyzed. Wind speed deviation and changeability depend on time and area. This situation requires a new tendency for wind-speed modeling and search for the atmospheric boundary layer modeling as a special consideration in wind-power research. There are many papers concerning these subjects. Wind speed and energy

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change with time and are not continual at the same area during the whole year. Wind speed is a regionalized variable measured at a set of irregular sites. Wind energy investigations mostly rely upon arithmetic average of the wind speed. However, many authors based the wind energy estimates on elaborated wind-speed statistics, including the standard deviation, skewness and kurtosis coefficient. Some researches advocate the use of three-parameter Weibull distribution in wind velocity applications. Their suggestions are taken as granted in many parts of the world for wind energy calculations Soukissian (2013).

This paper is intended as an introduction to the use of the Three-parameter Weibull distribution to model single-site hourly average wind speeds (i.e., wind speeds at a given wind farm). We derive a probability distribution for the electric power output of a wind turbine with given cut-in, cut-out and rated wind speeds. We also compute MLE method of this distribution and derive an expression for the capacity factor of the wind turbine Chang (2011).

Our focus will be abstract and mathematical rather than concrete and empirical and running some algorithms and statistical tests with standard software (e.g., Mat lab).

2. Weibull distribution

In the early 1920s, there were three groups of scientists working on the derivation of the distribution independently with different purposes. Waloddi Weibull was one of them Weibull (1939). The distribution bears his name because he promoted this distribution both internationally and interdisciplinary. His discoveries lead the distribution to be productive in engineering practice, statistical modeling and probability theory. Thousands of papers have been written on this distribution and it is still drawing broad attention. It is of importance to statisticians because of its ability to fit to data from various areas, ranging from life data to observations made in economics, biology or materials reliability theory. Recently empirical studies have shown that the Weibulll distribution is superior to the classical stable distributions inclusive the normal distributions, for fitting empirical economic data (Morgan *et al*. 2011 a,b,c).

The Weibull distribution is widely used in reliability. The Weibull distribution is the most widely used life time distribution. It is used as a model for life time of many manufactured items such as vacuum tubes, ball bearings and electrical insulation. It is also used in biomedical applications, for example as the distribution for life time to occurrence (diagnosis) of tumors in human populations or in animals meant for experimental research. Further the Weibull distribution gives a good match with the experimental data and it is the suitable model to estimate the wind energy potential of a particular location (D’Amico *et al*.2015 a,b).

2.1 Probability density function

The Weibull probability distribution which is a three parameter function can be expressed mathematically as

$$f(k,c,\varepsilon) = \left(\frac{k}{c-\varepsilon}\right)\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^{k-1} exp\left(-\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^{k}\right) \tag{1}$$

Where v is the wind speed, k is the non dimensional shape parameter, c is the scale parameter and ε is the location parameter. The dimensions of c and ε are same to $v\left(\frac{m}{s}\right)$. The location parameter ε is the minimum wind speed and $v \geq \varepsilon$.

Proposition 2.1:

The Weibull distribution with parameters $k, c, \varepsilon > 0$ has cumulative distribution function (cdf) given by

$$F(v; k, c, \varepsilon) = 1 - exp\left(-\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k\right) \quad , \quad v \geq 0 \tag{2}$$

Proof:

By the fundamental theorem of calculus

$$P(V \leq v) = \int_{-\infty}^{v} f(v; k, c, \varepsilon)\, dy = \int_{0}^{v} \left(\frac{k}{c-\varepsilon}\right)\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^{k-1} exp\left(-\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k\right) dy$$

$$= -\int_0^v \frac{d}{dy}(exp\left(-\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k\right))dy$$

$$= 1 - exp\left(-\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k\right)$$

Proposition 2.2:

Let $V \sim Weibull(k, c, \varepsilon)$. Then V has mean, variance respectively given by

(i)$E(V) = (c-\varepsilon)\Gamma\left(1+\frac{1}{k}\right) + \varepsilon$

(ii)$Var(V) = (c-\varepsilon)^2[\Gamma\left(1+\frac{2}{k}\right) - \Gamma^2(1+\frac{1}{k})]$

where $\Gamma(.)$ denotes the Gamma function Hong and Li (2014).

Proof of (i):

$$E(v) = \int_0^{\infty} v\ f(v)\, dv = \int_0^{\infty} v\left(\frac{k}{c-\varepsilon}\right)\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^{k-1} exp\left(-\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k\right) dv$$

We make the change of variable $y = \left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k$ to obtain

$$= (c-\varepsilon)\ \Gamma\left(1+\frac{1}{k}\right) + \varepsilon$$

Proof of (ii):

In the same way $E(V^2) = (c-\varepsilon)^2\Gamma\left(1+\frac{2}{k}\right) + \varepsilon^2 + 2\varepsilon(c-\varepsilon)\Gamma\left(1+\frac{1}{k}\right)$

Now $Var(V) = E(V^2) - [E(V)]^2$

$$=(c-\varepsilon)^2[\Gamma\left(1+\frac{2}{k}\right) - \Gamma^2(1+\frac{1}{k})]$$

Proposition 2.3:

Suppose $v_1, v_2, \ldots\ldots.., v_n$ are random samples from (1). Let r denote the sample correlation between the v_i and their ranks. Let CV and S respectively , denote the sample coefficient of variation and sample standard deviation. Then

$$\mathrm{r} = \left(\frac{\mu_V-\varepsilon}{\sigma_V}\right)\left(\frac{1}{2}-\frac{1}{2^{1+\frac{1}{k}}}\right)\sqrt{\frac{12(n-1)}{(n+1)}}$$

Proof:

The correlation between V_i and their ranks, say R_i is

$$r = corr(V_i, R_i) = \left(\int_\varepsilon^\infty v\, F_V(v)\ dF_V(v) - \frac{\mu_V}{2}\right)\sqrt{\frac{12(n-1)}{\sigma_V{}^2(n+1)}} \tag{3}$$

where $\mu_V = E(V)$ and $\sigma_V{}^2 = Var(V)$.

Using (1) and (2) we have

$$\int_\varepsilon^\infty v\, F_V(v)\ dF_V(v) = \frac{k}{c-\varepsilon}\int_\varepsilon^\infty v\left\{1-e^{-\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k}\right\}\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^{k-1} e^{-\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k} dv$$

Now let $v-\varepsilon = z$. Then we have

$$= \frac{k}{c-\varepsilon}\int_0^\infty (z+\varepsilon)\left\{1-e^{-\left(\frac{z}{c-\varepsilon}\right)^k}\right\}\left(\frac{z}{c-\varepsilon}\right)^{k-1} e^{-\left(\frac{z}{c-\varepsilon}\right)^k} dz$$

$$= \frac{k}{c-\varepsilon}\int_0^\infty (z+\varepsilon)\left(\frac{z}{c-\varepsilon}\right)^{k-1} e^{-\left(\frac{z}{c-\varepsilon}\right)^k} dz - \frac{k}{c-\varepsilon}\int_0^\infty (z+\varepsilon)\left(\frac{z}{c-\varepsilon}\right)^{k-1} e^{-2\left(\frac{z}{c-\varepsilon}\right)^k} dz$$

$$= \mathrm{I}_1 + \mathrm{I}_2$$

Setting $y = \left(\frac{z}{c-\varepsilon}\right)^k$ for I_1 & $y = 2\left(\frac{z}{c-\varepsilon}\right)^k$ for I_2 and $t = 1+\frac{1}{k}$, we have

$$\int_\varepsilon^\infty v\, F_V(v)\ dF_V(v) = \left((c-\varepsilon)\int_0^\infty y^{t-1}e^{-y}dy + \varepsilon\right) - \left(\frac{(c-\varepsilon)}{2^{\frac{1}{k}+1}}\int_0^\infty y^{t-1}e^{-y}dy + \frac{\varepsilon}{2}\right)$$

$$= (c-\varepsilon)\Gamma\left(1+\frac{1}{k}\right)\left(1-\frac{1}{2^{\frac{1}{k}+1}}\right) - \frac{\varepsilon}{2} \tag{4}$$

Using (4) in (3), the result is obtained.

Proposition 2.4:

The inverse Weibull cumulative function can be expressed in closed form.

Proof:

The p.d.f and cumulative distribution of Weibull are (1) and (2) respectively.
Equating (2) to u where 0<u<1, we get $F^{-1}(v) = (c-\varepsilon)[-\ln(1-u)]^{1/k} + \varepsilon$.
Which shows that the inverse Weibull cumulative function can be expressed in closed form.

Proposition 2.5:

Let $\alpha > 0\ and\ V \sim Weibull(k, c, \varepsilon)\ then\ \alpha V = Y \sim Weibull(k, \alpha c, \varepsilon)$

Proof:

Using Proposition 2.1, we see that

$$F_Y(v) = P(Y \leq v) = P\left(X \leq \frac{v}{\alpha}\right) = 1 - exp\left(-\left(\frac{v-\varepsilon}{\alpha(c-\varepsilon)}\right)^k\right)$$

This is the cdf of $Weibull(k, \alpha c, \varepsilon)$ random variable. Since the distribution function uniquely characterizes the law of the random variable, the conclusion follows immediately.

Proposition 2.6:

Let $V_1, V_2, \dots \dots .., V_n$ be independent random variables with $V_i \sim Weibull(k_i, c_i, \varepsilon)$ for $1 \leq i \leq n.$ Then

$$P(\min(V_1, V_2, \dots \dots .., V_n) > v) = \exp\left(-\sum_{i=1}^{n}\left(\frac{v-\varepsilon}{c_i-\varepsilon}\right)^{k_i}\right)$$

If $k_1 = k_2 = \cdots \dots .. \ k_n = k$ then $\min(V_1, V_2, \dots \dots .., V_n) \sim Weibull(k, c_{min}, \varepsilon)$ where $c_{min} - \varepsilon = \left(\sum_{i=1}^{n}(c_i-\varepsilon)^{-k}\right)^{-1/k}$

Proof:

Clearly $\min(V_1, V_2, \dots \dots .., V_n) > v \iff V_i > v \quad \forall\, i = 1,2, \dots .., n$
Since the V_i are independent

$$P(V_1 > v, V_2 > v, \dots .., V_n > v) = \prod_{i=1}^{n} P(V_i > v)$$

$$= \prod_{i=1}^{n}\left(1 - \left(1 - exp\left(-\left(\frac{v-\varepsilon}{c_i-\varepsilon}\right)^{k_i}\right)\right)\right)$$

$$= \prod_{i=1}^{n} exp\left(-\left(\frac{v-\varepsilon}{c_i-\varepsilon}\right)^{k_i}\right) \quad = \quad \exp\left(-\sum_{i=1}^{n}\left(\frac{v-\varepsilon}{c_i-\varepsilon}\right)^{k_i}\right)$$

This completes the proof of the first assertation. Now suppose that $k_i = k\,, 1 \leq i \leq n$. Then we may write the above as

$$= exp\left(-\frac{(v-\varepsilon)^k}{\left(\sum_{i=1}^{n}(c_i-\varepsilon)^{-k}\right)^{-1}}\right) \quad = \quad exp\left(-\frac{(v-\varepsilon)^k}{(c_{min}-\varepsilon)^k}\right)$$

We therefore have that

$$P(\min(V_1, V_2, \dots \dots .., V_n) \leq v) = 1 - P(\min(V_1, V_2, \dots \dots .., V_n) > v)$$

$$= 1 - exp\left(-\left(\frac{v-\varepsilon}{c_{min}-\varepsilon}\right)^k\right)$$

Which shows that $\min(V_1, V_2, \dots\dots.., V_n) \sim Weibull(k, c_{min}, \varepsilon)$.

3. Concept of Hazard rate

The parameter of interest here is the Hazard rate, h(v). This is the conditional probability that an equipment will fail in a given interval of unit time given that it has survived until that interval of time. It is, therefore, the instantaneous failure rate and can in general be thought of as a measure of the probability of failure, where this probability varies with the time the item has been in service. The three failure regimes are defined in terms of Hazard rate and not, as is a common misconception, in terms of failure rate

One of the most important quantities characterizing the life phenomenon in life testing analysis is the Hazard rate function defined by $h(v) = \frac{f(v)}{1-F(v)}$.This is one of the attractive properties that made the Weibull distribution so applicable. The Hazard rate function corresponding to Eqs. (1) and (2) is $h(v) = \frac{k}{c-\varepsilon}\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^{k-1}$. And the cumulative Hazard function is given by $H(v) = \int h(v)\,\mathrm{dv} = \left(\frac{v-\varepsilon}{c-\varepsilon}\right)^{k}$ Wayne Nelson(1972).

4. Estimation of the parameters

This section discusses the statistical fitting of wind speed measurement data to the 3-parameter Weibull distribution, which has been recognized as a suitable fit for wind speed distribution .To be more precise, what is meant of "wind speed" here is the periodic (e.g., per minute) measurement of wind speed at a particular location, height and in a certain direction. Clearly, wind has a velocity at any given time, i.e., a speed as well as a direction. Wind speed measurements, thus, correspond to the magnitude of wind velocity in the measurement direction, which is argued to closely resemble a Weibull distribution Edwards (2001) and Archer (2005)

Definiton4.1: Any function of the random samples $V_1, V_2, \dots.., V_n$ that are being observed say T($V_1, V_2, \dots.., V_n$) is called a Statistic.

Definiton 4.2: If a statistic is used to estimate an unknown parameter θ of a distribution then it is called an Estimator and a particular value of the estimator say $T_n(V_1, V_2, \dots.., V_n)$ is called an estimate of θ.The process of estimating an unknown parameter is known as estimation.

4.1 Some charact cs of estimators:

Various statistical properties of estimators can be used to decide which estimator is most appropriate in a given situation (Lun and Lam 2000, Mo *et al.* 2015 a,b).

4.2 Biased and unbiased estimator

- An estimator $\hat{\theta}$ is said to be unbiased for θ if E($\hat{\theta}$)= θ.

- If $\mathrm{E}(\hat{\theta}) \neq \theta$ then we say that $\hat{\theta}$ is biased.
- In general the bias of an estimator is $\mathrm{B}(\hat{\theta}) \equiv \mathrm{E}(\hat{\theta}) = \theta$.
- If $\mathrm{B}(\hat{\theta}) > 0$ then $\hat{\theta}$ over estimates θ.
- If $\mathrm{B}(\hat{\theta}) < 0$ then $\hat{\theta}$ under estimates θ.
- If $\hat{\theta}$ is unbiased then of course $\mathrm{B}(\hat{\theta}) = 0$.

4.3 Good estimators

In general a good estimator θ has the following properties.

- $\hat{\theta}$ is unbiased for θ
- $\hat{\theta}$ has small variance.

4.4 Unbiasedness

A statistic T is an unbiased estimator of the parameter θ iff $E(T) = \theta$

Example: Let X_1, X_2, X_3 be a random sample of size 3 from a Normal population of $N(\mu, \sigma^2)$. The statistic $T = \frac{1}{4}(X_1 + 2X_2 + X_3)$ is an unbiased estimate of μ.

Since $E(T) = E[\frac{1}{4}(X_1 + 2X_2 + X_3)]$

$$= \frac{1}{4}(\mu + 2\mu + \mu)$$
$$= \mu$$

4.5 Likelihood function

Let $(X_1, X_2, \dots, X_n): \Omega \to \mathbb{R}^n$ be a random variable with probability density function $f(x;\theta)$ for a k-tuple of parameters $\theta \epsilon \Theta \subset \mathbb{R}^k$. Recall that for a sample $X(\omega) = x \epsilon \mathbb{R}$, the likelihood function of θ, denoted by $l_x(\theta) = l(\theta)$ is defined by $l: \Theta \to \mathbb{R}$, $l(\theta) = f_\theta(x)$.

Let $\bar{\Theta}$ denote the topological closure of Θ in $\mathbb{R}$. If $\hat{\theta} \epsilon \bar{\Theta}$ satisfies $l(\hat{\theta}) = sup_{\theta \epsilon \bar{\Theta}} l(\theta)$, we say that $\hat{\theta}$ is a maximum likelihood estimate (MLE) of θ. If $\hat{\theta}: \Omega \to \mathbb{R}$, $\omega \to \hat{\omega}$, where $l_{X(\omega)}(\hat{\omega}) = sup_{\theta \epsilon \bar{\Theta}} l(\theta)$, then we say that $\hat{\theta}$ is a maximum likelihood estimator (MLE) of θ.

We define the log –likelihood function to be $\log l(\theta)$. We first note that this definition makes sense since we may assume that $l(\theta) > 0 \; for \; all \; \theta$, Shao and Jun (2003).

4.6 Maximum likelihood

It is not true that every probability density function has a nice closed form expression for the MLE $\hat{\theta}$. Unfortunately for us, this is also the case for the Weibull distribution. However, with the use of numerical software such as Mat lab, we can use an iterative scheme to approximate $\hat{\theta}$.

Let $V_1, V_2, \dots, V_n$ be independent identically distributed (i.i.d) $Weibull(k, c, \varepsilon)$ samples. Since the joint density function of independent random variables factors, we have that

$$l(k, c, \varepsilon) = \prod_{i=1}^{n} f(v_i; k, c, \varepsilon) = \prod_{i=1}^{n} \left(\frac{k}{c-\varepsilon}\right)\left(\frac{v_i - \varepsilon}{c - \varepsilon}\right)^{k-1} exp\left(-\left(\frac{v_i - \varepsilon}{c-\varepsilon}\right)^{k}\right)$$

$$=\left(\frac{k}{c-\varepsilon}\right)^n \, exp\left(-\frac{1}{(c-\varepsilon)^k}\sum_{i=1}^n (v_i-\varepsilon)^k\right)\frac{\prod_{i=1}^n (v_i-\varepsilon)^{k-1}}{(c-\varepsilon)^{n(k-1)}}$$

Taking the natural logarithm of both sides , we obtain that the log-likelihood function as

$$\log l(k,c,\varepsilon) = n \, log \, k - nk \, log \, (c-\varepsilon) - \frac{1}{(c-\varepsilon)^k}\sum_{i=1}^n (v_i-\varepsilon)^k + \sum_{i=1}^n (k-1)\log(v_i-\varepsilon)$$

4.7 Evaluation of the parameters using MLE method

Taking partial derivatives with respect to $c, k \; and \; \varepsilon$ we obtain (Majid *et al.* 2015 a,b)

$$\frac{\delta}{\delta c} log \, l(k,c,\varepsilon) = \frac{-nk}{(c-\varepsilon)} + \frac{k}{(c-\varepsilon)^{k+1}}\sum_{i=1}^n (v_i-\varepsilon)^k$$

$$\frac{\delta}{\delta k}\log \; l(k,c,\varepsilon) = \frac{n}{k} - n\log(c-\varepsilon) - \frac{1}{(c-\varepsilon)^k}\sum_{i=1}^n (v_i-\varepsilon)^k \log(v_i-\varepsilon)$$

$$+ \quad \sum_{i=1}^n (v_i-\varepsilon)^k \frac{\log(c-\varepsilon)}{(c-\varepsilon)^k} + \sum_{i=1}^n \log(v_i-\varepsilon)$$

$$\frac{\delta}{\delta \varepsilon}\log \; l(k,c,\varepsilon) = \frac{nk}{(c-\varepsilon)} + \frac{k}{(c-\varepsilon)^k}\sum_{i=1}^n (v_i-\varepsilon)^{k-1} \quad + k\sum_{i=1}^n \frac{(v_i-\varepsilon)^k}{(c-\varepsilon)^{k+1}} - (k-1)\sum_{i=1}^n \frac{1}{(v_i-\varepsilon)}$$

The log –likelihood function is in $C^{1,1}(0,\infty)$. We set the three preceding equations to 0. We have

$$0 = \frac{-nk}{(c-\varepsilon)} + \frac{k}{(c-\varepsilon)^{k+1}}\sum_{i=1}^n (v_i-\varepsilon)^k$$

$$\Rightarrow c = \left(\sum_{i=1}^n \frac{(v_i-\varepsilon)^k}{n}\right)^{1/k} + \; \varepsilon$$

Similarly we have

$$0 = \frac{n}{k} - n\log(c-\varepsilon) - \frac{1}{(c-\varepsilon)^k}\sum_{i=1}^n (v_i-\varepsilon)^k \log(v_i-\varepsilon) + \sum_{i=1}^n (v_i-\varepsilon)^k \frac{\log(c-\varepsilon)}{(c-\varepsilon)^k} + \sum_{i=1}^n \log(v_i-\varepsilon)$$

$$= \frac{n}{k} - n\log(c-\varepsilon) - \frac{1}{(c-\varepsilon)^k}\left(\sum_{i=1}^n (v_i-\varepsilon)^k \log(v_i-\varepsilon)\right) + \frac{n(c-\varepsilon)^k \log(c-\varepsilon)}{(c-\varepsilon)^k} \sum_{i=1}^n \log(v_i-\varepsilon)$$

$$= \frac{n}{k} - n \; \frac{\sum_{i=1}^n \log(v_i-\varepsilon)(v_i-\varepsilon)^k}{\sum_{i=1}^n (v_i-\varepsilon)^k} + \sum_{i=1}^n log(v_i-\varepsilon)$$

$$\Leftrightarrow \frac{\sum_{i=1}^n (v_i-\varepsilon)^{\mathrm{k}}\log(v_i-\varepsilon)}{\sum_{i=1}^n (v_i-\varepsilon)^{\mathrm{k}}} - \frac{1}{k} = \frac{\sum_{i=1}^n \log(v_i-\varepsilon)}{n}$$

In the same way

$$0=\frac{nk}{(c-\varepsilon)}+\frac{k}{(c-\varepsilon)^k}\sum_{i=1}^{n}(v_i-\varepsilon)^{k-1}+k\sum_{i=1}^{n}\frac{(v_i-\varepsilon)^k}{(c-\varepsilon)^{k+1}}-(k-1)\sum_{i=1}^{n}\frac{1}{(v_i-\varepsilon)}$$
$$=\frac{nk}{(c-\varepsilon)}+nk\sum_{i=1}^{n}\frac{1}{(v_i-\varepsilon)}+\frac{nk}{(c-\varepsilon)}-(k-1)\sum_{i=1}^{n}\frac{1}{(v_i-\varepsilon)}$$
$$=\frac{2kn}{(c-\varepsilon)}+\sum_{i=1}^{n}\frac{1}{(v_i-\varepsilon)}(nk-k+1)$$
$$\Leftrightarrow\sum_{i=1}^{n}\frac{1}{(v_i-\varepsilon)}=\frac{-2kn}{(c-\varepsilon)(nk-k+1)}$$

Our focus will be generated using some standard software (i.e., Matlab)

5. Single -site power distribution

In this section we introduce the aerodynamics at a very elementary and simplified level of wind turbines. Wind is the moving mass of air and as such, has kinetic energy. A portion of the kinetic energy is exploited to drive a wind turbine. In the wind energy E is equivalent to flux of the moving air mass m , Kinetic energy with a speed v as

$$E=\frac{1}{2}\,mv^2 \tag{5}$$

Since the measurement of m is almost impossible in aero dynamics. One can express it in terms of volume. It is preferable to convert the volume to specific mass $\rho=\frac{m}{v}$. Therefore Eq. (5) becomes $E=\frac{1}{2}\,\rho V v^2$. The volume can be expressed as V=AL in which A is vertical control cross section and L is the horizontal section. Physically $L=vt$. Therefore $V=A(vt)$. Hence $E=\frac{1}{2}\rho A t v^3$.Practically it is preferable to consider the wind energy per vertical unit area per time as unit wind energy $E_{u.}$ Using the above idea power in the wind is calculated by

$$P=\frac{1}{2}\rho A v^3$$

Where P is the energy in the wind

ρ is the density of the air taken as 1.225 kg/m^3

A is the cross section of the area through which it flows.

v is the mean value of the of the wind speed

Suppose we have a horizontal axis wind turbine which has infinitely many blades with no spacing between. If we assume that the air travels in a cylinder , called a stream tube and changes in velocity are continuous , and a few other ideal technical hypotheses, it can be shown that there is a technical limit to the amount of kinetic energy the Betz turbine can extract from the wind. This quantity $C=\frac{16}{27}\approx 0.59$ is known as Betz limit. We call C the power coefficient of the Betz turbine. Thus the amount of energy the Betz turbine extracts is $\frac{1}{2}C\rho Av^3$.

Modern wind turbines come reasonable close to the Betz limit with power coefficients upwards of $C_P \approx 0.5$ (Gasch and Jochen 2012). For the remainder of this paper, we will assume the electric power produced by a wind turbine is given by $\frac{1}{2}C_P\eta\rho Av^3$ where C_P is the power coefficient of the turbine , $\eta\epsilon(0,1)$ is some efficiency constant.

5.1 Power distribution of ideal wind turbine

Suppose $V \sim Weibull(k,c,\varepsilon)$. Let $P = \frac{1}{2}\rho AV^3$, where ρ and A are as above and are constants. In reality, ρ is stochastic, but we are assuming the variability of ρ is negligible. A wind turbine, though , does not operate at all wind speeds V. The implied wind speed of wind turbine with power coefficient C_P and efficiency coefficient η producing power P is given by

$$V_{turbine} = \begin{cases} 0 & V_{turbine} < v_{cut-in} \\ V & v_{cut-in} \le V \le v_{rated} \\ v_{rated} & v_{rated} < V < v_{cut-off} \\ 0 & V \ge v_{cut-off} \end{cases}$$

Where $v_{cut-in} < v_{rated} < v_{cut-off}$ are specified by the manufacturer. For example , the GE 1.5 ME SLE wind turbine has cut in wind speed $v_{cut-in} = 3.5\, m/s,$ rated wind speed $v_{rated} = 14\, m/s,$ and the cut-out wind speed $v_{cut-off} = 25\, m/.$
(see http://geosci.uchicago.edu/~moyer/GEOS24705/Readings/GEA14954C15-MW-Broch.pdf for the technical details of the 1.5 SLE model) (Genc *et al*. 2005 a,b). Analogously

$$P_{turbine} = \begin{cases} 0 & V_{turbine} < v_{cut-in} \\ \frac{1}{2}\rho AC_P\eta V^3 & v_{cut-in} \le V \le v_{rated} \\ \frac{1}{2}\rho AC_P\eta v_{rated}^3 & v_{rated} < V < v_{cut-off} \\ 0 & V \ge v_{cut-off} \end{cases}$$

It is evident $V_{turbine}$ and $P_{turbine}$ are discontinuous random variable , but we can still compute their distribution functions and moments. They will fail to have a probability density function, though, their laws are not absolutely continuous with respect to the Lebesgue measure: the law of $V_{turbine}$ assigns non zero probability to the singleton {0}, which is a set of Lebesgue measure zero (Kollu *et al*. 2012 a,b,c).

Proposition 5.1:
P has cdf

$$F_P(v) = 1 - exp\left(-\frac{1}{(c-\varepsilon)^k}\left(\frac{2(v-\varepsilon)}{\rho A}\right)^{k/3}\right)$$

and pdf

$$f_P(v) = \frac{k(v-\varepsilon)^{\frac{k}{3}-1}}{3(c-\varepsilon)^k}\left(\frac{2}{\rho A}\right)^{\frac{k}{3}} exp\left(-\frac{1}{(c-\varepsilon)^k}\left(\frac{2(v-\varepsilon)}{\rho A}\right)^{k/3}\right)$$

Proof:

By proposition 2.1 we have that

$$F_P(v) = \mathbb{P}(P \le v) = \mathbb{P}\left(\frac{1}{2}\rho A V^3 \le v\right) = \mathbb{P}\left(V \le \left(\frac{2v}{\rho A}\right)^{\frac{1}{3}}\right) = F\left(\left(\frac{2v}{\rho A}\right)^{\frac{1}{3}}\right)$$

$$= 1 - exp\left(\frac{-1}{(c-\varepsilon)^k}\left(\frac{2(v-\varepsilon)}{\rho A}\right)^{k/3}\right)$$

Differentiating both sides with respect to v, we obtain that P has probability density function f_P given by

$$f_P(v) = \frac{k}{3(c-\varepsilon)^k}\,\frac{2}{\rho A}\left(\frac{2(v-\varepsilon)}{\rho A}\right)^{\frac{k}{3}-1} exp\left(\frac{-1}{(c-\varepsilon)^k}\left(\frac{2(v-\varepsilon)}{\rho A}\right)^{k/3}\right)$$

$$= \frac{k(v-\varepsilon)^{\frac{k}{3}-1}}{3(c-\varepsilon)^k}\left(\frac{2}{\rho A}\right)^{\frac{k}{3}} exp\left(\frac{-1}{(c-\varepsilon)^k}\left(\frac{2(v-\varepsilon)}{\rho A}\right)^{k/3}\right).$$

Proposition 5.2:

$P_{turbine}$ has cdf

$$F_{P_{turbine}}(v) = \begin{cases} 0 & -\infty < v < 0 \\ 1 + e^{-\left(\frac{v_{cut-off}-\varepsilon}{(c-\varepsilon)}\right)^k} - e^{-\left(\frac{v_{cut-in}-\varepsilon}{(c-\varepsilon)}\right)^k} & 0 \le v \le \frac{1}{2}\rho A C_P \eta v_{cut-in}^3 \\ 1 - e^{-\frac{1}{(c-\varepsilon)^k}\left(\frac{2v}{\rho A C_P \eta}\right)^{\frac{k}{3}}} + e^{-\left(\frac{v_{cut-off}-\varepsilon}{(c-\varepsilon)}\right)^k} & \frac{1}{2}\rho A C_P \eta v_{cut-in}^3 \le v \le \frac{1}{2}\rho A C_P \eta v_{rated}^3 \\ 1 & \frac{1}{2}\rho A C_P \eta v_{rated}^3 \le v < \infty \end{cases}$$

Proof:

It is evident that $\mathbb{P}(P_{turbine} \le 0)$ is

$$\mathbb{P}\left(\{V \le v_{cut-in}\} \cup \{V \ge v_{cut-off}\}\right) = F(v_{cut-in}) + \left(1 - F(v_{cut-off})\right)$$

$$= \left(1 - e^{-\left(\frac{v_{cut-in}-\varepsilon}{(c-\varepsilon)}\right)^k}\right) + \left(1 - \left(1 - e^{-\left(\frac{v_{cut-off}-\varepsilon}{(c-\varepsilon)}\right)^k}\right)\right)$$

$$= 1 + e^{-\left(\frac{v_{cut-off}-\varepsilon}{(c-\varepsilon)}\right)^k} - e^{-\left(\frac{v_{cut-in}-\varepsilon}{(c-\varepsilon)}\right)^k}$$

So, $F_{P_{turbine}}(v) = 1 + e^{-\left(\frac{(v-\varepsilon)_{cut-off}}{(c-\varepsilon)}\right)^k} - e^{-\left(\frac{v_{cut-in}-\varepsilon}{(c-\varepsilon)}\right)^k}$, $0 \le v \le \frac{1}{2}\rho A C_P \eta v_{cut-in}^3$

For $0 < v = \frac{1}{2}\rho A C_P \eta v^3 \le \frac{1}{2}\rho A C_P \eta {v_r}^3$, note that $v \le \frac{1}{2}\rho A C_P \eta {v_r}^3 \Leftrightarrow v \le v_r$,

which implies that, $F_{P_{turbine}}(v) = \mathbb{P}(0 < P_{turbine} \le v) + \mathbb{P}(P_{turbine} = 0)$

$$= \mathbb{P}\left(\frac{1}{2}\rho A C_P \eta v_{cut-in}^3 \le P_{turbine} \le v\right) + 1 + e^{-\left(\frac{v_{cut-off}-\varepsilon}{(c-\varepsilon)}\right)^k} - e^{\left(\frac{v_{cut-in}-\varepsilon}{(c-\varepsilon)}\right)^k}$$

$$= \left(1 - e^{-\frac{1}{(c-\varepsilon)^k}\left(\frac{2v}{\rho AC_P\eta}\right)^{\frac{k}{3}}}\right) - \left(1 - e^{-\frac{1}{(c-\varepsilon)^k}\left(\frac{2.\frac{1}{2}\rho AC_P\eta(v_{cut-in}-\varepsilon)^3}{\rho AC_P\eta}\right)^{\frac{k}{3}}}\right) + 1 + e^{-\left(\frac{v_{cut-off}-\varepsilon}{(c-\varepsilon)}\right)^k} - e^{-\left(\frac{v_{cut-in}-\varepsilon}{(c-\varepsilon)}\right)^k}$$

$$= 1 - e^{-\frac{1}{(c-\varepsilon)^k}\left(\frac{2v}{\rho AC_P\eta}\right)^{\frac{k}{3}}} + e^{-\left(\frac{v_{cut-off}-\varepsilon}{(c-\varepsilon)}\right)^k}$$

For $\frac{1}{2}\rho AC_P\eta v_{rated}^3 \le v < \infty$, it is evident that $F_{P_{turbine}}(v) = 1$

Proposition 5.3:

$$\begin{aligned}\mathbb{E}[P_{turbine}] = \left(\frac{1}{2}\rho AC_P\eta\right)\Bigg[&(c-\varepsilon)^3\left(\Gamma\left(1+\frac{3}{k},\left(\frac{v_{rated}-\varepsilon}{c-\varepsilon}\right)^k\right) - \Gamma\left(1+\frac{3}{k},\left(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon}\right)^k\right)\right)\\ &+ 3\varepsilon(c-\varepsilon)^2\left(\Gamma\left(1+\frac{2}{k},\left(\frac{v_{rated}-\varepsilon}{c-\varepsilon}\right)^k\right) - \Gamma\left(1+\frac{2}{k},\left(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon}\right)^k\right)\right)\\ &+ 3\varepsilon^2(c-\varepsilon)\left(\Gamma\left(1+\frac{1}{k},\left(\frac{v_{rated}-\varepsilon}{c-\varepsilon}\right)^k\right) - \Gamma\left(1+\frac{1}{k},\left(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon}\right)^k\right)\right)\\ &+ \varepsilon^3\left(e^{-\left(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon}\right)^k} - e^{-\left(\frac{v_{rated}-\varepsilon}{c-\varepsilon}\right)^k}\right) + v_{rated}^3\left(e^{-\left(\frac{v_{rated}-\varepsilon}{(c-\varepsilon)}\right)^k} - e^{-\left(\frac{v_{cut-off}-\varepsilon}{(c-\varepsilon)}\right)^k}\right)\Bigg]\end{aligned}$$

Proof:

$$\begin{aligned}\mathbb{E}[P_{turbine}] &= \int_{-\infty}^{\infty} P_{turbine}\ d\left(F_{P_{turbine}}(v)\right)\\ &= \int_{v_{cut-in}}^{v_{rated}} \left(\frac{1}{2}\rho AC_P\eta v^3\right)\left(\frac{k}{c-\varepsilon}\right)\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^{k-1} exp\left(-\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k\right)dv\\ &+ \int_{v_{rated}}^{v_{cut-off}} \left(\frac{1}{2}\rho AC_P\eta v_{rated}\right)^m\left(\frac{k}{c-\varepsilon}\right)\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^{k-1} exp\left(-\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k\right)dv\end{aligned}$$

We make the change of variable $y = \left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k$

$$= \left(\frac{1}{2}\rho AC_P\eta\right)^m\left[\int_{\left(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon}\right)^k}^{\left(\frac{v_{rated}-\varepsilon}{c-\varepsilon}\right)^k}\left[(c-\varepsilon)y^{1/k}+\varepsilon\right]^3 \ e^{-y}\,dy + \int_{\left(\frac{v_{rated}-\varepsilon}{c-\varepsilon}\right)^k}^{\left(\frac{v_{cut-off}-\varepsilon}{c-\varepsilon}\right)^k}\left[(c-\varepsilon)y^{1/k}+\varepsilon\right]^3 e^{-y}dy\right]$$

$$\begin{aligned}= \left(\frac{1}{2}\rho AC_P\eta\right)\Bigg[&(c-\varepsilon)^3\left(\Gamma\left(1+\frac{3}{k},\left(\frac{v_{rated}-\varepsilon}{c-\varepsilon}\right)^k\right) - \Gamma\left(1+\frac{3}{k},\left(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon}\right)^k\right)\right)\\ &+ 3\varepsilon(c-\varepsilon)^2\left(\Gamma\left(1+\frac{2}{k},\left(\frac{v_{rated}-\varepsilon}{c-\varepsilon}\right)^k\right) - \Gamma\left(1+\frac{2}{k},\left(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon}\right)^k\right)\right)\\ &+ 3\varepsilon^2(c-\varepsilon)\left(\Gamma\left(1+\frac{1}{k},\left(\frac{v_{rated}-\varepsilon}{c-\varepsilon}\right)^k\right) - \Gamma\left(1+\frac{1}{k},\left(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon}\right)^k\right)\right)\\ &+ \varepsilon^3\left(e^{-\left(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon}\right)^k} - e^{-\left(\frac{v_{rated}-\varepsilon}{c-\varepsilon}\right)^k}\right) + v_{rated}^3\left(e^{-\left(\frac{v_{rated}-\varepsilon}{(c-\varepsilon)}\right)^k} - e^{-\left(\frac{v_{cut-off}-\varepsilon}{(c-\varepsilon)}\right)^k}\right)\Bigg]\end{aligned}$$

5.2 *Capacity factor*

We can define the capacity factor of a turbine to be the ratio

$$CF = \frac{\mathbb{E}(P_{turbine})}{\frac{1}{2}\rho A C_P \eta v_{rated}^3}$$

Now the closed form expression for CF:

$$\begin{aligned} CF = \left(\frac{1}{v_{rated}}\right)^3 \Bigg[&(c-\varepsilon)^3 \left(\Gamma\left(1+\frac{3}{k}, \left(\frac{v_{rated}-\varepsilon}{c-\varepsilon}\right)^k\right) - \Gamma\left(1+\frac{3}{k}, \left(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon}\right)^k\right)\right) \\ &+ 3\varepsilon(c-\varepsilon)^2 \left(\Gamma\left(1+\frac{2}{k}, \left(\frac{v_{rated}-\varepsilon}{c-\varepsilon}\right)^k\right) - \Gamma\left(1+\frac{2}{k}, \left(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon}\right)^k\right)\right) \\ &+ 3\varepsilon^2(c-\varepsilon) \left(\Gamma\left(1+\frac{1}{k}, \left(\frac{v_{rated}-\varepsilon}{c-\varepsilon}\right)^k\right) - \Gamma\left(1+\frac{1}{k}, \left(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon}\right)^k\right)\right) \\ &+ \varepsilon^3 \left(e^{-\left(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon}\right)^k} - e^{-\left(\frac{v_{rated}-\varepsilon}{c-\varepsilon}\right)^k} \right) \Bigg] + \left(e^{-\left(\frac{v_{rated}-\varepsilon}{(c-\varepsilon)}\right)^k} - e^{-\left(\frac{v_{cut-off}-\varepsilon}{(c-\varepsilon)}\right)^k} \right) \end{aligned}$$

5.3Turbine site matching

Suppose we have chosen a location where the wind speed V for a given month has a $Weibull(k, c, \varepsilon)$ distribution. Furthermore, suppose a manufacturer can manufacture wind turbines with fixed cut-in and cut-off speeds v_{cut-in} and $v_{cut-off}$ respectively, but can adjust the rated wind speed $v_r \epsilon [v_{cut-in} - \varepsilon, v_{cut-off} - \varepsilon)$. Suppose also that the power co efficient C_P, the efficiency coefficient η and the rotor area A are independent of v_r. What should be the rated wind speed if we want to maximize the capacity factor CF? What should be the rated wind speed if we want to maximize average power of the wind turbine $\mathbb{E}(P_{turbine})$? Are these two wind speeds equal? These questions can be answered by use of the calculus, Jowder (2009). Define a function

$$CF: [v_{cut-in} - \varepsilon, v_{cut-off} - \varepsilon) \to \mathbb{R} \text{ by}$$

$$\begin{aligned} CF = \left(\frac{1}{v}\right)^3 \Bigg[&(c-\varepsilon)^3 \left(\Gamma\left(1+\frac{3}{k}, \left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k\right) - \Gamma\left(1+\frac{3}{k}, \left(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon}\right)^k\right)\right) \\ &+ 3\varepsilon(c-\varepsilon)^2 \left(\Gamma\left(1+\frac{2}{k}, \left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k\right) - \Gamma\left(1+\frac{2}{k}, \left(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon}\right)^k\right)\right) \\ &+ 3\varepsilon^2(c-\varepsilon) \left(\Gamma\left(1+\frac{1}{k}, \left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k\right) - \Gamma\left(1+\frac{1}{k}, \left(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon}\right)^k\right)\right) \\ &+ \varepsilon^3 \left(e^{-\left(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon}\right)^k} - e^{-\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k} \right) \Bigg] + \left(e^{-\left(\frac{v-\varepsilon}{(c-\varepsilon)}\right)^k} - e^{-\left(\frac{v_{cut-off}-\varepsilon}{(c-\varepsilon)}\right)^k} \right) \end{aligned}$$

Then

$$CF'(v) = \frac{-3}{v^2} \left[(c-\varepsilon)^3 \left(\Gamma\left(1+\frac{3}{k}, \left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k\right) - \Gamma\left(1+\frac{3}{k}, \left(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon}\right)^k\right)\right) + 3\varepsilon(c-\varepsilon)^2 \left(\Gamma\left(1+\frac{2}{k}, \left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k\right) - \right.\right.$$

$\Gamma\left(1+\frac{2}{k},\left(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon}\right)^k\right)\Big) + 3\varepsilon^2(c-\varepsilon)\left(\Gamma\left(1+\frac{1}{k},\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k\right)-\Gamma\left(1+\frac{1}{k},\left(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon}\right)^k\right)\right)+\varepsilon^3\left(e^{-(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon})^k}-\right.$ $\left.e^{-(\frac{v-\varepsilon}{c-\varepsilon})^k}\right)\Big] + \left(\frac{1}{v}\right)^3\frac{k}{v-\varepsilon}\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k e^{-\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k}[(v-\varepsilon)^3+3\varepsilon(v-\varepsilon)^2+3\varepsilon^2(v-\varepsilon)+(v^3-\varepsilon^3)]$

We see that the last expression has a root precisely at $v = v_{cut-in}$ and since $CF'(v) < 0$ for $v > v_{cut-in}$ and when $k = 0$, it follows that the capacity factor is maximized for $v_{rated} = v_{cut-in}$.We now compute the value v_{rated} which maximizes $\mathbb{E}(P_{turbine})$. Define a function $P_{avg}: [v_{cut-in}, v_{cut-off}) \to \mathbb{R}$

$$\begin{aligned}P_{avg}(v) = \frac{1}{2}\rho A C_P \eta \Bigg[&(c-\varepsilon)^3\left(\Gamma\left(1+\frac{3}{k},\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k\right)-\Gamma\left(1+\frac{3}{k},\left(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon}\right)^k\right)\right)\\ &+3\varepsilon(c-\varepsilon)^2\left(\Gamma\left(1+\frac{2}{k},\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k\right)-\Gamma\left(1+\frac{2}{k},\left(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon}\right)^k\right)\right)\\ &+3\varepsilon^2(c-\varepsilon)\left(\Gamma\left(1+\frac{1}{k},\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k\right)-\Gamma\left(1+\frac{1}{k},\left(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon}\right)^k\right)\right)\\ &+\varepsilon^3\left(e^{-(\frac{v_{cut-in}-\varepsilon}{c-\varepsilon})^k}-e^{-(\frac{v-\varepsilon}{c-\varepsilon})^k}\right)+v^3\left(e^{-\left(\frac{v-\varepsilon}{(c-\varepsilon)}\right)^k}-e^{-\left(\frac{v_{cut-off}-\varepsilon}{(c-\varepsilon)}\right)^k}\right)\Bigg]\end{aligned}$$

Differentiating with respect to v,

$$\begin{aligned}P'_{avg}(v) = \frac{1}{2}\rho A C_P \eta \frac{k}{v-\varepsilon}\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k e^{-\left(\frac{v-\varepsilon}{c-\varepsilon}\right)^k}&[(v-\varepsilon)^3+3\varepsilon(v-\varepsilon)^2+3\varepsilon^2(v-\varepsilon)+(v^3-\varepsilon^3)]\\ +\frac{3}{2}\rho A C_P \eta v^2&\left(e^{-\left(\frac{v-\varepsilon}{(c-\varepsilon)}\right)^k}-e^{-\left(\frac{v_{cut-off}-\varepsilon}{(c-\varepsilon)}\right)^k}\right)\end{aligned}$$

Setting the RHS equal to 0, we see that P_{avg} has no critical points in the domain $[v_{cut-in} - \varepsilon, v_{cut-off} - \varepsilon)$. Rather, since $P_{avg} > 0$ on $[v_{cut-in} - \varepsilon, v_{cut-off} - \varepsilon)$,we only conclude that

$$sup_{v\epsilon[v_{cut-in}-\varepsilon, v_{cut-off}-\varepsilon)} P_{avg}(v) = \frac{1}{2}\rho A C_P \eta (v_{cut-off} - \varepsilon)^3$$

These results indicate that by having lowly rated wind turbines, we can maximize the capacity factor at the cost of minimizing expected power output. Conversely, we can increase expected power output at the cost of decreasing the capacity factor. It is not priori clear which approach is better, since we have not , nor will we, discussed the economics of wind power. One obvious downside to using lowly rated wind turbines is increased land usage, which would also raise costs if the wind form operator is leasing the land. (Zhou *et al*. 2010 a, b, c)

6. Conclusions

In this paper some mathematical properties of three parameter Weibull Distribution are addressed. Also Maximum likelihood method for estimating the parameters of the Weibull wind speed distributions are described. Also we have given a blue print for an analyst to estimate the wind recourses of a fixed location. With a location of, say hourly average wind speed for a year, we can use the maximum likelihood method to determine Weibull parameters for the monthly wind speed distributions. This study presents mathematical proofs of such estimates which is required for accurate wind power production simulation with a specific cut-in, cut-off and rated speeds and can also compute the capacity factor.

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References

Archer, C. and Jacobson, M. (2005), "Evaluation of global wind power", *Geophys. Res.*, **110**, D12110.

Chang, T.P. (2011), "Estimation of wind energy potential using different probability density functions", *Appl. Energy*, **88**(5),1848-1856

Cook, N.J. (2004), "Confidence limits for extreme wind speeds in mixed climates", *J. Wind Eng. Ind. Aerod.*, **92**, 41-51.

D'Amico, G., Petroni, F. and Prattico, F. (2015), "Wind speed prediction for wind farm applications by extreme value theory and copulas", *J. Wind Eng. Ind. Aerod.*, **145**, 229-236.

Edwards, P. and Hurst, R. (2001), "Level-crossing statistics of the horizontal wind speed in the planetary surface boundary layer", *Chaos*, **11**(3), 611-618.

Gasch, R. and Twelve, J. (2012), *Wind Power Plants: Fundamentals, Design, Construction, and Oper-Ation*, Springer,

Genc, A., Erisoglu, M., Pekgor, A., Oturanc, G. and Hepbasli, A. (2005), "Estimation of wind power potential using weibull distribution", *Energ. Source* , **27**(9), 809-822.

Hong, H.P. and Li, S.H. (2014), "Plotting positions and approximating first two moments of order statistics for Gumbel distribution: estimating quantiles of wind speed", *Wind Struct.*, **19**(4), 371-387.

http://geosci.uchicago.edu/~moyer/GEOS24705/Readings/GEA14954C15-MW-Broch.pdf

Jowder, F.A.L. (2009), "Wind power analysis and site matching of wind turbine generators in kingdom of bahrain", *Appl. Energy*, **86**, 538-545.

Kollu, R., Rayapudi, S.R., Narasimham, S.V.L. and Pakkurthi, K.M. (2012), "Mixture probability distribution functions to model wind speed distributions", *Int. J. Energy Environ. Eng.*, **27**(3).

Lun, I.Y.F. and Lam, J.C. (2000), "A study of Weibull parameters using long term wind observations", *Renew. Energ.*, **20**(2) ,145-153.

Majid, S., Siamak, K.H. and Mehrdad, B. (2015), "Estimation of Weibull parameters for wind energy application in Iran", *Wind Struct*., **21**(2), 203-221.

Mo, H.M., Hong, H.P. and Fan, F. (2015), "Estimating the extreme wind speed for regions in china using surface wind observations and real analysis data", *J. Wind Eng. Ind. Aerod.*, **143**, 19-33.

Morgan, E.C., Lackner, M., Vogel, R.M. and Baise, L.G. (2011), "Probability distributions for offshore wind speeds", *Energ. Convers. Manage.*, **52**, 15-26.

Nelson, W. (1972), "Theory and application of hazard plotting for censored failure data", *Technometrics*, **114**, 945-966.

Shao, J. (2003), *Mathematical Statistics (Second Edition)*, Springer.

Soukissian, T. (2013), "Use of multi-parameter distributions for offshore wind speed modeling: The Johnson SB distribution", *Appl. Energy*, 982-1000.

Weibull, W. (1939), "A statistical theory of strength of materials", *Ingenious Vetenskaps Akademien Handlingar*, 151.

Zhou, J., Erdem, E., Li, G. and Shi, J. (2010), "Comprehensive evaluation of wind speed distribution models: A case study for North Dakota sites", *Energy Convers. Manage.*, **51**(7), 1449-1458.

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