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Vibration analysis of a cracked beam with axial force and crack identification

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Abstract. A composite element method (CEM) is presented to analyze the free and forced vibrations of a cracked Euler-Bernoulli beam with axial force. The cracks are introduced by using Christides and Barr crack model with an adjustment on one crack parameter. The effects of the cracks and axial force on the reduction of natural frequencies and the dynamic responses of the beam are investigated. The time response sensitivities with respect to the crack parameters (i.e., crack location, crack depth) and the axial force are calculated. The natural frequencies obtained from the proposed method are compared with the analytical results in the literature, and good agreement is found. This study shows that the cracks in the beam may have significant effects on the dynamic responses of the beam. In the inverse problem, a response sensitivity-based model updating method is proposed to identify both a single crack and multiple cracks from measured dynamic responses. The cracks can be identified successfully even using simulated noisy acceleration responses.

Keywords: composite element; crack; axial force; response sensitivity; crack identification

1. Introduction

The presence of cracks in a structure may have a significant influence on the dynamic responses of the structure; it can even lead to the catastrophic failure of the structure. To predict the failure, vibration monitoring can be used to detect changes in the dynamic responses and/or dynamic characteristics of the structure. To this end, the knowledge of the effects of cracks on the vibrations of the structure is very important to engineers. To detect cracks as accurately as possible, it is necessary to consider the key factors that can affect the vibration responses and characteristics. It is known that the effect of axial force on dynamic responses and natural frequencies can be notable for structural components subjected to axial loads, such as bridge piers and building columns. Therefore, in order to predict the cracks accurately, it is very significant to develop efficient techniques for the vibration analysis of cracked beams with axial force.

Vibration analysis of cracked structures has been widely investigated in the last three decades. Dimarogonas (1996) and Ostachowicz and Krawczuk (2001) gave comprehensive reviews of the problems to the vibration of cracked structures and the crack models, respectively. Investigation of dynamic behaviour of cracked structures has attracted the attention of many researchers (Christides and Barr 1984, Kwon and Christy 1994, Ruotolo *et al*. 1996, Kisa and Brandon 2000, Chaudhari and Maiti 2000, Zhu *et al*. 2009, He and Lu 2010, Ayatollahi *et al*. 2010, Oz 2010).

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Several methods (Narkis 1994, Gounaris *et al*. 1996, Shifrin and Ruotolo 1999, Kisa and Brandon 2000) to determine the natural frequency changes due to a single crack and multiple cracks have been proposed.

The effects of axial forces on the dynamic response of the beam with or without cracks were not considered in the literatures mentioned above. Krawczuk and Ostachowicz (1993) obtained the natural frequencies of a cracked beam with constant axial force by finite element analysis; the effect of axial force was taken into account by use of the geometric stiffness matrix. Bokaian (1988) studied the effect of a constant axial compression load on the natural frequencies and mode shapes of a uniform single-span beam with different combinations of end conditions. Farghaly (1992) presented exact frequency and mode shape solutions for a uniform cantilever Euler-Bernoulli beam under axial load. The same problem but for the Timoshenko beam has been studied by Farghaly and Shebl (1995). Binici (2005) proposed an analytical method to obtain the eigen frequencies and mode shapes of beams with multiple cracks subjected to axial force. Lee (1995) examined the dynamic response of a rotating subject to an axial force and a moving load. The vibration analysis of axially loaded stepped beams was investigated in Refs. (Rosa 1996, Naguleswaran 1994, Kukla and Zamojska 2007). Da Silva *et al*. (2004) studied the behaviour of flush end-plate beam-to-column joints under bending and axial force. Yesilce and Catal (2009) investigated the free vibration analysis of Reddy-Bickford beams on elastic soil with/without axial force effect using the Differential Transform Method (DTM). Li *et al.* (2010) formulated the dynamic stiffness matrix for an axially loaded slender double-beam element. The Bernoulli-Euler beam theory is used to define the dynamic behaviors of the beams and the effects of the mass of springs and axial force are taken into account in the formulation.

There are also many studies that deal with the stability of cracked and uncracked structures. Chen and Chen (1988) investigated the stability of a rotating shaft with a single crack. Li (2001) used a transfer matrix approach to determine the buckling loads of multi-stepped beam-columns. Naguleswaran (2003) investigated the stability and vibration of a beam-column up to three step changes. Zhou and Huang (2006) studied the crack effect on the elastic buckling behavior of axially and eccentrically loaded columns.

The objective of this study is to introduce a new method for free and forced vibration analysis of axially loaded beams containing crack(s) by means of a composite element method and to identify the cracks using the measured structural dynamic responses. The finite beam element is formulated using the composite element method. The crack is introduced by using Christides and Barr (1984) crack model with an adjustment on the crack parameter. The natural frequency results obtained from the proposed method are compared against the results predicted by the analytical method and good agreement is found. The forced vibration analysis is conducted and the response sensitivities with respect to the crack parameters (i.e., crack location and crack depth) and with respect to the axial force are also calculated. Studies show that the natural frequencies and forced vibration responses of the beam are affected significantly by the cracks and the axial force. It is also found that the responses with respect to different parameters have different sensitivities. The inverse problem, i.e., to identify the location and severity of cracks from measured dynamic responses of the cracked beam is investigated using a response-based model updating method. A response sensitivity-based approach of crack identification is then presented in the identification of single and multiple crack damages. The parameters of the cracks can be identified successfully and it is found that the presented method is not sensitive to simulated measurement noise.

2. Forward analysis theory

Fig. 1 shows a simply supported cracked beam with a constant axial force. The cracks are assumed to be always open, and they do not change the mass of the beam. The differential equation for free vibration of the intact beam can be expressed as (Meirovitch 1975)

$$\frac{\partial^2}{\partial x^2} EI(x) \frac{\partial^2 y(x,t)}{\partial x^2} + P \frac{\partial^2 y(x,t)}{\partial x^2} + \rho A \frac{\partial^2 y(x,t)}{\partial t^2} = 0 \tag{1}$$

where E is the Young's modulus, I the moment of inertia of the beam, P the axial force acting on the beam, ρ the mass density, A area of the cross-section of the beam.

2.1 Crack model

According to Christides and Barr (1984), the variation of bending stiffness EI along the beam length takes up the form of

$$EI(x) = \frac{EI_0}{1 + (m-1)\exp(-2\alpha|x - x_L|/d)} \tag{2}$$

where E is the Young's modulus of the beam, $I_0 = wd^3/12$ is the second moment of area of the intact beam, $m = 1/(1-C_r)^3$, $C_r = d_c/d$ is the crack depth ratio and d_c and d are the depth of crack and the beam, respectively, x_L is the location of the crack. α is a constant which governs the rate of decay and it is estimated by Christides and Barr(1984) from experiments to be 0.667. However, according to the study by Lu and Law (2009), this constant needs to be adjusted to be 1.426 in the CEM and it is adopted in this paper.

2.2 Composite element incorporating a crack

2.2.1 The displacement field of the CEM

Composite element (Zeng 1998) is basically a combination of the conventional finite element method (FEM) and the classical theory (CT) of vibration. In the composite element method, the displacement field is written as the sum of the finite element displacement and the shape functions

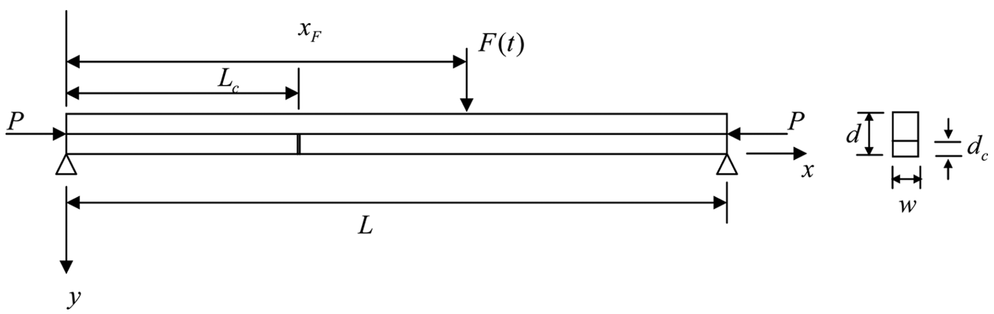

Fig. 1 A simply supported cracked beam with a constant axial force

from the classical theory. The displacement field of the CEM can be expressed as

$$u_{CEM}(x,t) = u_{FEM}(x,t) + u_{CT}(x,t) \tag{3}$$

where $u_{FEM}(x,t)$ and $u_{CT}(x,t)$ are the two parts of the CEM displacement field with the subscripts defining those of the FEM and CT, respectively.

Taking a planar beam element as an example, the first part of the CEM displacement field is expressed as the product of the shape function vector $N(x)$ and the nodal displacement vector q

$$u_{FEM}(x,t) = N(x)q(t) \tag{4}$$

where $q(t) = [v_1(t), \theta_1(t), v_2(t), \theta_2(t)]^T$ and 'v' and 'θ' represent the transverse and rotational displacements, respectively, and

$$\begin{aligned} N(x) &= \left[1-3\left(\frac{x}{L}\right)^2+2\left(\frac{x}{L}\right)^3, \frac{x}{L}-2\left(\frac{x}{L}\right)^2+\left(\frac{x}{L}\right)^3, 3\left(\frac{x}{L}\right)^2-2\left(\frac{x}{L}\right)^3, \left(\frac{x}{L}\right)^3-\left(\frac{x}{L}\right)^2\right] \\ &= [N_1(x), N_2(x), N_3(x), N_4(x)] \end{aligned} \tag{5}$$

The second part $u_{CT}(x,t)$ is obtained by the multiplication of analytical eigen function with a vector of coefficient c (also called the c degrees-of-freedom or c-coordinates)

$$u_{CT}(x,t) = \psi(x)c(t) \tag{6}$$

where $\psi(x) = [\phi_1, \phi_2, \ldots, \phi_N]$ and $c(t) = [c_1(t), c_2(t), \ldots, c_N(t)]$ ϕ_r (r=1,2,...N) is the eigen function of the beam.

Like the FEM, the CEM can be refined using h-refinement technique by increasing the number of finite elements. Moreover, it can also be refined through the c-refinement method, by increasing the number of analytical functions in the shape functions. As we can make use of the advantage of the c-refinement from the CEM, the beam only needs to be divided into a small number of finite elements. This will reduce the total number of degree-of-freedom in the finite element model.

The displacement field of the CEM for a uniform Euler-Bernoulli beam element can be written from Eqs. (3) to (6) as

$$u_{CEM}(x,t) = S(x)Q(t) \tag{7}$$

where $S(x) = [N_1(x), N_2(x), N_3(x), N_4(x), \phi_1(x), \phi_2(x), \ldots, \phi_N(x)]$ is the generalized shape function of the CEM, $Q(x) = [v_1(t), \theta_1(t), v_2(t), \theta_2(t), c_1(t), c_2(t), \ldots, c_N(t)]^T$ is the vector of generalized displacements, and N is the number of eigen functions used. These functions are chosen according to different boundary conditions of the beam. For instance, for simply-supported beam, they are selected as $\phi_i(x) = \sin\frac{i\pi x}{L}$, ($i$=1,2,...$N$). In the case of other boundary conditions, they can be chosen from the classical theory for transversal vibration of beam. The number of terms N is determined by a frequency convergence test (Lu and Law 2009). The frequency convergence criterion is defined as $\max_{i=1,2,3}\left|\Delta\omega_i^N/\omega_i^N\right|$, where ω_i^N is the estimation of the ith frequency with N-terms in the CT. $\Delta\omega_i^N = \omega_i^N - \omega_i^{N-1}$ is the difference of the ith frequency obtained with the N-terms and (N-1)-terms. It is found that when more than 33 terms are selected, the natural frequencies are converged (Lu and Law 2009). In this paper, 50 terms are chosen to ensure the requirement being satisfied. It should be pointed out that if more terms are used in the calculation, the natural frequencies of the beam are then more accurate.

2.2.2 The elemental stiffness, geometric and mass matrices

The elemental stiffness matrix of the cracked beam can be obtained from the following equation

$$K_e = \int_0^L \frac{d^2S^T}{dx^2} EI(x) \frac{d^2S}{dx^2} dx$$

$$= \begin{bmatrix} [k_{qq}] & [k_{qc}] \\ [k_{cq}] & [k_{cc}] \end{bmatrix} \tag{8}$$

where the submatrix $[k_{qq}]$ corresponds to the elemental stiffness matrix from the FEM for the cracked beam; the submatrix $[k_{qc}]$ corresponds to the coupling terms of the q-dofs and the c-dofs; submatrix $[k_{cq}]$ is a transpose matrix of $[k_{qc}]$, and the submatrix $[k_{cc}]$ corresponds to the c-dofs and is a diagonal matrix, $EI(x)$ is shown in Eq. (2).

The elemental geometric stiffness matrix due to a constant axial force P is expressed as

$$M_{Ge} = \int_0^L \frac{dS^T}{dx} \frac{dS}{dx} dx$$

$$= \begin{bmatrix} [k_{Gqq}] & [k_{Gqc}] \\ [k_{Gcq}] & [k_{Gcc}] \end{bmatrix} \tag{9}$$

where the submatrix $[k_{Gqq}]$ corresponds to the elemental geometric stiffness matrix from the FEM for the beam; the submatrix $[k_{Gqc}]$ corresponds to the coupling terms of the q-dofs and the c-dofs; submatrix $[k_{Gcq}]$ is a transpose matrix of $[k_{Gqc}]$, and the submatrix $[k_{Gcc}]$ corresponds to the c-dofs and is a diagonal matrix.

The elemental mass matrix is expressed as

$$K_e = \int_0^L S(x)^T \rho A S(x) dx$$

$$= \begin{bmatrix} [m_{qq}] & [m_{qc}] \\ [m_{cq}] & [m_{cc}] \end{bmatrix} \tag{10}$$

The submatrix $[m_{qq}]$ corresponds to the elemental mass matrix from the FEM for the cracked beam; the submatrix $[m_{qc}]$ corresponds to the coupling terms of the q-dofs and the c-dofs; submatrix $[m_{cq}]$ is the transpose matrix of $[m_{qc}]$, and the submatrix $[m_{cc}]$ corresponds to the c-dofs and is a diagonal matrix.

2.3 Free vibration analysis

After introducing the boundary conditions, the governing equation for free vibration of the cracked beam can be expressed as

$$(K - K_G - \omega^2 M)V = 0 \tag{11}$$

where K and M are system stiffness and mass matrix, respectively, K_G is the system geometric stiffness. ω is the circular frequency, from which the natural frequencies are identified. The ith normalized mode shapes of the stepped beam can be expressed as

$$\Phi_i = \sum_{i=1}^{4} N_i V_i + \sum_{i=1}^{N} \phi_i V_{i+4} \tag{12}$$

2.4 Forced vibration analysis

The equation of motion of the forced vibration of a cracked beam with n cracks subjected to axial force when expressed in terms of the composite element method is

$$M\ddot{Q} + C\dot{Q} + [K(x_{L_1}, d_{c_1}, \ldots, x_{L_i}, d_{c_i}, \ldots, x_{L_n}, d_{c_n}) - K_G]Q = f(t) \tag{13}$$

where C is the damping matrix which represents a Rayleigh damping model in this work as

$$C = a_1 M + a_2[K(x_{L_1}, d_{c_1}, \ldots, x_{L_i}, d_{c_i}, \ldots, x_{L_n}, d_{c_n}) - K_G] \tag{14}$$

where a_1 and a_2 are constants to be determined from two modal damping ratios. $f(t)$ is the generalized force vector. For an external force $F(t)$ acting at the location x_F from the left support, $f(t)$ can be expressed as

$$f(t) = [N_1(x_F) \;\; N_2(x_F) \;\; N_3(x_F) \;\; N_4(x_F) \;\; \phi_1(x_F) \;\; \ldots \;\; \phi_n(x_F)]^T F(x) \tag{15}$$

The generalized acceleration $\ddot{Q}$, velocity $\dot{Q}$ and displacement Q of the cracked beam can be obtained from Eq. (13) using direct integration method. The physical displacement, velocity and acceleration $\ddot{u}(x,t)$ can be obtained from

$$u(x,t) = [S(x)]^T Q \tag{16a}$$

$$\dot{u}(x,t) = [S(x)]^T \dot{Q} \tag{16b}$$

$$\ddot{u}(x,t) = [S(x)]^T \ddot{Q} \tag{16c}$$

2.5 Dynamic response sensitivities with respect to the axial force and crack parameters

Taking partial derivative of Eq. (13) with respect to the axial force, we have

$$M\frac{\partial \ddot{Q}}{\partial P} + C\frac{\partial \dot{Q}}{\partial P} + [K(x_{L_1}, d_{c_1}, \ldots, x_{L_i}, d_{c_i}, \ldots, x_{L_n}, d_{c_n}) - K_G]\frac{\partial Q}{\partial P} = a_2\frac{\partial K_G}{\partial P}\dot{Q} + \frac{\partial K_G}{\partial P}Q \tag{17}$$

where

$$\frac{\partial K_G}{\partial P} = \int_0^L \frac{dS^T}{dx}\frac{dS}{dx}dx \tag{18}$$

Let $R = \frac{\partial Q}{\partial P}$, Eq. (17) can rewritten as

$$M\ddot{R}+C\dot{R}+[K(x_{L_1},d_{c_1},\ldots,x_{L_i},d_{c_i},\ldots,x_{L_n},d_{c_n})-K_G]R = a_2\frac{\partial K_G}{\partial P}\dot{Q}+\frac{\partial K_G}{\partial P}Q \tag{17a}$$

Since the dynamic response and the partial differential $\frac{\partial K_G}{\partial P}$ has been obtained from Eqs. (13) and (18), respectively, the right-hand-side of Eq. (17a) is known and can be treated as 'input force'. Note that Eq. (17a) is similar to Eq. (13), and the dynamic response sensitivity (i.e., the generalized acceleration response sensitivity, velocity response sensitivity and displacement response sensitivity) with respect to the axial force can be obtained by direct integration again. The physical acceleration response sensitivity with respect to axial force $\frac{\partial \ddot{u}(x,t)}{\partial P}$ is obtained from

$$\frac{\partial \ddot{u}(x,t)}{\partial P} = [S(x)]^T\frac{\partial \ddot{Q}}{\partial P} \tag{19}$$

Similarly, the generalized dynamic response sensitivity with respect to the depth and the location of the *i*th crack can be obtained from the following equations

$$M\frac{\partial \ddot{Q}}{\partial d_{c_i}}+C\frac{\partial \dot{Q}}{\partial d_{c_i}}+K(x_{L_1},d_{c_1},\ldots,x_{L_i},d_{c_i},\ldots,x_{L_n},d_{c_n})\frac{\partial Q}{\partial d_{c_i}} = -a_2\frac{\partial K(x_{L_1},d_{c_1},\ldots,x_{L_i},d_{c_i},\ldots,x_{L_n},d_{c_n})}{\partial d_{c_i}}\dot{Q}$$
$$-\frac{\partial K(x_{L_1},d_{c_1},\ldots,x_{L_i},d_{c_i},\ldots,x_{L_n},d_{c_n})}{\partial d_{c_i}}Q \tag{20}$$

$$M\frac{\partial \ddot{Q}}{\partial x_{L_i}}+C\frac{\partial \dot{Q}}{\partial x_{L_i}}+K(x_{L_1},d_{c_1},\ldots,x_{L_i},d_{c_i},\ldots,x_{L_n},d_{c_n})\frac{\partial Q}{\partial x_{L_i}} = -a_2\frac{\partial K(x_{L_1},d_{c_1},\ldots,x_{L_i},d_{c_i},\ldots,x_{L_n},d_{c_n})}{\partial d_{c_i}}\dot{Q}$$
$$-\frac{\partial K(x_{L_1},d_{c_1},\ldots,x_{L_i},d_{c_i},\ldots,x_{L_n},d_{c_n})}{\partial x_{c_i}}Q \tag{21}$$

The physical acceleration response sensitivity with respect to crack parameters $\frac{\partial \ddot{u}(x,t)}{\partial d_{c_i}}$, and $\frac{\partial \ddot{u}(x,t)}{\partial x_{L_i}}$ can be obtained from Eqs. (22) and (23), respectively.

$$\frac{\partial \ddot{u}(x,t)}{\partial d_{c_i}} = [S(x)]^T\frac{\partial \ddot{Q}}{\partial d_{c_i}} \tag{22}$$

$$\frac{\partial \ddot{u}(x,t)}{\partial x_{L_i}} = [S(x)]^T\frac{\partial \ddot{Q}}{\partial x_{L_i}} \tag{23}$$

3. Inverse problem

In the identification, the vector of parameter to be updated is $\theta = [X\ d_c]^T$, where $X = [x_{L_1}, x_{L_2}, \ldots, x_{L_n}]$ and $d_c = [d_{c_1}, d_{c_2}, \ldots, d_{c_n}]$ are the vectors of locations and the crack depths of the n cracks, respectively. The acceleration responses at several different locations of the beam can be used to identify the crack parameters. The 'measured' response is simulated by adding different levels of artificial normally distributed random noise into the one calculated from Eq. (16(c)).

$$\hat{\ddot{u}} = \ddot{u}_{cal} + Ep * N_{oise} * \mathrm{var}(\ddot{u}_{cal}) \tag{24}$$

where $\hat{\ddot{u}}$ is the vectors of polluted acceleration; E_p is the noise level; N_{oise} is a standard normal distribution vector with zero mean and unit standard deviation; var(•) is the variance of the time history; $\ddot{u}_{cal}$ is the vector of calculated acceleration.

The sensitivity-based approach (Lu and Law 2007) is adopted for the updating of the vector of parameters θ.

$$\delta\ddot{u} = Z\delta\theta \tag{25}$$

where $\delta\theta$ is the vector of perturbations in the updating parameters, $\delta\ddot{u} = \hat{\ddot{u}} - \ddot{u}$ is the differences in the vector of polluted generalized acceleration $\hat{\ddot{u}}$ and the vector of calculated acceleration $\ddot{u}$. Matrix Z consists of the response sensitivity, which is the first derivative of the dynamic response with respect to the updating parameters. These derivatives are calculated from Eqs. (20) and (21). For example, if the acceleration of ith degree-of-freedom is used, the detailed sensitivity matrix is shown as

$$Z = \begin{bmatrix} \frac{\partial \ddot{u}^i(t_1)}{\partial x_{L_1}} & \frac{\partial \ddot{u}^i(t_1)}{\partial x_{L_2}} & \cdots & \frac{\partial \ddot{u}^i(t_1)}{\partial x_{L_n}} & \frac{\partial \ddot{u}^i(t_1)}{\partial d_{c_1}} & \frac{\partial \ddot{u}^i(t_1)}{\partial d_{c_2}} & \cdots & \frac{\partial \ddot{u}^i(t_1)}{\partial d_{c_n}} \\ \frac{\partial \ddot{u}^i(t_2)}{\partial x_{L_1}} & \frac{\partial \ddot{u}^i(t_2)}{\partial x_{L_2}} & \cdots & \frac{\partial \ddot{u}^i(t_2)}{\partial x_{L_n}} & \frac{\partial \ddot{u}^i(t_2)}{\partial d_{c_1}} & \frac{\partial \ddot{u}^i(t_2)}{\partial d_{c_2}} & \cdots & \frac{\partial \ddot{u}^i(t_2)}{\partial d_{c_n}} \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \frac{\partial \ddot{u}^i(t_f)}{\partial x_{L_1}} & \frac{\partial \ddot{u}^i(t_f)}{\partial x_{L_2}} & \cdots & \frac{\partial \ddot{u}^i(t_f)}{\partial x_{L_n}} & \frac{\partial \ddot{u}^i(t_f)}{\partial d_{c_1}} & \frac{\partial \ddot{u}^i(t_f)}{\partial d_{c_2}} & \cdots & \frac{\partial \ddot{u}^i(t_f)}{\partial d_{c_n}} \end{bmatrix} \tag{26}$$

where t_1 is the beginning of the time history and t_f is the end of the time history. Eq. (25) can be solved from the damped least-squares method (DLS) (Lu and Law 2007)

$$\delta\theta = (Z^T Z + \lambda I)^{-1} Z^T \delta\ddot{u} \tag{27}$$

where λ is the non-negative damping coefficient governing the participation of least-squares error in the solution. When the parameter λ approaches zero, the estimated vector $\delta\vartheta$ approaches to the solution obtained from the simple least-squares method from Eq. (25). L-curve method (Hansen 1994) is used in this paper to obtain the optimal regularization parameter λ.

The updated crack parameter is written as

$$\theta_{j+1} = \theta_j + \delta\theta_j \tag{28}$$

where subscript "j" indicates the iteration number during the computation.

4. Applications

4.1 Validation of free vibration analysis for a beam with cracks from the proposed method in comparison with existing result

A free-free steel beam studied by Sinha *et al*. (2002) is used to verify the correctness of the

Table 1 Comparison of natural frequencies (Hz) of the steel free-free beam with one crack

Mode	d_{c1} = 4 mm at x_1 = 430 mm			d_{c1} = 8 mm at x_1 = 430 mm			d_{c1} = 12 mm at x_1 = 430 mm		
	Experi-ment	Sinha et al. (2002)	Proposed	Experi-ment	Sinha et al. (2002)	Proposed	Experi-ment	Sinha et al. (2002)	Proposed
1	74.688	74.406	74.735	74. 063	73. 628	74. 216	72. 813	72. 958	73. 302
2	205.625	204.183	205.619	202.500	201.283	203.233	197.188	198. 928	197.355
3	405.625	405.368	405.845	404.688	404.557	404.561	403.125	403.916	403.361
4	666.250	668.429	668.821	662.813	665.356	664.266	655.938	662.874	662.513

proposed method. Sinha et al. obtained the natural frequencies of beam using finite element method with experimental verification. The material properties of the beam are: Young's modulus 203.91 Gpa, mass density 7800 kg/m^3, The width, height and length of the beam are 25.3 mm, 25.3 mm and 1330 mm, respectively. The location of the crack is 430 mm from the left end and the depth of the crack varies from 4 mm to 12 mm. In the establishment of composite element model of the beam, the beam is treated as one Euler beam element and fifty shape functions are used for the free vibration of the beam. It is accomplished using the MATLAB software (2002). The modal frequencies of the beam obtained from the proposed method are compared with those in the existing literatures as shown in Table 1. One can find that results from the proposed method are better than those from Sinha *et al*. (2002) except in the last modal frequency for the first crack state and the fundamental modal frequency for the third crack state. The natural frequencies predicted from the proposed method agree satisfactorily with the experimental values.

4.2 Validation of free vibration analysis for a cracked beam with axial force from the proposed method in comparison with existing result

A simply supported beam with a single crack is studied to check the correctness of the proposed method. The material properties and dimensions of the beam are: Young's modulus E = 200 GPa, mass density ρ = 7850 kg/m^3, width w = 20 mm, depth d = 20 mm, total length L = 2000 mm. Again, the whole beam is treated as one Euler beam element and fifty shape functions are used in the composite element model of the beam. The crack depth in the beam varies in two stages of 2 mm and 6 mm. The axial force has three different magnitude, natural f namely, -0.3P_{cr}, 0, 0.3P_{cr}, where $P_{cr} = \frac{\pi^2 EI}{L^2}$ is the critical load for buckling. In Table 2, the requencies of the beam obtained from the proposed model are compared with those by Binici (2005), in which the natural frequencies of the cracked beam with axial force are obtained analytically in a close form. The results show that the CEM approach is effective and accurate for free vibration analysis of a cracked beam with axial force.

4.3 Further validation of free vibration analysis for a beam with multiple cracks and with axial force

The proposed method is further verified by a free vibration analysis of a beam with two cracks and with axial force. A simply supported beam with multiple cracks and subjected to axial force is studied for further validation of the proposed method (Binici 2005). The results from the proposed method and those by Binici (2005) are listed in Table 3. From this table, one can see the natural frequencies from the proposed method agree very well with those by Binici (2005).

Table 2 Comparison of the first two non-dimension natural frequencies of a simply supported beam for different crack locations with different relative crack depth and different magnitude of axial forces

			$P=-0.3P_{cr}$					$P=0$					$P=0.3P_{cr}$				
			0.1L	0.2L	0.3L	0.4L	0.5L	0.1L	0.2L	0.3L	0.4L	0.5L	0.1L	0.2L	0.3L	0.4L	0.5L
dc/d=0.1	ω_1/ω_{10}	Present	1.1402	1.1396	1.1389	1.1383	1.1381	0.9998	0.9992	0.9984	0.9978	0.9976	0.8361	0.8355	0.8347	0.8340	0.8338
		Binici (2005)	1.1394	1.1394	1.1394	1.1379	1.1364	1.0030	1.0015	0.9955	0.9955	0.9939	0.8364	0.8348	0.8318	0.8314	0.8302
		Error	0.0008	0.0002	-0.0005	0.0004	0.0017	-0.0032	-0.0023	0.0029	0.0023	0.0037	-0.0003	0.0007	0.0029	0.0026	0.0036
	ω_2/ω_{20}	Present	1.0334	1.0290	1.0290	1.0334	1.0369	0.9970	0.9935	0.9935	0.9970	1.0000	0.9593	0.9545	0.9545	0.9593	0.9617
		Binici (2005)	1.0333	1.0288	1.0288	1.0330	1.0364	0.9970	0.9924	0.9924	0.9964	1.0000	0.9591	0.9545	0.9545	0.9591	0.9621
		Error	0.0001	0.0002	0.0002	0.0004	0.0005	0.0	0.0011	0.0011	0.0006	0.0	0.0002	0.0	0.0	0.0003	-0.0004
dc/d=0.3	ω_1/ω_{10}	Present	1.1381	1.1320	1.1010	1.1010	1.0800	0.9971	0.9796	0.9603	0.9498	0.9411	0.8293	0.8100	0.7698	0.7646	0.7602
		Binici (2005)	1.1385	1.1308	1.100	1.0923	1.0846	0.9923	0.9769	0.9615	0.9462	0.9385	0.8231	0.8077	0.7769	0.7615	0.7608
		Error	-0.0004	0.0012	0.001	0.0077	-0.0046	0.0048	0.0027	-0.0012	0.0036	0.0026	0.0062	0.0023	-0.0067	0.0031	-0.0006
	ω_2/ω_{20}	Present	1.0104	0.9721	0.9800	1.0110	1.0305	0.9713	0.9323	0.9404	0.9717	1.0000	0.9306	0.900	0.9009	0.9302	0.9617
		Binici (2005)	1.0045	0.9712	0.9773	1.0076	1.0258	0.9688	0.9348	0.9397	0.9712	0.9924	0.9318	0.8979	0.8985	0.9348	0.9591
		Error	0.0059	0.0009	0.0027	0.0024	0.0047	0.0025	-0.0025	0.0017	0.0005	0.0076	-0.0008	0.0021	0.0024	-0.0046	0.0026

Table 3 Comparisons of results with Binici (2005) for the simply supported beam with two cracks

Parameters					Results from the proposed method		Results from Binici (2005)	
x_{L_1}/L	x_{L_2}/L	d_{c_1}/L	d_{c_2}/L	P/P_{cr}	ω_1/ω_{10}	ω_2/ω_{20}	ω_1/ω_{10}	ω_2/ω_{20}
0.1	0.4	0.3	0.5	0.1	0.7939	0.9210	0.7917	0.9223
0.1	0.4	0.3	0.5	0.2	0.7252	0.9071	0.7244	0.9085
0.1	0.4	0.3	0.5	0.3	0.6475	0.8926	0.6501	0.8944
0.1	0.4	0.2	0.4	0.1	0.8327	0.9436	0.8331	0.9420
0.1	0.4	0.2	0.4	0.2	0.7716	0.9302	0.7700	0.9285
0.1	0.4	0.2	0.4	0.3	0.7002	0.9162	0.7012	0.9148
0.2	0.3	0.3	0.5	0.1	0.8131	0.8394	0.8105	0.8371
0.2	0.3	0.3	0.5	0.2	0.7469	0.8236	0.7442	0.8217
0.2	0.3	0.3	0.5	0.3	0.6738	0.8079	0.6713	0.8060
0.2	0.3	0.2	0.4	0.1	0.8552	0.8791	0.8528	0.8773
0.2	0.3	0.2	0.4	0.2	0.7928	0.8645	0.7910	0.8627
0.2	0.3	0.2	0.4	0.3	0.7250	0.8488	0.7239	0.8479

Note: $P_{cr}=\pi^2 EI/L^2$, $\omega_{10}=\left(\frac{\pi}{L}\right)^2\sqrt{\frac{EI}{\rho A}}$, $\omega_{20}=\left(\frac{2\pi}{L}\right)^2\sqrt{\frac{EI}{\rho A}}$

4.4 Dynamic response and response sensitivity with respect axial force

The forced vibration analysis for the cracked beam with axial force is conducted in this section. The effects of the presence of crack on the dynamic response of the beam is investigated and the axial force as well. The parameters of the simply supported beam under study are taken as: $E=28$ GPa, $w=200$ mm, $d=200$ mm, $L=8.0$ m, mass density $\rho=2500$ kg/m^3, Three cases are investigated in the following.

4.4.1 Effect of crack on the dynamic response

An impulsive force is assumed to act at mid-span of the beam with a magnitude of 100N, the force starts to act on the beam from the beginning and lasts for 0.1 second. The time step is 0.002 s in calculating the dynamic response, the time duration in the calculation of dynamic response is 8 seconds. Rayleigh damping model is used to obtain the system damping matrix and the two damping constant are taken as 0.01 and 0.02, respectively.

The acceleration responses at the 1/4 span of the beam for different crack depths are shown in Figs. 2(a) and (b). The crack is assumed to be fixed at the mid-span of the beam. From these figures, one

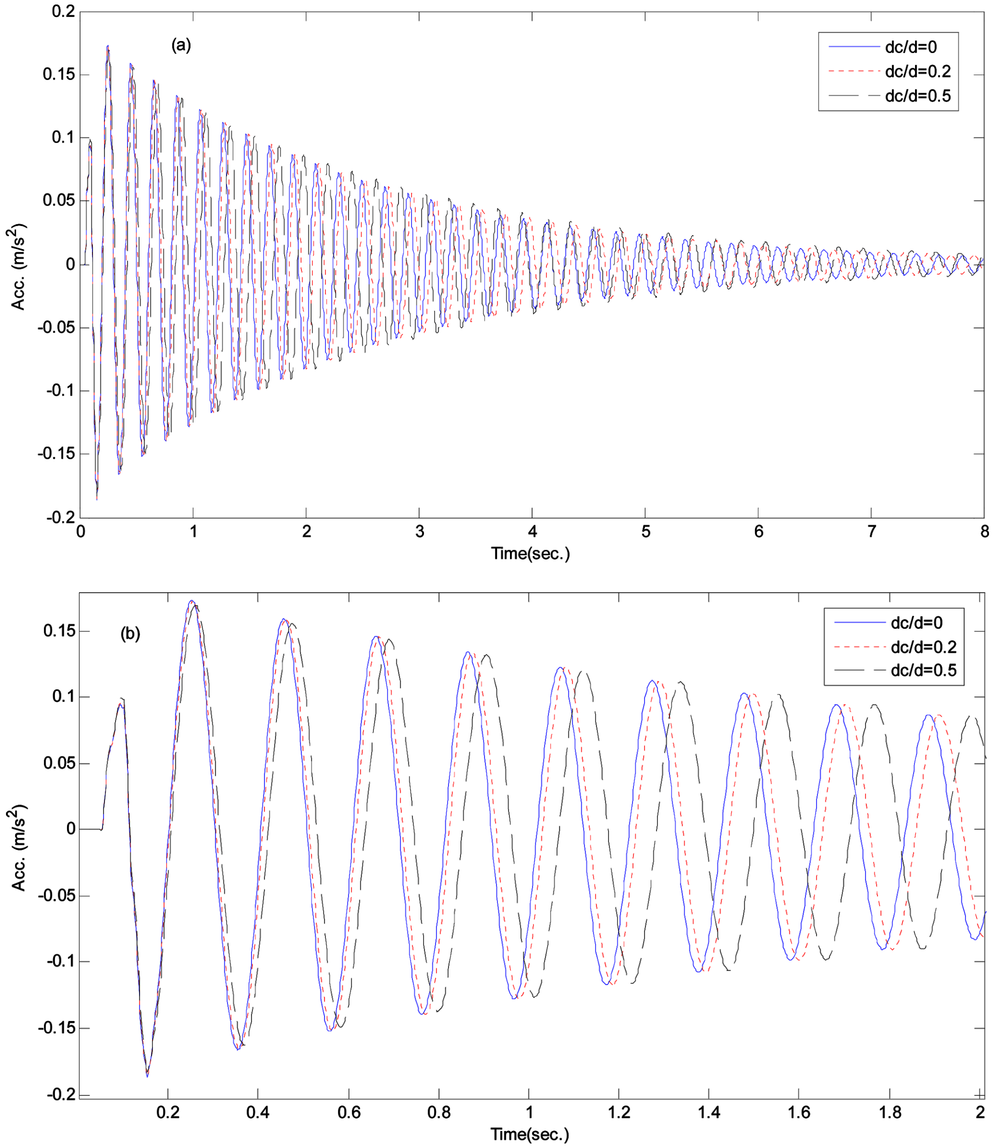

Fig. 2 (a) Comparison on the dynamic responses for different crack depths and (b) a close view

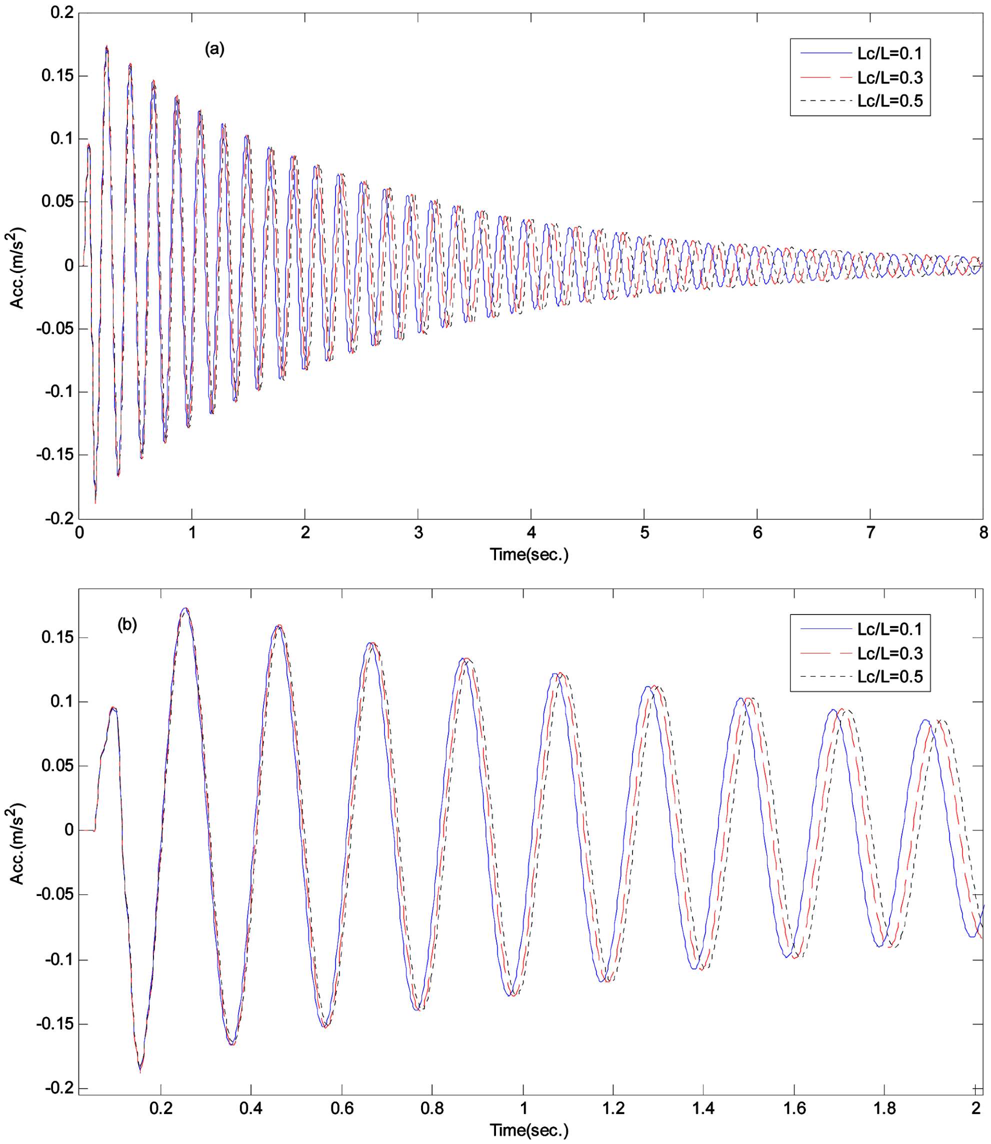

Fig. 3 (a) Comparison on the dynamic responses for different crack locations and (b) a close view

can see that when the crack depth increases, the natural frequencies of the beam decrease, so there is a shift in the acceleration response for different crack depths.

The acceleration response at the 1/4 span of the beam for different crack locations with a fixed crack depth $d_c/d = 0.3$ are shown in Fig. 3(a) and (b). The crack is assumed to be at the 0.1 L, 0.3L and 0.5L of the beam, respectively. From these figures, one also can see that there is a shift on the acceleration response. This reveals that the difference of crack location may have different effect on the natural frequencies.

4.4.2 Effect of axial force on the dynamic response

The effect of axial force on the dynamic response of the beam is investigated in this section. The

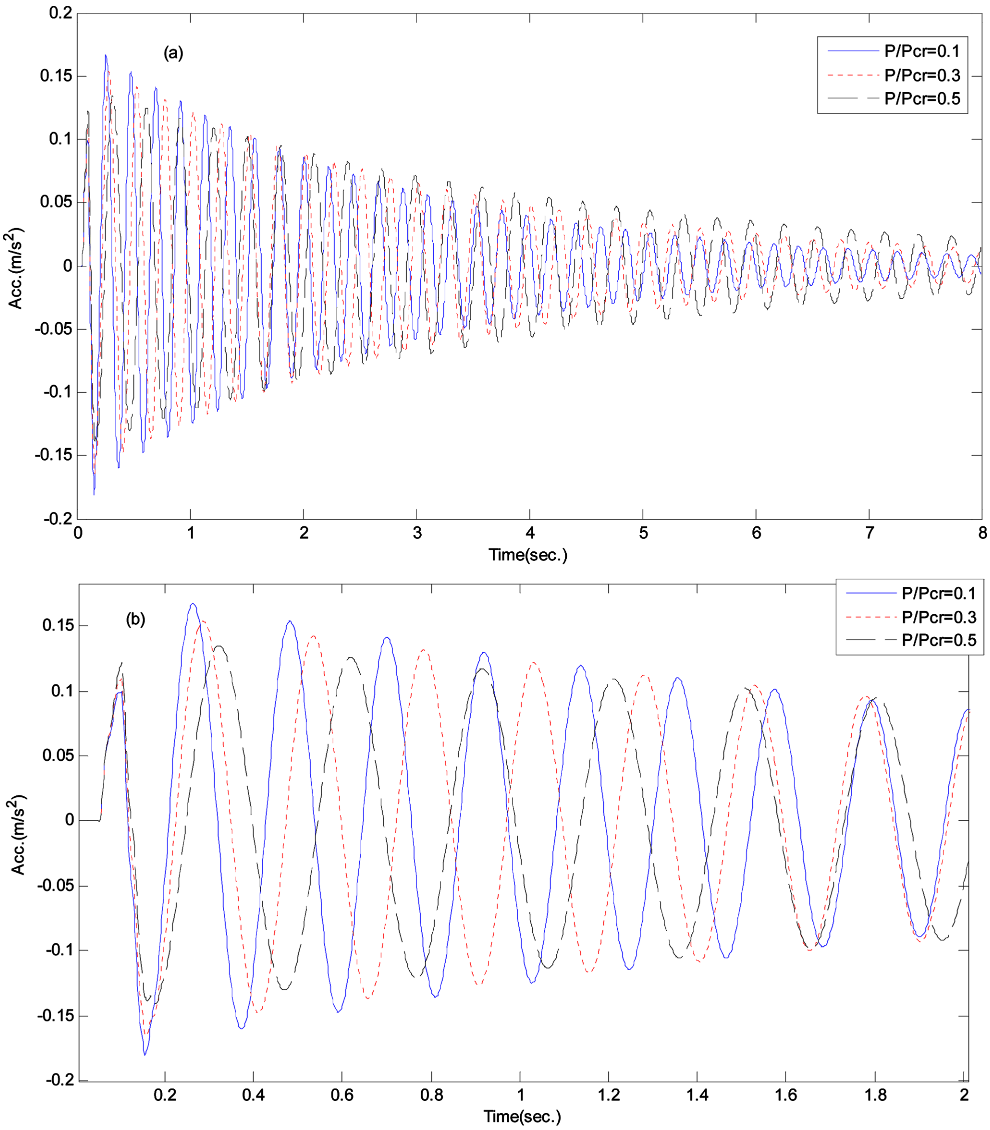

Fig. 4 (a) Comparison on the dynamic responses for different axial forces and (b) a close view

same impulsive force as the Case above is used. Figs. 4(a) and (b) show the comparison on the acceleration response at the 1/4 span of the beam for different axial force levels with $P = 0.1P_{cr}\,N$, $P = 0.3\,P_{cr}\,N$, $P = 0.5P_{cr}\,N$, respectively. From these figures, one can find with the increase in the magnitude of the axial force, the effect of axial force on the dynamic response of the beam becomes significant.

4.4.3 Dynamic response sensitivities with respect to different parameters

For a given excitation force $F(t)$ acting at the location x_F from the left support of the beam, the generalized force vector $f(t)$ can be obtained from Eq. (15), and the generalized acceleration $\ddot{Q}$, velocity $\dot{Q}$ and displacement Q of the cracked beam can be obtained from Eq. (13) by direct

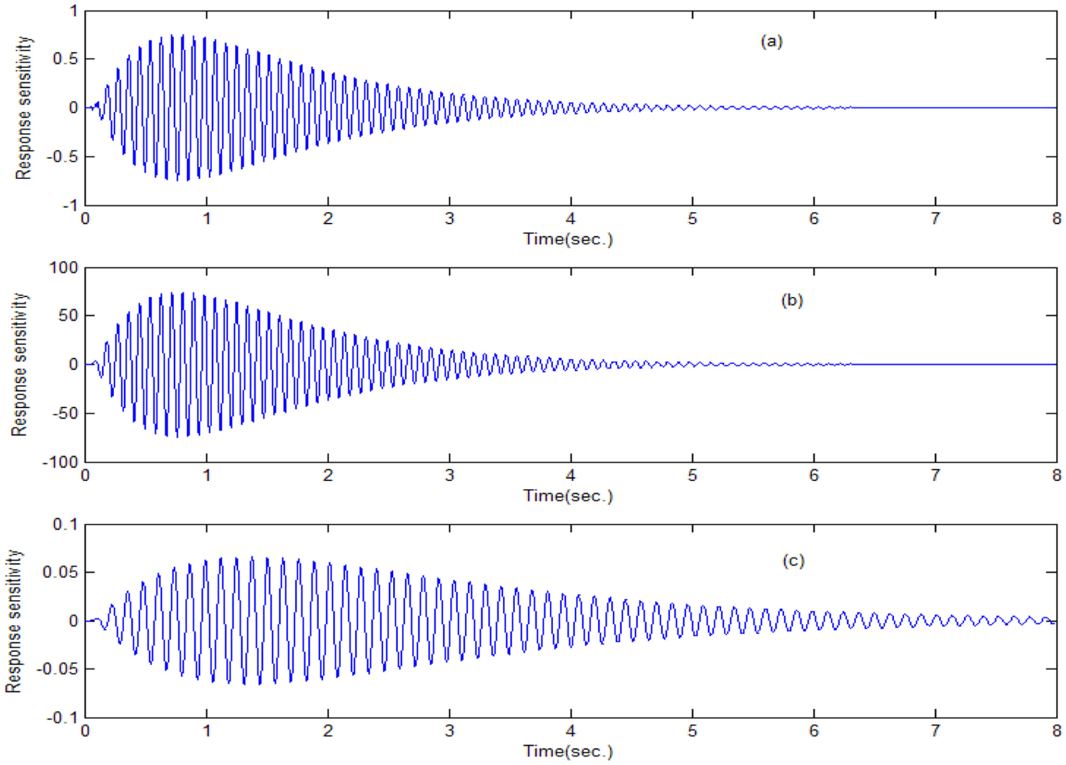

Fig. 5 Dynamic response sensitivity with respect to different parameters ((a) with respect to crack location, (b) with respect to crack depth, and (c) with respect to axial force)

integration method. The physical acceleration $\ddot{u}(x,t)$ is obtained from Eq. (16). As the dynamic responses $\ddot{Q}$, $\dot{Q}$ and Q have been obtained from Eq. (13), the response sensitivities with respect to crack depth, crack location and axial force can be calculated, respectively, according to Eqs. (17), (22) and (23). Figs. 5(a), (b) and (c) show the sensitivities of acceleration at the mid-span with respect to the crack location of and the crack depth and axial force, respectively. It is noted that the magnitude of the response sensitivity of the crack depth is almost 100 times of that of the crack location and is almost 1000 times of that of the axial force.

4.5 Crack identification from response sensitivity-based updating method

The simply supported beam in the first example is used for identifying the cracks in the beam. In the inverse problem, the measured acceleration time histories from an impulsive external force at three different locations, namely, L/4, L/2 and 3L/4 of the beam are used to identify the crack parameters. Five and ten percent artificial random noises are added to the calculated acceleration separately to simulate the 'measured' response. Convergence of computation is considered achieved when the norm of relative difference between two sets of successively identified parameters equals 1.0×10^{-5}. Two cases are studied. The first case deals with a single crack identification. The crack is assumed to locate at 700 mm from the left support of the beam with a depth of 3 mm. The second case deals with identification of multiple cracks. Besides the first crack, another crack is assumed to locate at 1200 mm from the left support of the beam with a depth of 5 mm.

Table 4 The identified results

	True	Identified values		
		Noise free	5% noise	10% noise
Single Crack (initial guess: x_L = 1000 mm, d_c = 0)				
Crack location x_L (mm)	700	700.4	705.3	706.4
Crack depth d_c (mm)	3	2.99	2.98	2.95
No. of iteration required	N/A	21	23	27
Optimal regularization parameter	N/A	3.22×10^{-5}	3.53×10^{-5}	3.74×10^{-5}
Two Cracks (initial guess: $x_{L1} = x_{L2}$ = 1000 mm, $d_{c1} = d_{c2}$ = 0)				
Crack locations x_{L1}/x_{L2} (mm)	700/1200	700.9/1202.4	705.8/1205.1	707.6/1209.7
Crack depths d_{c1}/d_{c2} (mm)	3/5	3.02/4.97	2.95/4.94	2.93/4.88
No. of iteration required	N/A	25	28	30
Optimal regularization parameter	N/A	4.32×10^{-5}	4.49×10^{-5}	5.68×10^{-5}

4.5.1 Identification of single crack

The identified results shown in Table 4 indicate that the parameters of the crack can be identified with very good accuracy under different noise levels. This indicates the identified results are not sensitive to the artificial noise.

4.5.2 Identification of two cracks

Table 4 also shows the identified results for the two cracks. Although the results are not as good as those for the single crack, the identified parameters of the two cracks are still satisfactory under different noise levels with a maximum relative error or around1.1% in the crack location and 2.3% in the crack depth.

5. Conclusions

In the present study, a composite element method is proposed to determine the natural frequencies of axially loaded beams and to calculate forced vibration responses. The time response sensitivities with respect to the crack parameters, i.e., crack location, crack depth and the axial force, are conducted. The results obtained from the proposed method are in great agreement with those in the literature. The results from forced vibration analysis show that different crack parameters may have different effects on the dynamic responses of the beam. With the increase in the magnitude of the axial force, the effect of the axial force on the dynamic response of the beam becomes significant. It is found that the magnitude of the response sensitivity varies differently for different parameters, such as the crack location, crack depth and axial force. A response sensitivity-based crack identification method is presented and verified by two numerical simulations. Very good identified results can be obtained even using noisy acceleration measurements.

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