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On the static and dynamic stability of beams with an axial piezoelectric actuation

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Abstract. The present contribution is concerned with the static and dynamic stability of a piezo-laminated Bernoulli-Euler beam subjected to an axial compressive force. Recently, an inconsistent derivation of the equations of motions of such a smart structural system has been presented in the literature, where it has been claimed, that an axial piezoelectric actuation can be used to control its stability. The main scope of the present paper is to show that this unfortunately is impossible. We present a consistent theory for composite beams in plane bending. Using an exact description of the kinematics of the beam axis, together with the Bernoulli-Euler assumptions, we obtain a single-layer theory capable of taking into account the effects of piezoelectric actuation and buckling. The assumption of an inextensible beam axis, which is frequently used in the literature, is discussed afterwards. We show that the cited inconsistent beam model is due to inadmissible mixing of the assumptions of an inextensible beam axis and a vanishing axial displacement, leading to the erroneous result that the stability might be enhanced by an axial piezoelectric actuation. Our analytical formulations for simply supported Bernoulli-Euler type beams are verified by means of three-dimensional finite element computations performed with ABAQUS.

Keywords: smart beam; piezoelectric actuation; Bernoulli-Euler beam; buckling load; static stability; dynamic stability.

1. Introduction

In the last decade, considerable advances have been achieved concerning the use of piezoelectricity in controlling structural displacements, see Rao and Sunar (1994), Rao and Sunar (1999) and Tzou (1998) for some reviews. Controlling structural displacements can be considered as an important step towards the newly formulated general goal of developing ageless structures, Shoureshi (2003). Particularly, the topic of a complete compensation of force-induced displacements, which is also denoted as shape control, has gained considerable interest. For a review on shape control by piezoelectric actuation, see e.g., Irschik (2002). See Irschik and Pichler (2004) for a class of 3D-solutions of shape control within the framework of linear elastodynamics. Recently, our group has also started to study the compensation

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of structural vibrations produced by superimposed rigid-body motions, see Irschik, *et al*. (2003) and Irschik, *et al*. (2004). The problem of compensating coupled bending-torsional motions of beams by piezoelectric actuation in the presence of rigid-body motions has been successfully tackled in the doctoral thesis of Zehetner (2005). In our studies, it turned out to be not possible to influence static or dynamic buckling forces of a beam by an axial piezoelectric actuation, i.e., an actuating normal force, be the beam at rest, or be it floating, such as in the case of structures with superimposed rigid body motions. Independently, the goal of enhancing the buckling capacity of beams by attached piezoelectric layers has been formulated in the literature, and successful results have been reported by Wang (2002) and Wang (2005). In the latter references, beams with one end fixed in axial direction, the other end being free to move axially under the action of an external compressive force, were considered. External follower forces applied at the axially movable end of the beam were included in Wang and Quek (2002) in order to include flutter instability. In extension of the theory presented in Wang (2002), Wang and Queck (2002) and Wang (2005) it has been shown by Chen, *et al*. (2002) that piezoelectric actuation may significantly affect the dynamic instability of beams due to harmonically varying external compressive forces. A further study on the buckling control of structures by piezoelectric actuation was presented by Wang and Varadan (2003). In Chen, *et al*. (2002), Wang (2002), Wang and Queck (2002), Wang and Varadan (2003) and Wang (2005), a piezoelectric actuation force that may even cancel the normal force due to the external compressive force appears in the respective field equations for the transverse beam displacement. This fact has been criticized in a paper by de Faria (2004) on the buckling enhancement of laminated beams with piezoelectric actuators, where a short theoretical explanation for the critical remarks was presented. The main goal of de Faria (2004) was to study buckling due to a given axial support motion. In the following, we do not consider the latter case. Rather, we remain within the framework of the structural systems considered in Chen, *et al*. (2002), Wang (2002), Wang and Queck (2002), Wang and Varadan (2003) and Wang (2005), namely a slender one-span beam with one axially fixed end, the other end being axially moving under the action of an external compressive force, where we restrict to constant or time-varying non-follower forces. Exemplarily, we treat a simply supported beam with axially constant mass and stiffness, the actuating layers being attached on both sides of the beam and extending over its whole span. Our scope is to demonstrate that static and dynamic buckling cannot be influenced by a piezoelectric actuating normal force. A first account of this work has been presented in Zehetner and Irschik (2005).

The material presented subsequently is organized as follows. In the first part of the paper, we lay some emphasis on the concise derivation of a consistent linear beam theory capable of taking into account the effects of piezoelectric actuation and static as well as dynamic buckling. We consider a composite beam in a state of plane bending and start with an exact description of the kinematics of the beam axis. We then introduce the Bernoulli-Euler assumptions leading to a single-layer theory for the composite beam, and we derive the consequences of the Bernoulli-Euler constraints within the non-linear theory of continuum mechanics. The corresponding equations of motion, fitting to the latter results, are derived from the D'Alembert principle in the version of Lagrange. Using an order of magnitude approximation, we arrive at a v.Karman type beam theory with effective mass and stiffness. Exemplarily, we treat the case of a symmetrically built-up, three-layer, one-span and simply supported beam, one end being axially free, while the other end is axially movable under the action of a compressive force of the non-follower type. Introducing Hooke's law extended to the case of piezoelectric or thermal actuation, assuming a symmetric actuation in the outer layers, and performing a further simplification, which is again motivated by an order-of-magnitude argument, we end up with a suitable theory. We denote this theory as extensible theory for the sake of comparison, since it allows considering the influence of an actuating piezoelectric normal force upon the axial displacement of the beam axis. Assuming furthermore that the beam axis is inextensible, we arrive at the so-called inextensible theory, in which the axial displacement follows by direct integration from deflection of the beam axis. We point out that in both, the extensible and the inextensible theory, there is no influence of a piezoelectric actuating normal force to be found upon the deflection and thus upon buckling, the corresponding results for static and dynamic buckling being equal according to the two theories. We then show that the results given in Chen *et al*. (2002), Wang (2002), Wang and Queck (2002), Wang and Varadan (2003) and Wang (2005), which claim that a considerable influence of piezoelectric actuation upon buckling can be achieved, may be obtained in the framework of our derivations, when we inconsistently mix the assumptions of an inextensible beam axis with the assumption that the axial displacement of the beam axis vanishes. This latter inconsistence, which has been explicitly stated in Chen, *et al*. (2002), indeed represents a subtle but far-reaching difference to the inextensible theory. We present the consequences of this inconsistent assumptions of vanishing axial displacement and axial strain upon the buckling load of the simply supported beam, results that have been presented before in Chen, *et al*. (2002), Wang (2002), Wang and Queck (2002), Wang and Varadan (2003) and Wang (2005). Using a well-known analogy between thermal and piezoelectric actuation, we finally discretize the beam under consideration by means of three-dimensional finite elements, thus omitting any assumptions introduced by the various beam theories. We demonstrate that the numerical computations do not show any influence of the actuation upon the static buckling load, in contrast to what has been claimed in Chen, *et al*. (2002), Wang (2002), Wang and Queck (2002), Wang and Varadan (2003) and Wang (2005). Together with some theoretical arguments, we thus give strong evidence that the buckling regime of beams with one axially moving end under the action of a compressive load can not be influenced by a piezoelectric actuating normal force. Hence, the formulations presented in Chen, *et al*. (2002), Wang 2002, Wang and Queck (2002), Wang and Varadan (2003) and Wang (2005) must be considered as valuable but inconsistent attempts towards an unfortunately non-reachable goal.

2. Kinematics

The Bernoulli-Euler beam theory is chosen as the kinematical basis of the following considerations. It is accordingly assumed that plane cross-sections originally normal to the beam axis perform a rigid body motion during the deformation and remain normal to the deflected beam axis, see e.g., Ziegler (1998). When applied to the smart composite beam, these assumptions result in an effective single-layer theory, which has a quite wide range of applicability, despite neglecting the influence of shear upon the deformation. For shear deformable beam theories, in which the Bernoulli-Euler requirements are released, see e.g., Huang (1961), Li (1998), Pai and Nayfeh (1992) and Smolenski (1999). Relationships between the Bernoulli-Euler theory and shear deformable beam theories were investigated by Irschik (1991), Irschik (1993) and Reddy (1997). In the present paper, also having in mind the cited relationships to shear deformable theories, we are interested in a concise derivation of a beam theory based on the Bernoulli-Euler assumptions, since it has been claimed in the framework of the latter kinematical assumptions that stability could be influenced by piezoelectric actuation, see Chen, *et al*. (2002), Wang (2002), Wang and Queck (2002), Wang and Varadan (2003) and Wang (2005).

Plane elastic deformations of initially straight beam-type structures are studied in the following. The deformations are assumed to take place in the (x, y)-plane of a Cartesian coordinate system, the x-coordinate (axial coordinate) being taken along the length of the undeformed beam axis and the

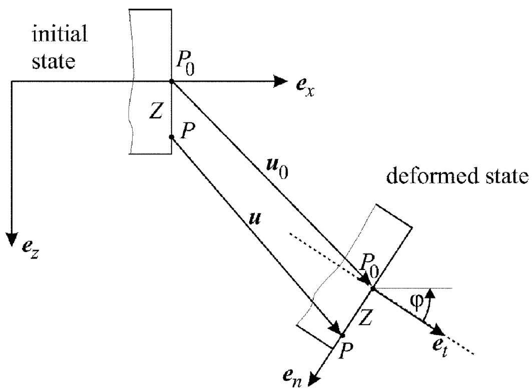

Fig. 1 Kinematics of a beam segment

z-coordinate denoting the transverse coordinate in thickness direction, see Fig. 1. The beam is taken to be built up symmetrically with respect to this plane, and it consists of layers made of non-polarizable substrate materials and of some piezoelectric layers. The various layers are assumed to be perfectly bonded with negligible thickness of the bonding layers.

Utilizing a Cartesian base ($\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$), a point P_0 of the beam axis is identified in the reference configuration by the place $\boldsymbol{X}_o = (X\ \ Y\ \ 0)^T$ and in the current configuration by $\boldsymbol{x}_o = \boldsymbol{X}_o + \boldsymbol{u}_o$, where $\boldsymbol{u}_o$ denotes the displacement vector, $\boldsymbol{u}_o = (u(\boldsymbol{X}_o, t))\ 0\ w(\boldsymbol{X}_o, t))^T$, see Fig. 1. The axial displacement of the beam axis is denoted by u, and w is the transverse deflection.

The place of an arbitrary point P of the cross-section in the reference configuration can be described by the vector

$$\boldsymbol{X} = \boldsymbol{X}_o + Z\boldsymbol{e}_z = (X\ \ Y\ \ Z)^T \tag{1}$$

In Eq. (1) $\boldsymbol{e}_z$ denotes the unit normal vector of the undeformed beam axis, and Z is the (transverse) thickness coordinate. The unit tangent vector of the deformed beam axis $\boldsymbol{e}_t$ is given by

$$e_t = \frac{\boldsymbol{F}_o\boldsymbol{e}_x}{\|\boldsymbol{F}_o\boldsymbol{e}_x\|} = \frac{1}{C}(1 + u'\ \ 0\ \ w')^T \tag{2}$$

The Finger deformation gradient of the beam axis is $\boldsymbol{F}_o = grad\ \boldsymbol{x}_o$. Accordingly, we have $C = (w'^2 + (1+u')^2)^{1/2}$ in Eq. (2), where a prime denotes the derivative with respect to the axial coordinate X of the undeformed configuration. The bending angle φ of the deformed beam axis with respect to the initial straight axis then is given by, see Fig. 1,

$$\varphi(\boldsymbol{X}_o, t) = \tan^{-1}\left(\frac{\boldsymbol{e}_t \cdot \boldsymbol{e}_z}{\boldsymbol{e}_t \cdot \boldsymbol{e}_x}\right) = \tan^{-1}\left(\frac{w'}{1 + u'}\right) \tag{3}$$

According to the above stated Bernoulli-Euler hypothesis, the cross-section of the beam during deformation is supposed to stay normal to the beam axis and to perform rigid body motions only. In the deformed state, the unit normal vector $\boldsymbol{e}_n$ of the axis becomes

$$\boldsymbol{e}_n = \boldsymbol{Q}\boldsymbol{e}_z = (-\sin\varphi\ \ 0\ \ \cos\varphi)^T \tag{4}$$

where $\boldsymbol{Q}$ denotes the tensor describing the rigid body rotation of the beam cross-section,

$$\boldsymbol{Q} = \begin{pmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{pmatrix} \tag{5}$$

Using Eqs. (2) and (4) yields the relation

$$\boldsymbol{e}_t^T \boldsymbol{e}_n = (1 + u')\sin\varphi - w'\cos\varphi = 0 \tag{6}$$

In the deformed state, the position of point P on the cross section then can be described by

$$\boldsymbol{x} = \boldsymbol{X}_o + \boldsymbol{u}_o + Z\boldsymbol{e}_n = (X + u - Z\sin\varphi \;\; Y \;\; w + Z\cos\varphi)^T \tag{7}$$

compare Fig. 1, and the corresponding deformation gradient tensor becomes

$$\boldsymbol{F} = grad\, \boldsymbol{x} = \begin{pmatrix} 1 + u' - Z\varphi'\cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ w' - Z\varphi'\sin\varphi & 0 & \cos\varphi \end{pmatrix} \tag{8}$$

With Eqs. (1) and (7) the displacement vector $\boldsymbol{u} = \boldsymbol{x} - \mathbf{X}$ of an arbitrary point P follows to

$$\boldsymbol{u} = \boldsymbol{u}_o + Z(\boldsymbol{e}_n - \boldsymbol{e}_z) = (u - Z\sin\varphi \;\; 0 \;\; w + Z(\cos\varphi - 1))^T \tag{9}$$

Generally, the Green strain tensor is given by $\boldsymbol{E} = (\boldsymbol{F}^T\boldsymbol{F} - \boldsymbol{I})/2$, where $\boldsymbol{I}$ is the identity tensor. For a plane deformation according to the considered Bernoulli-Euler-type theory, inserting Eq. (8) and using Eq. (6), one obtains

$$\boldsymbol{E} = \begin{pmatrix} E_{xx} & 0 & E_{xz} \\ 0 & 0 & 0 \\ E_{xz} & 0 & 0 \end{pmatrix} \tag{10}$$

where the axial strain E_{xx} and shear strain E_{xz} are

$$E_{xx} = u' - Z\varphi'(w'\sin\varphi + (1 + u')\cos\varphi) + \frac{1}{2}(u'^2 + w'^2 + Z^2\varphi'^2) \tag{11}$$

$$E_{xz} = -\frac{1}{2}((1 + u')\sin\varphi + w'\cos\varphi) = 0 \tag{12}$$

So far, our considerations are exact consequences of the Bernoulli-Euler kinematical assumptions. The latter impose constraints upon the three-dimensional field theory of continuum mechanics, these constraints leading to Eqs. (11) and (12).

3. Simplified nonlinear case

Thin structures such as beams are usually so flexible that the exact nonlinear strains, Eqs. (11) and (12), can be reduced to the simplified form of small finite deformations, see Yu (1996) for a discussion. The corresponding theory dates back to a famous heuristic formulation by v. Karman (1910). To obtain the axial and transverse strain according to the v.Karman-theory of moderate but finite beam deflections from the above exact relations, we adopt an order-of magnitude procedure described by Crespo da Silva (1988), assuming that

$$u = \varepsilon^2 \bar{u} = O(\varepsilon^2), \quad w = \varepsilon \bar{w} = O(\varepsilon), \quad \text{with} \quad \bar{u} \sim \bar{w} \tag{13}$$

where ~ stands for same order of magnitude, and ε is a small bookkeeping parameter. Particularly, Taylor's series expansion of the bending angle φ of the beam axis, Eq. (3), gives

$$\varphi = -\varepsilon \bar{w}' + O(\varepsilon^3) = -w' + O(\varepsilon^3) \tag{14}$$

Terms of $O(\varepsilon^3)$ are neglected in the following, thus $\varphi \simeq -w'$. Noting that since $\sin(w') = w' + O(\varepsilon^3)$ and $\cos(w') = 1 - w'^2/2 + O(\varepsilon^4)$, the strain components in Eqs. (11) and (12) reduce to

$$E_{xx} = u' - Zw'' + \frac{1}{2}w'^2, \quad E_{xz} = 0 \tag{15}$$

which correspond to the classical postulate by von Karman, see also Yu (1996). Within the same order-of-magnitude considerations, the displacement vector for the considered Bernoulli-Euler beam, Eq. (9), simplifies to

$$\boldsymbol{u} = (u - Zw' \quad 0 \quad w)^T \tag{16}$$

Eqs. (15) and (16) constitute the kinematics of the moderate-deformation beam theory under consideration.

4. D'Alembert's principle

The equations of motion corresponding to the above kinematical relations are derived utilizing D'Alembert's principle in the version of Lagrange, see e.g., Ziegler (1998). The latter balances the virtual work of internal, external and inertial forces of the continuum occupying the beam volume V in the reference configuration,

$$\delta W_i = -\int_V S_{ij} \delta E_{ij} dV \tag{17}$$

$$\delta W_e = \int_A \boldsymbol{f}_A^T \delta \boldsymbol{x} dA + \int_V \boldsymbol{f}_V^T \delta \boldsymbol{x} dV \tag{18}$$

$$\delta W_a = \int_V \rho \frac{d^2}{dt^2} \boldsymbol{x} \delta \boldsymbol{x} dV \tag{19}$$

in the form

$$\delta W_i + \delta W_e = \delta W_a \tag{20}$$

In Eq. (17) Einstein's summation convention is used, and stresses are denoted by S_{ij}. Strictly speaking, they are components of the 2. Piola-Kirchhoff stress tensor. In Eqs. (18) and (19) $\boldsymbol{f}_A$ represents a distributed boundary force per unit area, $\boldsymbol{f}_V$ is a distributed body force per unit volume in the reference configuration, and ρ is the mass density. The variation sign is denoted by δ. Note that the integrations in Eqs. (17)-(19) are with respect to the place in the undeformed configuration, A denoting the cross-sectional area.

Inserting Eq. (15) into Eq. (17) yields the virtual work of internal forces appropriate for a Bernoulli-Euler beam theory,

$$\delta W_i = -\int_L \left(N\delta\left(u' + \frac{1}{2}w'^2\right) - M\delta w''\right) dX \tag{21}$$

with normal force N and bending moment M,

$$N = \int_A S_{xx} dA, \qquad M = \int_A S_{xx} Z dA \tag{22}$$

respectively. Partial integration of Eq. (21) gives

$$\delta W_i = \int_L (N'\delta u + ((Nw')' + M'')\delta w) dX - (N\delta u + (M' + Nw')\delta w - M\delta w')\big|_0^L \tag{23}$$

with the notation $(\cdot)\big|_0^L = (\cdot)\big|_{X=L} - (\cdot)\big|_{X=0}$. Assuming that there is a distributed force per length acting on the beam axis of the form $\boldsymbol{f}^e = (f_x^e, 0, f_z^e)^T$, and that single forces $\boldsymbol{F}_x^e = (F_x^e, 0, F_z^e)^T$ and a single moment M^e act at the beam-ends, the virtual work of external forces in Eq. (18) can be can be replaced by

$$\delta W_e = \int_L (f_x^e \delta u + f_z^e \delta w) dX + (F_x^e \delta u + F_z^e \delta w - M^e \delta w')\big|_0^L \tag{24}$$

In the following, we remain in the framework of Bernoulli-Euler beam theory, in which the effect of shear and rotatory inertia is neglected. The reason is that our paper is concerned with a comparison to other Bernoulli-Euler type contributions for slender beams, see the literature cited in section 2. Later, we will check the outcomes of our formulation by 3D finite element computations, in which the cited effects are taken into account. For the Bernoulli-Euler theory the virtual work of inertial forces can be expressed by

$$\delta W_a = \int_L \mu(\ddot{u}\delta u + \ddot{w}\delta w) dX \tag{25}$$

where μ is the mass per length,

$$\mu = \int_A \rho dA \tag{26}$$

Inserting Eqs. (23)-(25) into D'Alembert's principle, Eq. (20), yields the equations of motion for a Bernoulli-Euler beam in the framework of a v.Karman-type approximation,

$$-N' + \mu\ddot{u} = f_x^e \tag{27}$$

$$-M''-(Nw')' + \mu\ddot{w} = f_z^e \tag{28}$$

together with the boundary conditions

$$\begin{aligned} X = 0, L: \quad N &= F_x^e \quad \text{or} \quad u = 0 \\ M' + Nw' &= F_z^e \quad \text{or} \quad w = 0 \\ M &= M^e \quad \text{or} \quad w' = 0 \end{aligned} \tag{29}$$

Note that relations (27)-(29) are yet independent from the material behavior.

5. Simply supported beam: extensible beam theory

In the following we study the example of a simply supported beam subjected to an axial compressive force $F_x^e = -F$ of the non-follower type, see Fig. 2. The cross-section is assumed to be symmetric with respect to the axes y and z. On the top and bottom, actuating layers made of piezoelectric material are assumed to be perfectly bonded to a substrate material.

For this special case of loading and boundary conditions, the equilibrium equations in longitudinal (x) direction follow from Eqs. (27) and (29),

$$\begin{aligned} -N' + \mu\ddot{u} &= 0 \\ X = 0: \quad u &= 0 \\ X = L: \quad N &= -F \end{aligned} \tag{30}$$

while in transversal (z) direction Eqs. (28) and (29) yield

$$\begin{aligned} -M'' - (Nw')' + \mu\ddot{w} &= 0 \\ X = 0, L: \quad w = 0, \quad M &= 0 \end{aligned} \tag{31}$$

Integration of Eq. (30) results in

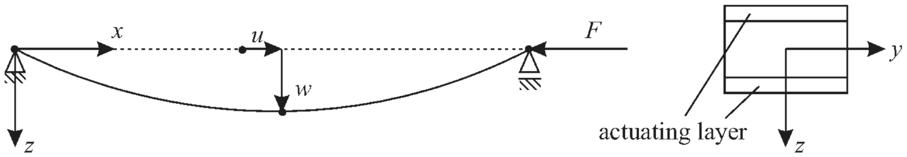

Fig. 2 Simply supported beam

$$N = -F - \int_x^L \mu \ddot{u}(\xi) d\xi \tag{32}$$

Inserting Eq. (32) into Eq. (31) gives

$$-M'' - \mu\ddot{u}w' + Fw'' + w''\int_X^L \mu\ddot{u}(\xi)d\xi + \mu\ddot{w} = 0 \tag{33}$$

Assuming that

$$-\mu\ddot{u}w' + w''\int_X^L \mu\ddot{u}(\xi)d\xi \ll \mu\ddot{w} \tag{34}$$

which can be justified by a order-of-magnitude study similar to the one above, the transversal equilibrium equation for the simply supported beam in Fig. 2 reads

$$\begin{gathered} -M'' + Fw'' + \mu\ddot{w} = 0 \\ X = 0, L: \qquad w = 0 \qquad M = 0 \end{gathered} \tag{35}$$

The latter relations form the basis of what we subsequently will call the extensional beam theory for the sake of comparison. The extensional theory follows from the von Karman theory by the simplification indicated in Eq. (34).

6. Constitutive relations

As already mentioned, the equations of motion given in sections 4 and 5 are yet independent of the material behavior. The uni-axial constitutive relation for linear elastic materials with actuation can be written in the form of an extended Hooke law

$$S_{xx} = Y_e(E_{xx} - E_{xx}^0) \tag{36}$$

with the (effective) Young modulus Y_e and the actuating eigenstrain E_{xx}^0. Strictly speaking, we deal with a St.Venant-Kirchhoff material in the presence of eigenstrain. For thermoelastic materials, there is $E_{xx}^0 = \alpha T$, with the thermal expansion coefficient α and temperature difference T. For layers of piezoelastic material with the thickness direction z, eigenstrain can be set to $E_{xx}^0 = d_{31}E_z$, where d_{31} is a piezoelectric coefficient and E_z is the electric field in thickness direction of the layer. For this analogy between thermal and piezoelectric actuation, see e.g., Vinson (1993). Inserting Eq. (36) into Eq. (22) yields the stress resultants

$$N = A_{11}\left(u' + \frac{w'^2}{2}\right) - N^a, \qquad M = -D_{11}w'' - M^a \tag{37}$$

where A_{11} and D_{11} are longitudinal and transversal stiffness,

$$A_{11} = \int_A Y_e dA, \quad D_{11} = \int_A Y_e Z^2 dA \tag{38}$$

N^a and M^a are the actuating normal force and bending moment given by

$$N^a = \int_A Y_e E_{xx}^0 dA, \quad M^a = \int_A Y_e E_{xx}^0 Z dA \tag{39}$$

Substituting Eq. (37) into the equilibrium conditions for the simply supported beam, Eqs. (30) and (35) yield the equations of motion for coupled plane extensional-flexural beam vibrations. In axial direction, one obtains

$$-\left(A_{11}\left(u' + \frac{w'^2}{2}\right)\right)' + \mu\ddot{u} = -N^{a'}$$
$$X = 0: \qquad u = 0$$
$$X = L: \qquad A_{11}\left(u' + \frac{w'^2}{2}\right) = -F + N^a \tag{40}$$

and in transversal direction there is

$$(D_{11}w'')'' + Fw'' + \mu\ddot{w} = -M^{a''}$$
$$X = 0, L: \qquad w = 0 \qquad D_{11}w'' = -M^a \tag{41}$$

In the following, Eqs. (40) and (41) will be denoted as extensional beam theory. Note that according to the above order of magnitude studies there is $u' + w'^2/2 = O(\varepsilon^2)$. However, since the axial stiffness A_{11} usually is very large, the product $A_{11}(u' + w'^2/2)$ must not be neglected.

7. Inextensible beam theory

In the literature, the approximate model of an inextensible beam axis is frequently used for the considered beam problems with an axially movable end, where it is assumed that the axial deformation is small compared to flexural deformation, see e.g. Ziegler 1998. In the framework of this theory it is assumed that the arc length of the beam axis in the deformed state is equal to the length of the beam axis in the undeformed initial state. Thus the axial strain of the beam axis is set to zero, with Eq. (15) there is

$$E_{xx}(Z = 0) = u' + \frac{1}{2}w'^2 = 0 \tag{42}$$

such that the axial displacement u of the beam axis can be computed from the deflection w simply by integration,

$$u(X) = -\frac{1}{2}\int_0^X w'^2 d\xi \tag{43}$$

The theory based on Eq. (43) is denoted as inextensible beam theory in the following.

Inserting Eq. (42) into Eq. (21) and integrating by parts yields the virtual work of internal forces for an inextensible beam,

$$\delta W_i = \int_L M''\delta w dX - (M'\delta w - M\delta w')|_0^L \tag{44}$$

i.e., the virtual work of the normal force vanishes. Inserting Eq. (43) into Eq. (24) gives the virtual work of external forces,

$$\delta W_e = \int_L \left(f_z^e \delta w - f_x^e \int_0^X w'\delta w' d\xi \right) dX - F_x^e \int_L w'\delta w' d\xi + (F_z^e \delta w - M^e \delta w')\Big|_0^L \tag{45}$$

For the simply supported beam in Fig. 2, the axial compressive force $F_x^e = -F$ is considered as external force on the right end, and $f_z^e = f_x^e = F_z^e = M^e = 0$. Thus, the virtual work of external forces can be expressed by

$$\delta W_e = F\int_L w'\delta w' dX \tag{46}$$

Substituting Eq. (43) into Eq. (25) and neglecting higher order terms yields the virtual work of inertial forces

$$\delta W_a = \int_L \mu \ddot{w} \delta w dX \tag{47}$$

Introducing Eqs. (44), (46) and (47) into D'Alembert's principle, Eq. (20), one finally obtains the transversal equilibrium equations for the model of an inextensible Bernoulli-Euler beam,

$$\begin{gathered} -M'' + Fw'' + \mu\ddot{w} = 0 \\ X = 0, L: \quad w = 0, \quad M = 0 \end{gathered} \tag{48}$$

8. Discussion of relations between extensible and inextensible beam theory

From Eq. (48) it is seen that the basic assumption of the inextensible beam model, Eq. (43), yields the same transversal equation of motion, as has been obtained above for the extensible beam model, Eq. (35). Substituting the constitutive relation for the bending moment, Eq. (37), then again leads to the partial differential Eq. (41) for the deflection w, such that the inextensible beam theory yields the same result for the latter as the extensible theory. Thus, with respect to the deflection, there is complete coincidence between the extensible and the inextensible theory, both theories showing no influence of the actuating normal force N^a upon the deflection w, and hence upon buckling. A boundary value problem has to be solved for the axial displacement u in the extensional theory, Eq. (40), the latter showing an influence of the actuating normal force N^a upon the axial displacement u, while u in the inextensible theory can be computed from the deflection w directly by integration, see Eq. (43). We note that care has to be taken with respect to Eq. (43), since the assumption of inextensibility appears to be admissible only for special boundary conditions. For instance in case that the axial displacement of the beam is prohibited on both ends, i.e., for $u|_{X=0} = u|_{X=L} = 0$, the inextensible model would lead to a contradiction. On the other hand side, if the axial displacement is only constrained on one boundary, the other end being movable, the inextensional theory should result in a reasonable estimate for the axial displacement u. In general, there will be a difference in u, also in the absence of piezoelectric actuation. This becomes obvious, e.g., for the example of a beam with tensile normal force acting at the movable end as the only source of deformation. For this problem the deflection is zero all over the beam, $w = 0$. Thus, the inextensible model yields $u = 0$, while in the framework of the extensible theory u becomes a function of F.

In the following two sections, static and dynamic stability investigations are performed for the simply supported beam in Fig. 2.

9. Static buckling of simply supported beam

We first consider the quasi-static case, neglecting inertial effects. Additionally it is assumed that the stiffness D_{11} is constant along the beam and that actuation in the outer layers is symmetric with respect to the axis, such that the actuating moment in Eq. (39) vanishes, $M^a = 0$. The latter assumption allows studying the buckling phenomenon. With these assumptions, Eq. (41) yields the ordinary differential equation for the deflection w,

$$D_{11} w'''' + F w'' = 0$$

$$X = 0, L: \qquad w = 0, \qquad w'' = 0 \tag{49}$$

It is well known that Eq. (49) has the trivial solution $w = 0$, and that buckling occurs when the compressive force F arrives at the critical value

$$F_c = \frac{D_{11}\pi^2}{L^2} \tag{50}$$

see e.g., Ziegler (1998). Note that the critical load is a function of bending stiffness and length, but that there is no influence of the actuating normal force N^a.

10. Dynamic buckling of simply supported beam

In the following, it is assumed, that the beam in Fig. 2 is loaded by the harmonic axial force

$$F = F_0 + F_d \cos\theta t \tag{51}$$

with the static load F_0, the amplitude F_d of the dynamic load and the excitation frequency θ. For the same reasons as mentioned in section 9, we set $M^a = 0$. Inserting the approximation.

$$w = \left(a \sin\frac{\theta t}{2} + b \cos\frac{\theta t}{2}\right) \sin\frac{\pi X}{L} \tag{52}$$

a and b being constants, into Eq. (41), again assuming that there is no actuating moment, yields the boundary of the main region of instability, see e.g., Bolotin (1961),

$$\frac{\theta}{2\Omega} = \sqrt{1 \pm \mu_e} \tag{53}$$

where Ω is the eigenfrequency of a beam loaded by the constant axial compressive force F_0, and μ_e is

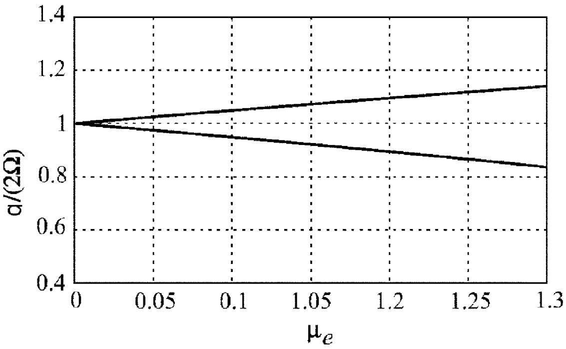

Fig. 3 Main region of instability

an excitation parameter,

$$\Omega = \frac{\pi^2}{L^2}\sqrt{\frac{D_{11}}{\mu}}\sqrt{1 - F_0\frac{L^2}{D_{11}\pi^2}} \tag{54}$$

$$\mu_e = \frac{F_d L^2}{2(D_{11}\pi^2 - F_0 L^2)} \tag{55}$$

The result of Eq. (53) is shown in Fig. 3. As it is obvious from Eqs. (53)-(55), the regions of instability are not dependent on the actuating normal force N^a.

11. Inconsistent modeling of a simply supported beam

As has been noted above, an influence of the actuating normal force N^a upon the deflection w has been observed in Chen *et al*. (2002), Wang (2002), Wang and Queck (2002), Wang and Varadan (2003) and Wang (2005), in contrast to our above derivations. When studying the reasons for this, it turns out that in these papers the assumptions that the axial displacement vanishes,

$$u = 0, \qquad u' = 0 \tag{56}$$

and that the beam axis is inextensible, see Eq. (42), have been mixed in an inadmissible way. We shortly derive the consequence of this assumption in the following. In Chen, *et al*. (2002) the work of internal forces has been derived based on the assumption that the axial displacement u vanishes, see Eq. (9) of Chen, *et al*. (2002). Inserting the assumption of Eq. (56) into Eqs. (21) and (37) we obtain

$$\delta W_i = -\int_L (N w' \delta w' - M \delta w'') dX, \qquad N = A_{11}\frac{w'^2}{2} - N^a \tag{57}$$

Neglecting higher order terms the virtual work of internal forces becomes

$$\delta W_i = \int_L (N^a w' \delta w' + M \delta w'') dX \tag{58}$$

On the other hand, in Chen, *et al*. (2002) the assumption of an inextensible beam axis according to Eq. (42) has been used to express the work of external forces, obtaining the expression given in Eq. (46), compare Eq. (13) in Chen, *et al*. (2002).

Inserting Eqs. (46), (47) and (58) into D'Alembert's principle, Eq. (20), yields

$$\int_L [M\delta w'' + (F - N^a)w'\delta w']dX = \int_L \mu\ddot{w}\delta w dX \tag{59}$$

Substituting furthermore the bending moment from Eq. (37), assuming that the actuating layers are symmetric such that $M^a = 0$, performing partial integrations, and neglecting higher order terms finally gives

$$(D_{11}w'')'' + (F - N^a)w'' + \mu\ddot{w} = 0 \tag{60}$$

see e.g., Eq. (2) of Wang and Varadan (2003). Formulations corresponding to Eq. (60) have been used in Chen, *et al*. (2002), Wang (2002), Wang and Queck (2002), Wang and Varadan (2003) and Wang (2005) for stability investigations, which will be discussed in the following. Before, we note that the assumption of a vanishing axial displacement must not be confused with the inextensible theory, since the combination of the assumptions in Eqs. (42) and (56) imply that the deflection w is also zero.

In the following, we denote the model resting upon Eqs. (56)-(60) as inconsistent beam model, a notation that will be subsequently confirmed by three-dimensional finite element computations.

Based on Eq. (60) and when the beam stiffness is constant, the critical static buckling load according to the inconsistent model becomes

$$F_c = \frac{D_{11}\pi^2}{L^2} + N^a \tag{61}$$

The buckling load in Eq. (61) appears as a linear function of the actuating normal force N^a, which is physically incorrect, as it can be easily shown by the choice of

$$N^a = -\frac{D_{11}\pi^2}{L^2} \tag{62}$$

which yields a critical force of zero. Now assume, for the moment being zero, that the axial force F in Fig. 2 is zero, and that the actuating normal force is realized according to Eq. (62) by thermal expansion strains. According to Eqs. (61) and (62), the beam then would be in a state of thermal buckling. This is incorrect, as in reality, the beam would perform an axial contraction due to the thermal expansion strains in Eqs. (62), but it would not buckle, see e.g., Boley and Weiner (1960). As was shown in the latter reference, thermal buckling can only occur, if both ends of the beam would be axially fixed, and when $N^a > 0$.

For the dynamic case, in analogy to section 10, the boundary of the main region of instability obtained from the inconsistent model, Eq. (60), becomes

$$\frac{\theta}{2\Omega} = (1 + \Omega^*)\sqrt{1 \pm \mu_e(1 + \mu_e^*)} \tag{63}$$

with

$$\Omega^* = \sqrt{\frac{D_{11}\pi^2 - (F_0 - N^a)L^2}{D_{11}\pi^2 - F_0 L^2}} - 1 \tag{64}$$

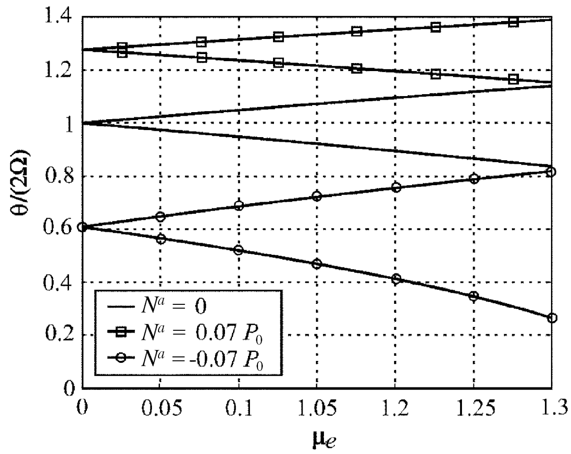

Fig. 4 Main region of instability for the inconsistent model

$$\mu^* = 1 - \frac{D_{11}\pi^2 - F_0 L^2}{D_{11}\pi^2 + (N^a - F_0)L^2} \tag{65}$$

The result of Eq. (63) is shown in Fig. 4 for three values of the actuating force, compare Chen, *et al.* (2002). As a result of inconsistent modeling, the main region of stability can be arbitrarily tuned by the actuating normal force, which appears to be physically incorrect from reasons similar to the one presented for the static critical load above.

12. Finite element verification

In order to give further evidence for the correctness of the above reasoning about the inconsistency of the model based on Eqs. (56)-(60), finite element computations have been performed using *ABAQUS*, Version 6.4. Three dimensional hexahedral elements have been used to discretize the beam shown in Fig. 5, which is composed of three aluminum layers with the dimensions

$$L = 1\text{m}, \qquad h_1 = 4 \cdot 10^{-3}\text{m}, \qquad Y_e = 7.22 \cdot 10^{10}\text{N/m}^2, \qquad \nu = 0.34$$

$$b = 4 \cdot 10^{-2}\text{m}, \qquad h_2 = 1 \cdot 10^{-3}\text{m}, \qquad \rho = 2.7 \cdot 10^3\text{kg/m}^3 \tag{66}$$

where L is the span, b the width, h_1 and h_2 the height of layers 1 and 2, Y_e is Young's modulus, ρ density and ν denotes Poisson's ratio. The thermal expansion coefficient for layers 1 and 2 has been chosen as

$$\alpha_1 = 0, \qquad \alpha_2 = 24 \cdot 10^{-5} \tag{67}$$

such that layer 1 can be interpreted as passive, and layers 2 as actuating layers. Instead of piezoelectric actuation, for convenience we use the analogy between thermal expansion strains and piezoelectric actuation, see Vinson (1993). With Eq. (39), thermal eigenstrain $E^0_{xx} = \alpha_2 T$ and temperature difference T, the actuating normal force reads

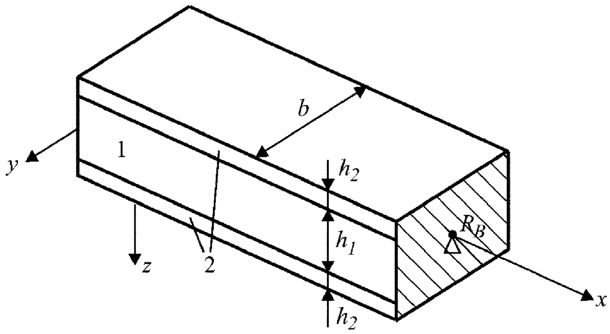

Fig. 5 Numerical example

$$N^a = 2bh_2Y_e\alpha_2T \tag{68}$$

The boundary conditions have been realized by attaching rigid plates to the front ends, which is sketched in Fig. 5 by a hatched area on the right end. According to Fig. 2 the boundary conditions of simple support are applied on the reference points R_A and R_B of the left and right plate. In order to obtain the critical loads as a function of the actuating force, the temperature T has been applied to layers 2, and a buckling analysis has been performed. The result is shown in Fig. 6, where the normalized forces

$$\bar{F}_c = F_c\frac{L^2}{D_{11}\pi^2}, \qquad \bar{N}^a = N^a\frac{L^2}{D_{11}\pi^2} \tag{69}$$

have been introduced. The continuous line shows the result of the extensible and the inextensible theories, Eq. (50), and the dashed line represents the erroneous result of the inconsistent modeling, Eq. (61). The finite element solution, which is marked by circles, shows an excellent coincidence with Eq. (50). The three-dimensional finite element computations thus clearly demonstrate that the buckling load of the simply supported beam under consideration can not be influenced by a thermal or piezoelectric actuating normal force, as is correctly predicted by both, the extensible and the inextensible beam

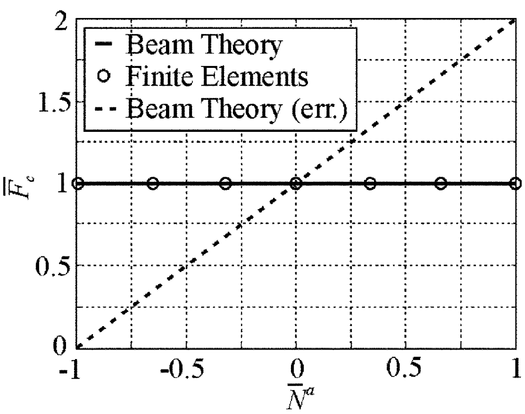

Fig. 6 Normalized static critical load

theory, and which is in contrast to what has been claimed in Chen, *et al*. (2002), Wang (2002), Wang and Queck (2002), Wang and Varadan (2003) and Wang (2005).

13. Conclusions

In the present paper, we have discussed the equations of motion of a simply supported laminated beam under the action of a compressive force and a piezoelectric actuation. The influence of piezoelectricity on the regions of stability has been studied within the framework of a simplification of the Bernoulli-Euler-v.Karman beam theory, which we have denoted as extensible beam theory above. Additionally, we have considered the model of an inextensible beam axis. An obviously inconsistent version of the equations of motion, which can be found in literature, has been discussed, and the inappropriateness of the latter theory has been demonstrated. It has been shown in the framework of both, the extensible and the inextensible theory, that, in contrast to what is suggested by the inconsistent formulation, buckling cannot be influenced by means of piezoelectric actuating normal forces. This result has been verified by means of three-dimensional finite element computations utilizing a well-known analogy between piezoelectric and thermal actuation.

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