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Nonlocal elasticity approach for free longitudinal vibration of circular truncated nanocones and method of determining the range of nonlocal small scale

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Abstract. The free longitudinal vibration of a circular truncated nanocone is investigated based on the nonlocal elasticity theory. Exact analytical formulations for tapered nanostructures are derived and the nonlinear differential governing equation of motion is developed. The nonlocal small scale effect unavailable in classical continuum theory is addressed to reveal the long-range interaction of atoms implicated in nonlocal constitutive relation. Unlike most previous studies applying the truncation method to the infinite higher-order differential equation, this paper aims to consider all higher-order terms to show the overall nonlocality. The explicit solution of nonlocal stress for longitudinal deformation is determined and it is an infinite series incorporating the classical stress derived in classical mechanics of materials and the infinite higher-order derivative of longitudinal displacement. Subsequently, the first three modes natural frequencies are calculated numerically and the significant effects of nonlocal small scale and vertex angle on natural frequencies are examined. The coupling phenomenon of natural frequency is observed and it is induced by the combined effects of nonlocal small scale and vertex angle. The critical value of nonlocal small scale is defined, and after that a new proposal for determining the range of nonlocal small scale is put forward since the principle of choosing the nonlocal small scale is still unclear at present. Additionally, two different types of nonlocal effects, namely the nonlocal stiffness weakening and strengthening, reversed phenomena existing in nanostructures are observed and verified. Hence the opposite nonlocal effects are resolved again clearly. The nano-engineers dealing with a circular truncated nanocone-based sensors and oscillators may benefit from the present work.

Keywords: nonlocal elasticity theory; nonlocal small scale; longitudinal vibration; circular truncated nanocone; nonlocal weakening; nonlocal strengthening

1. Introduction

In nano-science and nano-engineering, the free vibration of nano-material and nano-structure as a common phenomenon is easily seen. Many nano-engineering devices, for instance the nano-electronic-mechanical systems (NEMS) contain various kinds of nano-structural models such as nanowire, nanotube, nanobeam, nanoplate and nanocone, etcetera. It is indispensable to understand the dynamical behaviors of these nano-structures for designing and optimizing such models or components in nano-technology. Nowadays, the studies on the nanowire, nanotube, nanobeam and nanoplate are popular and there are a remarkably increasing number of research papers (Lim 2009, Lim 2010, Aifantis 2011, Li *et al*. 2011a, Li *et al*. 2011b, Thai and Vo 2012, Li 2013, Li *et al*. 2013, Anjomshoa, Jomehzadeh *et al*. 2014, Li 2014a, Sciarra and Barretta 2014, Li, Chen *et al*. 2015, Li *et al*. 2015, Lim, Zhang *et al*. 2015, Challamel *et al*. 2016, Dastjerdi *et al*. 2016, Li 2016, Liu *et al*. 2016, Tuna and Kirca 2016, Shen *et al*. 2017) for these topics. However, research and publication on circular truncated nanocone via an appropriate continuum approach are rather scarce and it is really necessary to reveal the longitudinal vibration of circular truncated nanocones in order to promote its application to nano-engineering devices.

It is well known that the mechanical properties at nano-scale are quite different from the corresponding counterpart at classical macro-scale due to the scale effect. It is essential to take the scale effect into consideration for a nano-scale structure because of the higher ratio of surface to volume. There have been several kinds of elasticity theories which are capable of describing the nano-structural behaviors, of which the strain gradient theory and nonlocal theory are the most popular approaches. The original form of strain gradient theory was put forward more than a hundred years ago and then it was developed systematically by Aifantis *et al*. (e.g., Xu *et al*. 2013, Xu *et al*. 2014, Aifantis 2016, Kateb *et al*. 2016). Unlike the strain gradient, the nonlocal theory is stress gradient. As a modified continuum mechanics theory, the nonlocal theory was firstly proposed by Eringen and Edelen (1972). Unlike the classical continuum constitutive relation which regards the stress at a point is related to the strain at that single point only, the nonlocal theory assumes that the stress at a point is a function of strains and history of deformations at all points in the domain. Actually, the nonlocal theory implicates some information about the forces between

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atoms/molecules, and the internal characteristic length scale is also introduced into the nonlocal constitutive relation as a material parameter to capture the scale effect and long-range interaction of nano-structures. Consequently, the nonlocal elasticity theory has been developed and applied in nano-mechanics extensively (e.g., Yang and Lim 2009, Lim 2010, Guo and Yang 2012, Anjomshoa *et al*. 2014, Li 2014a, Zenkour *et al*. 2014, Li *et al*. 2015, Lim *et al*. 2015, Zenkour *et al*. 2015, Challamel *et al*. 2016, Dastjerdi *et al*. 2016, Tuna and Kirca 2016, Li *et al*. 2017, Shen *et al*. 2017) due to its inherent suitable for modeling the long-range interaction via the internal characteristic length scale. At present, some different theoretical branches concerning the nonlocal nano-mechanics are formed. Particularly, one of the most popular branches is the nonlocal stress model, which was developed by Lim (2009, 2010). The nonlocal stress model starts from the nonlocal energy variational principle and considers the higher-order nonlocal stress effect of nano-structures and it concludes some new effective nonlocal phenomena which are consistent with experimental observations (Lim 2009, 2010). Since its advent, Lim's nonlocal stress model has been utilized to study various nano-mechanical issues widely and much related work has been published (e.g., Li 2011, Li *et al*. 2013, Li 2016). Subsequently, Lim, Zhang *et al*. (2015) further proposed a new higher-order nonlocal elasticity and strain gradient theory based on Lim's previous nonlocal stress model. The nonlocal strain gradient theory considers both the higher-order gradients of nonlocal stress and the nonlocality of strain gradient and it is regarded as an extension of the Eringen's original nonlocal theory (Eringen and Edelen 1972) and the Mindlin's strain gradient theory (Mindlin 1965). On the other hand, Aifantis (2011) analyzed the correlation between the gradient theory and nonlocal theory, in memory of the late A.C. Eringen (1921-2009), and the results were strongly motivated by Eringen's studies. Anjomshoa *et al*. (2014) developed a finite element approach according to the nonlocal elasticity theory to investigate the buckling property of nano-scaled multi-layered graphene nanoplate embedded in polymer matrix, and its correctness was verified by a comparison with molecular dynamics and analytical solutions. Sciarra and Barretta (2014) proposed a new variational formulation of nonlocal Euler-Bernoulli nanobeams and it was concluded that the rigidity of nanostructures may be flexible or stiffer, which is identical to the earlier work (Li 2014b, Li *et al*. 2015). Besides Li *et al*. (2015) addressed the free vibration of micro-scale and nano-scale beams using the nonlocal elasticity theory and eigenvalue method to reveal the size-dependent behaviors, and further the mechanisms of size dependence and physical meaning of small scale parameter were presented. Some comparisons with classical theory, molecular dynamic simulation and surface effects theory validate the work by Li *et al*. (2015). Unlike the nonlocal differential model, a rencent work by Tuna and Kirca (2016) determined the closed-form analytical solutions of Eringen's original integral model for static bending of Euler-Bernoulli and Timoshenko beams, respectively. In their work, the present Fredholm type integral governing constitutions were transformed to Volterra integral equations of the second kind, and the Laplace transformation was also applied to the corresponding equations. Dastjerdi, Lotfi *et al*. (2016) presented the effect of eccentric vacant defect on bending analysis of graphene sheets using the nonlocal elasticity theory and the Mindlinnanoplate theory. The subject was modeled as an anti-symmetric problem and a new semi-analytical polynomial method was employed to solve the problem. The derived results were compared with the results of ABAQUS finite element tool to prove the validity of their work.

Although there is a great deal of work concerning with the nanowire, nanotube, nanobeam and nanoplate (e.g., Yang and Lim 2009, Lim 2010, Aifantis, 2011, Anjomshoa *et al*. 2014, Sciarra and Barretta 2014, Li *et al*. 2015, Lim *et al*. 2015, Challamel *et al*. 2016, Dastjerdi *et al*. 2016, Tuna and Kirca 2016, Shen *et al*. 2017), there has been less work, to the authors' best knowledge, considers the free longitudinal vibration of circular truncated nanocone based on the nonlocal elasticity theory. The carbon nanocone is a transition nanostructure from a graphene sheet to a fullerene, which was firstly discovered by Ge and Sattler (1994). The exciting findings were reported about the excellent structural and electrical properties of carbon nanocone (Ge and Sattler 1994, Krishnan *et al*. 1997). Regarding the mechanical properties, for example, the nanocone has been applied extensively in nano-devices and nano-composites as a basic element due to its high mechanical strength. Krishnan *et al*. (1997) observed experimentally five types of nanocones distinguished by five cone angles of 19.2°, 38.9°, 60°, 86.6°, and 123.6°, respectively. Guo and Yang (2012) investigated the axial vibration of carbon nanocones via a non-uniform rod model using a modified Wentzel–Brillouin–Kramers method. However, more work was concerned with the free transverse vibration of nanocones in recent years. For example, Firouz-Abadi *et al*. (2011) presented the free transverse vibration of nanocones based on the nonlocal elasticity theory for the first time. The effects of small-scale and geometrical parameters of nanocones on the natural frequencies were investigated. Subsequently, Fotouhi *et al*. (2013) further extended the topic to the free transverse vibration of nanocones embedded in an elastic medium by employing the nonlocal shell model. Hu *et al*. (2012) studied the transverse dynamics of single-walled carbon nanocones via Timoshenko beam model and molecular dynamics simulations. They examined effects of apex angles, top radii and lengths on fundamental frequencies of transverse vibrations of cantilevered single-walled carbon nanocones. Hence it is necessary to investigate the nonlocal scale property and vertex angle effect of nanocone for its potential applications.

In this paper, the nonlocal elasticity approach involving the small scale effect for longitudinal vibration of a circular truncated nanocone is constructed. The exact expression of nonlocal longitudinal stress is determined and it can be written as an infinite series. The cross section of the circular truncated nanocone is a function of axial coordinate, which results in a nonlinear partial differential governing equation of motion. The theoretical formulations are derived and they are solved by Galerkin method. The aim of this study is to reveal the nonlocal small scale and vertex angle effect in free longitudinal vibration of circular truncated nanocone because they should be taken into account in modeling and designing such nanostructures. In the numerical results, some new observations are shown and some unclear issues such as how to determine the range of nonlocal small scale are solved. The work may provide some useful reference to current nano-engineering.

2. Problem model and governing equation

Considering the continuum model for a circular truncated nanocone shown in Fig. 1, the force equilibrium equation for an element shown in Fig. 2 can be obtained as

$$(F+dF)-F=\rho A(x)dx\frac{\partial^2 u(x,t)}{\partial t^2} \tag{1}$$

where dx is length of the circular truncated nanocone element, $A(x)$ is the area of the cross section with an axial coordinate x apart from the left end of circular truncated nanocone, ρ is the mass density of the nanocone, $u(x,t)$ is the longitudinal displacement of the nanocone, t is time, $F+dF$ and F are the internal axial forces at the left and right sides of the element, respectively. Note that the force analysis methods employed in Fig. 1 and Eq. (1) are the same as the classical continuum mechanics. However, the nonlocal elasticity idea will be introduced and then drawn into the continuum model.

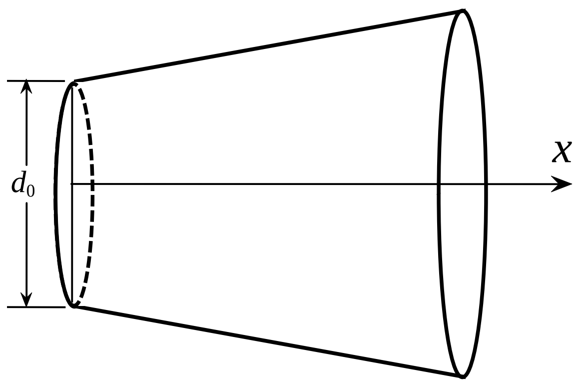

Fig. 1 Sketch of a circular truncated nanocone

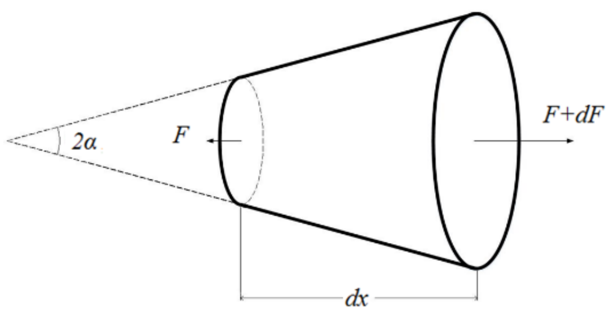

Fig. 2 Force equilibrium for an element of nanocone

In the nonlocal elasticity theory, the stress at a point $A$is expressed as a function of strains at not only A but also all points and the history of strains in the continuum domain (Eringen and Edelen 1972). However, the contributions of other points are weaker with increasing the distances between other points and the reference point A. The nonlocal differential stress-strain relation was derived from its original integral form equivalently by Eringen (1983) as

$$\sigma-(e_0a)^2\frac{d^2\sigma}{dx^2}=E\varepsilon \tag{2}$$

where σ is the nonlocal stress, ε is the strain in nonlocal field and it is the same as the classical or local counterpart, E is the elastic modulus of the material, e_0 is a material constant in nonlocal elasticity theory, and a is an internal characteristic length scale (e.g., lattice parameter, particle diameter, etc.). Note that e_0 and a represent the intrinsic scale parameters revealing the nonlocal small scale effect.

In fact, the classical continuum theory is a particular case of nonlocal theory, and it is recovered from the nonlocal theory by taking e_0=0.

Because $F=\sigma A$, we have

$$\frac{d^2F}{dx^2}=\frac{d^2(\sigma A)}{dx^2}=\frac{d}{dx}\left(\frac{d(\sigma A)}{dx}\right)=A\frac{d^2\sigma}{dx^2}+2\frac{d\sigma}{dx}\frac{dA}{dx}+\sigma\frac{d^2A}{dx^2} \tag{3a}$$

Multiplying the cross sectional area Aat both sides of Eq. (2) yields

$$\sigma A-(e_0a)^2A\frac{d^2\sigma}{dx^2}=EA\varepsilon \tag{3b}$$

As a result, the nonlocal force equilibrium formula for longitudinal vibration of circular truncated nanocone arrives at

$$F-(e_0a)^2\left(\frac{d^2F}{dx^2}-\sigma\frac{d^2A}{dx^2}-2\frac{d\sigma}{dx}\frac{dA}{dx}\right)=EA\frac{\partial u}{\partial x} \tag{3c}$$

where $\varepsilon=\frac{\partial u}{\partial x}$ is adopted and it is consistent with theclassical strain-displacement relation. Note that the displacement u is a function with respect to x and t, hence the partial differential operator is used in Eq. (3(c)).

A first partial derivative of Eq. (3(c)) with respect to $x$is employed and one obtains

$$\frac{dF}{dx}-(e_0a)^2\left(\frac{d^3F}{dx^3}-\sigma\frac{d^3A}{dx^3}-3\frac{d\sigma}{dx}\frac{d^2A}{dx^2}-2\frac{d^2\sigma}{dx^2}\frac{dA}{dx}\right)=EA\frac{\partial^2u}{\partial x^2}+E\frac{dA}{dx}\frac{\partial u}{\partial x} \tag{4}$$

From Eq. (1), one obtains that $\frac{dF}{dx}=\rho A\frac{\partial^2 u}{\partial t^2}$. Combining the expression with Eq. (4), we have

$$\begin{aligned}&\rho A\frac{\partial^2u}{\partial t^2}-(e_0a)^2\left(\rho A\frac{\partial^4u}{\partial x^2\partial t^2}+\rho\frac{d^2A}{dx^2}\frac{\partial^2u}{\partial t^2}+2\rho\frac{dA}{dx}\frac{\partial^3u}{\partial x\partial t^2}-\sigma\frac{d^3A}{dx^3}\right.\\&\left.-3\frac{d\sigma}{dx}\frac{d^2A}{dx^2}-2\frac{d^2\sigma}{dx^2}\frac{dA}{dx}\right)=EA\frac{\partial^2u}{\partial x^2}+E\frac{dA}{dx}\frac{\partial u}{\partial x}\end{aligned} \tag{5}$$

When the cross sectional area A is reduced to a constant in Eq. (5), the linear partial differential equation for longitudinal vibration of a nanorod is then recovered (e.g., see Eq. (32) in Li *et al*. 2017, Eq. (16) in Li *et al*. 2017). On the other hand, the conception of nonlocal stress shouldbe defined here. As aforementioned, the classical model and classical stress can be recovered from the nonlocal theory in case of e_0=0. Consequently, it is possible to have classical and nonlocal stresses in one nonlocal model. In other words, the nonlocal stress is not only the non-classical stress but also including the classical stress. Using an iterative method, the series expression of nonlocal stress can be determined from Eq. (2) directly

$$\sigma = E\sum_{k=0}^{\infty}\left(e_0 a\right)^{2k}\frac{\partial^{2k+1}u}{\partial x^{2k+1}} = E\frac{\partial u}{\partial x} + E\sum_{k=1}^{\infty}\left(e_0 a\right)^{2k}\frac{\partial^{2k+1}u}{\partial x^{2k+1}} \quad (6)$$

It is found that the nonlocal stress can be treated as the combination of classical stress derived in mechanics of materials and an infinite series of higher-order derivative relating to longitudinal displacement with respect to axial coordinate, or the non-classical stress. The result shown in Eq. (6) verifies the definition of nonlocal stress. According to the constitution of nonlocal theory, the higher-order derivative of Eq. (6) represents the long-range force between atoms/molecules. We assume the vertex angle of circular truncated nanocone is 2α, and diameter of the left-most end section (x=0) is d_0. Accordingly, the area of arbitrary cross section can be described as

$$A(x) = \frac{\pi}{4}\left(d_0 + 2x\tan\alpha\right)^2 \quad (7)$$

Substituting Eqs. (6) and (7) into Eq. (5), one arrives at

$$\begin{aligned}&\left(d_0+2x\tan\alpha\right)^2\left(\rho\frac{\partial^2 u}{\partial t^2}-E\frac{\partial^2 u}{\partial x^2}\right)-4E\tan\alpha\left(d_0+2x\tan\alpha\right)\frac{\partial u}{\partial x}\\&=\left(e_0a\right)^2\left\{\rho\left(d_0+2x\tan\alpha\right)^2\frac{\partial^4 u}{\partial x^2\partial t^2}+8\rho\tan^2\alpha\frac{\partial^2 u}{\partial t^2}-24E\tan^2\alpha\sum_{k=0}^{\infty}\left(e_0a\right)^{2k}\frac{\partial^{2k+2}u}{\partial x^{2k+2}}\right.\\&\left.+8\left(d_0+2x\tan\alpha\right)\tan\alpha\left[\rho\frac{\partial^3 u}{\partial x\partial t^2}-E\sum_{k=0}^{\infty}\left(e_0a\right)^{2k}\frac{\partial^{2k+3}u}{\partial x^{2k+3}}\right]\right\}\end{aligned} \quad (8)$$

Eq. (8) is the nonlinear partial differential governing equation for longitudinal vibration of circular truncated nanocones considering the nonlocal elasticity field effect, and its non-dimensional normalized form is

$$\begin{aligned}&\left(1+2\bar{x}\tan\alpha\right)^2\left(\frac{\partial^2\bar{u}}{\partial\bar{t}^2}-\frac{\partial^2\bar{u}}{\partial\bar{x}^2}\right)-4\tan\alpha\left(1+2\bar{x}\tan\alpha\right)\frac{\partial\bar{u}}{\partial\bar{x}}\\&=\tau^2\left[\left(1+2\bar{x}\tan\alpha\right)^2\frac{\partial^4\bar{u}}{\partial\bar{x}^2\partial\bar{t}^2}+8\tan^2\alpha\frac{\partial^2\bar{u}}{\partial\bar{t}^2}-24\tan^2\alpha\sum_{k=0}^{\infty}\tau^{2k}\frac{\partial^{2k+2}\bar{u}}{\partial\bar{x}^{2k+2}}\right.\\&\left.+8\left(1+2\bar{x}\tan\alpha\right)\tan\alpha\left(\frac{\partial^3\bar{u}}{\partial\bar{x}\partial\bar{t}^2}-\sum_{k=0}^{\infty}\tau^{2k}\frac{\partial^{2k+3}\bar{u}}{\partial\bar{x}^{2k+3}}\right)\right]\end{aligned} \quad (9)$$

where the dimensionless variables are defined as

$$\bar{x}=\frac{x}{d_0},\ \bar{t}=t\sqrt{\frac{E}{\rho d_0^2}},\ \bar{u}=\frac{u}{d_0},\ \tau=\frac{e_0a}{d_0} \quad (10)$$

in which τ is a non-dimensional nonlocal small scale parameter. It is obvious that τ=0 corresponds to e_0=0 and the classical continuum theory is recovered with τ=0 in non-dimensional physical analyses.

It is noted that the derived nonlinear partial differential equation of motion includes the entire nonlocal effects since the full expression of Eq. (6) is adopted, where Eq. (6) is from Eq. (2) and the latter is the original nonlocal differential constitutive relation. Therefore, the overall nonlocality is taken into account in the mathematical model constructed in this paper. However, the common truncation method was utilized in most previous studies (Lim 2009, Lim 2010, Anjomshoa *et al*. 2014, Li 2014a) where only a few front terms in infinite series were retained but the other terms were omitted without certification. Suppose the longitudinal displacement is $\bar{u}\left(\bar{x},\bar{t}\right)=\bar{U}\left(\bar{x}\right)e^{i\bar{\omega}_n\bar{t}}$, where $\bar{\omega}_n$ is the normalized vibration natural frequency and the relation between normalized and physical frequency is $\bar{\omega}_n=\omega_n\sqrt{\rho d_0^2/E}$. Substitution of the longitudinal displacement expression into Eq. (9), one can obtain the following ordinary differential governing equation in spatial domain as

$$\begin{aligned}&\left(1+2\bar{x}\tan\alpha\right)^2\left(\frac{d^2\bar{U}}{d\bar{x}^2}+\bar{\omega}_n^2\bar{U}\right)+4\tan\alpha\left(1+2\bar{x}\tan\alpha\right)\frac{d\bar{U}}{d\bar{x}}\\&=\tau^2\left[\bar{\omega}_n^2\left(1+2\bar{x}\tan\alpha\right)^2\frac{d^2\bar{U}}{d\bar{x}^2}+8\bar{\omega}_n^2\bar{U}\tan^2\alpha+24\tan^2\alpha\sum_{k=0}^{\infty}\tau^{2k}\frac{d^{2k+2}\bar{U}}{d\bar{x}^{2k+2}}\right.\\&\left.+8\left(1+2\bar{x}\tan\alpha\right)\tan\alpha\left(\bar{\omega}_n^2\frac{d\bar{U}}{d\bar{x}}+\sum_{k=0}^{\infty}\tau^{2k}\frac{d^{2k+3}\bar{U}}{d\bar{x}^{2k+3}}\right)\right]\end{aligned} \quad (11)$$

In order to determine the solution, we further assume $\bar{U}=Ce^{\lambda\bar{x}}$ where C is a constant and λ is the dispersion parameter (Li *et al*. 2011a, Li 2013). Substituting the expression into Eq. (11), we derive a characteristic algebra equation as

$$\begin{aligned}&\left(1+2\bar{x}\tan\alpha\right)^2\left(\lambda^2+\bar{\omega}_n^2\right)+4\lambda\tan\alpha\left(1+2\bar{x}\tan\alpha\right)\\&=\tau^2\left[\bar{\omega}_n^2\lambda^2\left(1+2\bar{x}\tan\alpha\right)^2+8\left(\bar{\omega}_n^2+\frac{3\lambda^2}{1-\tau^2\lambda^2}\right)\tan^2\alpha\right.\\&\left.+8\lambda\left(1+2\bar{x}\tan\alpha\right)\tan\alpha\left(\bar{\omega}_n^2+\frac{\lambda^2}{1-\tau^2\lambda^2}\right)\right]\end{aligned} \quad (12)$$

3. Results and discussion

In this section, taking a circular truncated nanocone with fixed-free boundary constraint as an example, we suppose the circular truncated nanocone is fixed at left end and free at right end. For this purpose, the longitudinal displacement at the fixed end should be zero, and the external axial force at the free end should also be zero. Hence the boundary conditions are written as

$$\begin{aligned}&\bar{U}\left(0\right)=0\\&\frac{d\bar{U}\left(1\right)}{d\bar{x}}=0\end{aligned} \quad (13)$$

Combining Eqs. (11)-(13), one can determine the natural frequencies for free longitudinal vibration of fixed-free circular truncated nanocones numerically. We can also apply the Galerkin method to Eq. (11) and the expression of non-dimensional longitudinal displacement in spatial domain can be given by

$$\bar{U}=\sum_{m=1}^{M}\left[q_m\phi_m(\bar{x})\right] \tag{14}$$

where q_m is the undetermined coefficient. According to the boundary conditions expressed in Eq. (13), we can use the trial solution which satisfies the boundary conditions as

$$\phi_m(\bar{x})=\sin\frac{(2m-1)\pi\bar{x}}{2} \tag{15}$$

Substituting Eq. (14) into the ordinary differential governing equation and multiplying the expression of ϕ_n at both ends, then integrating the results from 0 to 1 yields

$$\begin{aligned}&\bar{\omega}_n^2\left\{\tau^2\sum_{m=1}^{M}\int_0^1\left[q_m\phi_n\left(1+2\bar{x}\tan\alpha\right)^2\frac{d^2\phi_m}{d\bar{x}^2}\right]d\bar{x}+8\tau^2\tan^2\alpha\sum_{m=1}^{M}\int_0^1 q_m\phi_n\phi_m d\bar{x}\right.\\&\left.+8\tau^2\tan\alpha\sum_{m=1}^{M}\int_0^1\left[q_m\phi_n\left(1+2\bar{x}\tan\alpha\right)\frac{d\phi_m}{d\bar{x}}\right]d\bar{x}-\sum_{m=1}^{M}\int_0^1\left[q_m\phi_n\left(1+2\bar{x}\tan\alpha\right)^2\phi_m\right]d\bar{x}\right\}\\&+24\tau^2\tan^2\alpha\sum_{m=1}^{M}\int_0^1\left(q_m\phi_n\sum_{k=0}^{\infty}\tau^{2k}\frac{d^{2k+2}\phi_m}{d\bar{x}^{2k+2}}\right)d\bar{x}-\sum_{m=1}^{M}\int_0^1\left[q_m\phi_n\left(1+2\bar{x}\tan\alpha\right)^2\frac{d^2\phi_m}{d\bar{x}^2}\right]d\bar{x}\\&+8\tau^2\tan\alpha\sum_{m=1}^{M}\int_0^1\left[q_m\phi_n\left(1+2\bar{x}\tan\alpha\right)\sum_{k=0}^{\infty}\tau^{2k}\frac{d^{2k+3}\phi_m}{d\bar{x}^{2k+3}}\right]d\bar{x}\\&-4\tan\alpha\sum_{m=1}^{M}\int_0^1\left[q_m\phi_n\left(1+2\bar{x}\tan\alpha\right)\frac{d\phi_m}{d\bar{x}}\right]d\bar{x}=0\end{aligned} \tag{16}$$

where n=1,2,3…M.

For each value of n, we can determine an equation including q_1, q_2, …,q_M. The number of such equations is M. Hence the equations can be expressed as

$$\bar{\omega}_n^2\left[\mathbf{M}\right]\left\{q\right\}+\left[\mathbf{K}\right]\left\{q\right\}=\mathbf{0} \tag{17}$$

where $\{q\}=\{q_1, q_2, q_3, \ldots, q_M\}^{\mathrm{T}}$. In the following numerical results we take M=5. According to Krishnan *et al*. (1997), vertex angle of nanocones 2α includes five values, i.e.. 19.2°, 38.9°, 60°, 86.6°, and 123.6°. Therefore, effects of the non-dimensional nonlocal small scale on the first three modes natural frequencies for longitudinal vibration of fixed-free circular truncated nanocones are shown in Figs. 3-7 for α=9.6°, 19.45°, 30°, 43.3°, 61.8°, respectively.

The coupling phenomenon of non-dimensional natural frequencies is observed subjected to the combined effects of nonlocal small scale and vertex angle from Figs. 3-7. It is seen that the vertex angle plays a significant role in free longitudinal vibration behaviors of circular truncated nanocones. The first three modes frequencies consist of distinguishing values for different vertex angles. The second and third modes frequencies are coupled for α=9.6°, 19.45°, 30° and 43.3° with a relative larger non-dimensional nonlocal small scale (e.g., $\tau\geq$0.25 for α=9.6°), while the first, second and third modes frequencies are coupled together with increasing the vertex angle continuously, e.g., for α=61.8°. Hence the coupling phenomenon of natural frequencies is common relatively for larger vertex angle nanocones, which provides a reference for understanding and managing the mechanical properties of circular truncated nanocones. In fact, the radius effect is involved in the remarkable effect of the vertex angle and it cannot be neglected in studying the nanocone elements based nanoscale equipments. On the other hand, the nonlocal small scale effect based long-range force of circular truncated nanocones also plays an important role and it has a remarkable influence on natural frequencies.

Firstly, the first three modes natural frequencies change significantly and even get coupled with the increasing of non-dimensional nonlocal small scale. The values of nonlocal small scale at which the frequencies get coupled can be determined from Figs. 3-7. For examples, the 2nd and 3rd natural frequencies get coupled after the threshold of nonlocal small scale τ=0.25 for α=9.6°; the 2nd and 3rd frequencies get coupled after the threshold of nonlocal small scale τ=0.23 for α=19.45°; the 2nd and 3rd frequencies get coupled after the threshold τ=0.22 for α=30°; the 2nd and 3rd frequencies get coupled after the threshold τ=0.217 for α=43.3°; the 2nd and 3rd frequencies get coupled after τ=0.21 for α=61.8°. It is seen that the nonlocal small scale threshold becomes smaller with increasing the vertex angles, and moreover, not only the 2nd and 3rd but also the 1st and 2nd natural frequencies get coupled for α=61.8°, which means the natural frequencies get coupled more easily for circular truncated nanocone with larger vertex angle. Additionally, the 1st and 2nd frequencies get coupled twice for α=61.8°, and the nonlocal small scale thresholds are τ=0.17 and τ=0.2, respectively.

Secondly, the 1st natural frequency becomes zero under a certain value of nonlocal small scale. For instance, the nonlocal small scale τ=0.27, 0.25 and 0.24 change the 1st frequency to zero for circular truncated nanocones with α=30°, 43.3° and 61.8°, respectively. Such values can be defined as critical values of nonlocal small scale in free longitudinal vibration behaviors of circular truncated nanocones. That means, what is the maximum nonlocal small scale we can select in a specific study, since we know the minimum nonlocal small scale is zero (τ=0 as the minimum nonlocal small scale is always correct because it corresponds to the classical elasticity while τ>0 represents the existence of nonlocal small scale effect). Actually, it is still unclear that how to choose the range of non-dimensional nonlocal small scale when applying the nonlocal theory to nano-mechanics. It is surprised that different range adopted in different published papers without certification (e.g., τ=0~0.06 in Wang *et al*. (2008), τ=0~0.02 in Lim (2009, 2010), τ=0~0.8 in Guo and Yang (2012), τ=0~0.15 in Li (2013), τ=0~0.1 in Yu and Lim (2014), τ=0~0.2 in Lim *et al*. (2015), τ=0~0.2 in Li *et al*. (2017)). It is no doubt that the range of nonlocal small scale cannot be chosen arbitrarily. Through this study, we can propose a method for determining the range of nonlocal small scale. The procedure of the method is to measure the critical value of nonlocal small scale first and then determine the range of nonlocal small scale. Accordingly, the proper range of nonlocal small scale is from zero to its critical value. As far as the research object concerned in this paper, the reasonable intervals of nonlocal small scale are [0, 0.27], [0, 0.25] and [0, 0.24] for α=30°, 43.3° and 61.8°, respectively. Of course, the range of nonlocal small scale may change for other research topics or objects but it can be determined using the same method.

Thirdly, the non-monotonicity of variations of natural frequencies with respect to nonlocal small scale is observed. The natural frequencies may decrease with increasing the nonlocal small scale (e.g., from τ=0 to τ=0.22 for 2nd natural frequencies with α=9.6°), while may increase with increasing the nonlocal small scale (e.g., from τ=0.22 to τ=0.3 for 2nd natural frequencies with α=9.6°). Such observation is consistent with some previous studies (Li 2014b, Li *et al*. 2015, Lim *et al*. 2015). The nonlocal weakening and nonlocal strengthening phenomena are commonly seen in nonlocal elastic models for different nano-materials and it has been a contentious subject because the weakening and strengthening phenomena are opposite (Li *et al*. 2015, Lim *et al*. 2015, Shen and Li 2017). The nonlocal weakening (e.g., Wang *et al*. 2008, Firouz-Abadi *et al*. 2011, Li *et al*. 2017) means the stiffness of nanostructures becomes smaller under the nonlocal small scale effect than that without the nonlocal small scale effect, namely, the nanostructural rigidity based on nonlocal elasticity gets smaller than the nanostructural rigidity based on classical elasticity. Consequently, the deformation of nanostructures becomes larger and the vibration frequency becomes smaller with an increase of the nonlocal small scale τ. The nonlocal strengthening (e.g. Lim 2009, 2010, Li 2014a, Yu and Lim 2014, Shen and Li 2017) means the nanostructural rigidity becomes larger under the nonlocal small scale effect than that based on classical elasticity. Consequently, the deformation becomes smaller and the frequency becomes larger with increasing the nonlocal small scale τ. The dispute now is resolved and both the nonlocal weakening and strengthening have been proved to be reasonable and they are related to different types of surface effects of nano-materials or structures. (Li 2014b, Shen and Li 2017). In fact, in 1983 Eringen pointed out that "nonlocal theory accounts for surface physics, an important assert not included in classical theories" based on the derivation of original nonlocal constitutive relations.

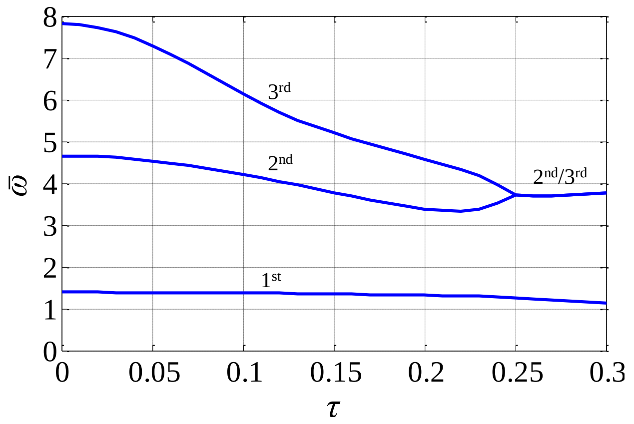

Fig. 3 Variation of the first three modes non-dimensional natural frequencies with respect to non-dimensional nonlocal small scale for α=9.6°

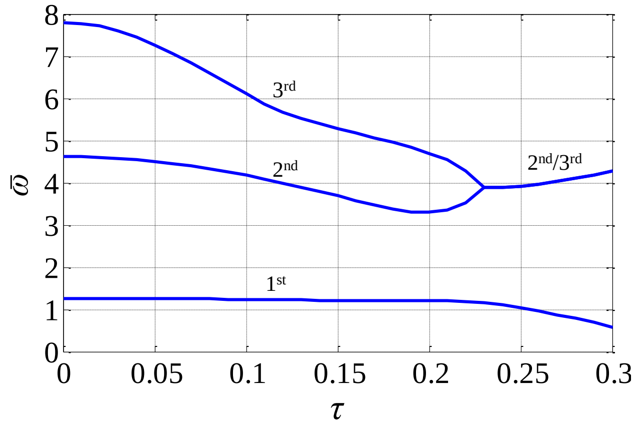

Fig. 4 Variation of the first three modes non-dimensional natural frequencies with respect to non-dimensional nonlocal small scale for α=19.45°

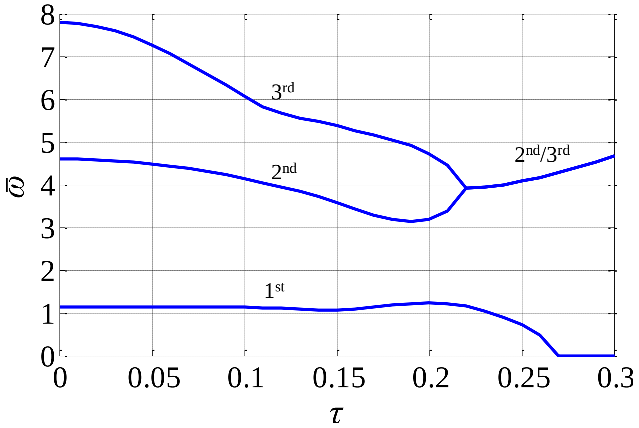

Fig. 5 Variation of the first three modes non-dimensional natural frequencies with respect to non-dimensional nonlocal small scale for α=30°

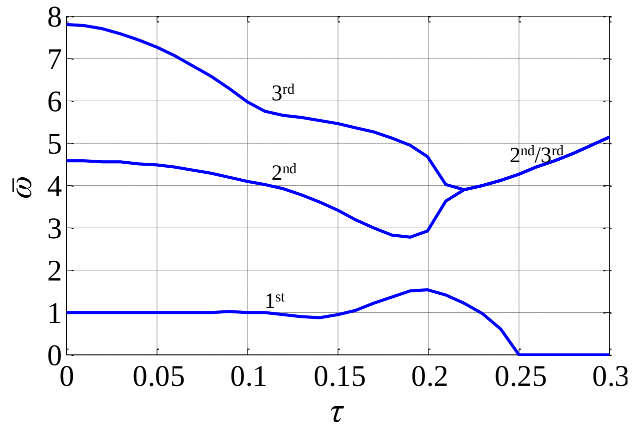

Fig. 6 Variation of the first three modes non-dimensional natural frequencies with respect to non-dimensional nonlocal small scale for α=43.3°

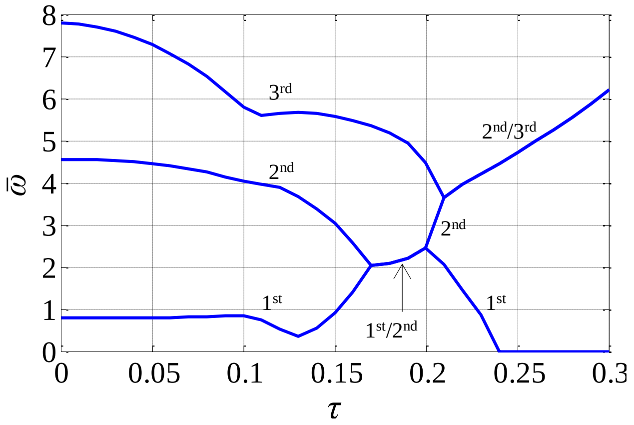

Fig. 7 Variation of the first three modes non-dimensional natural frequencies with respect to non-dimensional nonlocal small scale for α=61.8°

This is the echo of the judging studies for different nonlocal effects and it provides the foundation to understand why the nonlocal weakening and strengthening effects associate with different types of surface effects, namely the attractive or repulsive, respectively (Li 2014b). The present study indicates and confirms the existence of different nonlocal effects in nanostructures. The correspondence relation between free longitudinal vibration frequencies and the nonlocal small scale as well as vertex angle is helpful for designing and optimizing the NEMS or other nano-devices where the circular truncated nanocone acts as a basic component.

4. Conclusions

This work is concerned with the free longitudinal vibration of a circular truncated nanocone used frequently in NEMS. The nonlocal elasticity theory is employed and the simplified mathematical model is constructed using the nonlinear differential governing equation. The nonlocal longitudinal stress can be expressed as an infinite series containing infinite higher-order derivatives. The first, second and third modes natural frequencies are calculated using the Galerkin method. It is concluded that both nonlocal small scale and vertex angle play remarkable roles.

The coupling of natural frequency is found and it is more easily seen in circular truncated nanocones with larger vertex angle. The threshold of nonlocal small scale at which the frequencies get coupled is proposed, and the threshold becomes smaller with increasing the vertex angles. Subsequently, we define the critical value of nonlocal small scale that makes the first mode frequency vanish. Based on this, we solve the unclear issue about how to determine the range of nonlocal small scale. On the other hand, from the numerical results it is indicated the larger nonlocal small scale may result in either higher natural frequency or lower one and it depends on different surface properties of nanostructures. Consequently, two different types of nonlocal effects including nonlocal stiffness weakening and strengthening are verified. Accordingly, the correlation between the nonlocal theory and surface physics declared by Eringen is confirmed. The present study is useful for understanding and controlling the longitudinal vibration behaviors of circular truncated nanocones since they are common elements in the current nano-engineering. For example, we should pay particular attention to the coupling of natural frequencies for larger-vertex-angled nanocone-based nanostructures.

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