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# Variability of thermal properties for a thermoelastic loaded nanobeam excited by harmonically varying heat

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**Abstract.** This work produces a new model of nonlocal thermoelastic nanobeams of temperature-dependent physical properties. A nanobeam is excited by harmonically varying heat and subjected to an exponential decaying time varying load. The analytical solution is obtained by means of Laplace transform method in time domain. Inversions of transformed solutions have been preceded by using calculus of residues. Effects of nonlocal parameter, variability thermal conductivity, varying load and angular frequency of thermal vibration on studied fields of nanobeam are investigated and discussed.

**Keywords:** nanobeam; varying load; harmonically heating; variability thermal conductivity; nonlocal thermoelasticity

## 1. Introduction

In most thermoelastic problems, physical material properties of structures are considered to be constants. It is obvious that material properties of different structures are sensitive to presence of thermal filed. Most material properties like Young's modulus, thermal conductivity, etc. are no longer remaining constants at high temperature. In addition, temperature-dependent material properties affect thermo-mechanical behavior of structures (Noda 1991). Therefore, temperature dependency of material properties should be taken under consideration to obtain more accurate and efficient solutions of thermoelasticity problems.

Uma *et al.* (2001) investigated effect of grain structure on thermal conductivities of undoped polysilicon layers at high temperatures and electrical-resistance thermometry. They found that the layer thermal conductivities depend strongly on the details of deposition process through grain size distribution. These results are considered useful for thermal design of microelectronic and micro-electro-mechanical devices that use polycrystalline silicon layers. Along similar lines, studies on the temperature and pressure dependence of thermal conductivity and thermal capacity are also relevant to modern technology. Also, the study may find applications in design and improvement of resonator devices under loading environment.

The fields of micro-electro-mechanical systems (MEMS) have become quickly and gone into many resistances and correspondence applications (Younis 2011). It is essential for MEMS designers to estimate mechanical properties of flexible micro- or nano-components keeping in mind end goal for predicting amount of deflection due to applied mechanical or thermal loads and the other way around to forestall cracking-fracture, improve performance and to increase lifetime of MEMS gadgets (Allameh 2003).

The definition of stress is considered as a basic difference between classical and nonlocal elasticity theories. In classical elasticity theory the stress at a point is a function of strain at the same point. Whereas in nonlocal elasticity theory of Eringen (2002), the stress at any point is a function of strains at all points. Therefore, nonlocal elasticity theory (Eringen 1983) has many applications in various fields of physics and engineering (Zenkour and Abouelregal 2014, Ebrahimi *et al.* 2015a,b, Ebrahimi and Salari 2015a,c).

Lord and Shulman (1967) (LS) presented the generalized theory of thermoelasticity with one relaxation time for isotropic materials. The heat equation is of hyperbolic type and consequently ignores the paradox of infinite velocity of propagation inherent in both coupled (CTE) and uncoupled thermoelasticity theories. Tzou (1995a,b, 1996) presented a connection between heat flux and temperature gradient in his dual-phase-lags (DPLs) theory.

Yanping and Yilong (2010) considered static deflections of micro-cantilever beams subjected to transverse loading using neural network method. Thai (2012) investigated the bending, buckling, and vibration responses of nanobeams using Eringen's nonlocal differential constitutive relations based on nonlocal shear deformation beam theory. Sharma and Kaur (2015) investigated dynamic response of thermoelastic microbeam resonators under time-varying transverse loads using thermoelasticity theory. Ebrahimi and Salari (2015b) discussed thermal effect on vibration analysis of functionally graded size-dependent nanobeams under various types of thermal loads. Radebe and Adali

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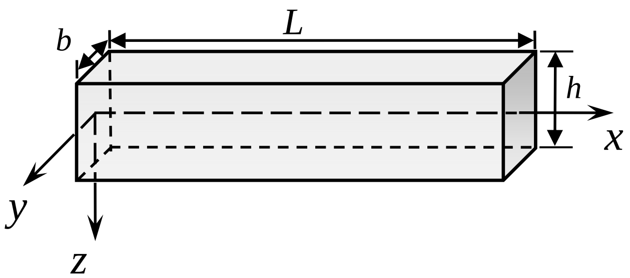



Fig. 1 Schematic diagram for a nanobeam

(2015) studied deflection of a nanobeam subject to load and material uncertainties by convex modeling.

Ebrahimi and Shafiei (2016) investigated the size dependent vibration of a rotating functionally graded nanobeam based on Eringen's nonlocal theory. Ebrahimi and Shaghaghi (2016) investigated both the small scale and thermal effects on vibration of preloaded nanobeams with non-ideal boundary conditions. Akbas (2016) used the modified couple stress theory to present forced vibration of a simply-supported viscoelastic nanobeam excited by a transverse triangular force impulse modulated by a harmonic motion. Barretta *et al*. (2016) discussed the application of an enhanced version of the nanotechnology model of Eringen differential. Ghafarian and Ariaei (2016) investigated the analysis of free vibration of a rotating multiple nanobeams using the nonlocal theory proposed by Eringen. Recently, Arefi and Zenkour (2017a,b,c) discussed the thermoelastic vibration and bending analyses of piezomagnetic three-layer nanobeams and magneto-electro nanobeams rest on visco-Pasternak foundation.

It has been observed that many investigators studied transverse vibrations, thermoelastic damping and frequency shift in nanobeams due to mechanical shocks, laser heating, electrostatic loads and moving loads. In this work, an attempt has been made to study dynamic response of homogeneous, isotropic, thermoelastic nanobeam resonators under time-varying mechanical transverse loads excited by harmonically varying heat. The nonlocal thermoelastic model based on dual-phase-lags (Zenkour *et al.* 2013, Abbas and Zenkour 2014, Abouelregal and Zenkour 2014a, Zenkour and Abouelregal 2015, Ezzat and El-Bary 2016, Zenkour 2016) is produced to solve a generalized thermoelastic problem of nanobeam with variable thermal conductivity. The boundary first end of the nanobeam is subjected to both thermal and mechanical loads. The effects due to nonlocal, the point load and angular frequency of thermal vibration parameters will be studied. Numerical results are presented to discuss small-scale effect on nanobeam resonator.

## 2. Nonlocal basic equations

General nonlocal constitutive relations of Eringen (1972) for a nanobeam may be written as

$$[1-(ae_0)^2\nabla^2]\tau_{kl}=\sigma_{kl} \tag{1}$$

The stress-strain relations are represented as

$$\sigma_{ij}=2\mu e_{ij}+\lambda e\delta_{ij}-\gamma(T-T_0)\delta_{ij} \tag{2}$$

The heat conduction equation which includes dual phase-lag effects takes the form (Tzou 1995a, b, Ozisik and Tzou 1994)

$$\begin{gathered}\left(1+\tau_\theta\frac{\partial}{\partial t}\right)\left(K\theta_{,i}\right)_{,i}=\left(1+\tau_q\frac{\partial}{\partial t}+\frac{1}{2}\tau_q^2\frac{\partial^2}{\partial t^2}\right)\\ \left(\rho C_E\frac{\partial\theta}{\partial t}+\gamma T_0\frac{\partial e}{\partial t}-\rho Q\right)\end{gathered} \tag{3}$$

where $0\le\tau_\theta<\tau_q$.

In case of thermo-physical properties $K$, $C_E$ and $\rho$ depend on the space coordinates only, Eq. (3) will be appeared as linear partial differential equation. However, if one of these thermo-physical properties is temperature-dependent, Eq. (3) will be appeared as a nonlinear partial differential equation.

Eq. (3) describes different thermoelasticity theories according to values of $\tau_q$ and $\tau_\theta$ as

- nonlocal CTE theory: $\tau_\theta=\tau_q=0$.
- nonlocal LS theory with one relaxation time (Lord and Shulman 1967): $\tau_\theta=0,\ \tau_q=\tau_0$.
- nonlocal DPL theory (Tzou 1995a,b 1996): $\tau_q\ge\tau_\theta>0$.

The corresponding local thermoelasticity theories are recovered by setting $ae_0=0$ in the above equations. It should be noted that to obtain the other material properties of isotropic, linear thermoelastic materials the following relations maybe used

$$\lambda=\frac{\nu E}{(1+\nu)(1-2\nu)},\quad \mu=\frac{E}{2(1+\nu)} \tag{4}$$

## 3. Problem formulation

A schematic diagram of a thin elastic nanobeam is illustrated in Fig. 1. In equilibrium, the nanobeam is considered to be unstrained, unstressed and at reference temperature $T_0$. Bending vibrations of nanobeam about $x$-

axis of small amplitude is consistent with linear Euler-Bernoulli theory. In this theory, any plane cross-section in the beginning perpendicular to axis of nanobeam remains plane and perpendicular to neutral surface through the bending. Hence, the displacement field of any point of nanobeam can be expressed as

$$u_1 = u = -z\frac{\partial w}{\partial x}, \quad u_2 = 0, \quad u_3 = w(x,t) \tag{5}$$

According to Eringen's nonlocal theory of thermoelasticity (Eringen 1972, 1983), with the aid of Eq. (1), one-dimensional constitutive relation can be simplified to

$$\sigma_x - \xi\frac{\partial^2\sigma_x}{\partial x^2} = -E\left(z\frac{\partial^2 w}{\partial x^2} + \alpha_T\theta\right) \tag{6}$$

It can be observed that that when the parameter $a$ is neglected, i.e., elements of a medium are considered to be continuously distributed, then $\xi = 0$, and Eq. (6) is reduced to constitutive equation of the classical case.

The flexural moment of cross-section is expressed as

$$M = \int_{-h/2}^{h/2} z\sigma_x \mathrm{d}z \tag{7}$$

Upon using Eqs. (6) and (7), we obtain

$$M - \xi\frac{\partial^2 M}{\partial x^2} = -EI\left(\frac{\partial^2 w}{\partial x^2} + \alpha_T M_T\right) \tag{8}$$

where

$$M_T = \frac{12}{h^3}\int_{-h/2}^{h/2} z\theta(x,z,t)\mathrm{d}z \tag{9}$$

If the nanobeam is transversely loaded by $q(x,t)$, then equation of transverse motion will be the following form (Zhang *et al.* 2005)

$$\frac{\partial^2 M}{\partial x^2} = -q(x,t) + \rho A\frac{\partial^2 w}{\partial x^2} \tag{10}$$

The flexural moment can be determined from Eqs. (8) and (10) as

$$M = \xi\left(\rho A\frac{\partial^2 w}{\partial t^2} - q\right) - EI\left(\frac{\partial^2 w}{\partial x^2} + \alpha_T M_T\right) \tag{11}$$

Eliminating the moment $M$ from Eq. (10) with the aid of Eq. (11), we obtain equation of motion of nanobeam as

$$\left[\frac{\partial^4}{\partial x^4} + \frac{\rho A}{EI}\frac{\partial^2}{\partial t^2}\left(1 - \xi\frac{\partial^2}{\partial x^2}\right)\right]w - \frac{1}{EI}\left(1 - \xi\frac{\partial^2}{\partial x^2}\right)q + \alpha_T\frac{\partial^2 M_T}{\partial x^2} = 0 \tag{12}$$

## 4. Temperature dependency of thermal conductivity

Let us assumed that $K$ is linearly varying with temperature according to the relation (Berman 1953)

$$K = K(\theta) = K_0(1 + K_1\theta) \tag{13}$$

In case of temperature-independent material properties, we take $K_1 = 0$. Using Kirchhoff's transformation (Berman 1953)

$$\psi = \frac{1}{K_0}\int_0^\theta K(\theta)\mathrm{d}\theta \tag{14}$$

and substituting of Eq. (13) into Eq. (14) gives (Berman 1953)

$$\psi = \theta\left(1 + \frac{1}{2}K_1\theta\right) \tag{15}$$

It follows from Eq. (14) that

$$\nabla\psi = \frac{K(\theta)}{K_0}\nabla\theta \tag{16}$$

and

$$\frac{\partial\psi}{\partial t} = \frac{K(\theta)}{K_0}\frac{\partial\theta}{\partial t} \tag{17}$$

Using Eqs. (16) and (17) into Eq. (3) gives

$$\left(1 + \tau_\theta\frac{\partial}{\partial t}\right)\nabla^2\psi = \left(1 + \tau_q\frac{\partial}{\partial t} + \frac{1}{2}\tau_q^2\frac{\partial^2}{\partial t^2}\right)\left(\frac{1}{k}\frac{\partial\psi}{\partial t} + \frac{\gamma T_0}{K_0}\frac{\partial e}{\partial t} - \rho Q\right) \tag{18}$$

Substituting Eq. (5) into Eq. (18) in absence of heat sources ($Q = 0$) yields

$$\left(1 + \tau_\theta\frac{\partial}{\partial t}\right)\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2}\right) = \left(1 + \tau_q\frac{\partial}{\partial t} + \frac{1}{2}\tau_q^2\frac{\partial^2}{\partial t^2}\right)\left[\frac{1}{k}\frac{\partial\psi}{\partial t} - \frac{\gamma T_0}{K_0}z\frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial x^2}\right)\right] \tag{19}$$

Eqs. (12) and (19) describe the nonlocal thermoelasticity theory with DPLs.

## 5. General solution

To solve governing system equations, we let the solution of temperature increment is sinusoidal varying as

$$\{\psi,\theta\}(x,z,t) = \{\Psi,\Theta\}(x,t)\sin\left(\frac{\pi z}{h}\right) \tag{20}$$

Using Eq. (20) in governing Eqs. (11), (12) and (19), we obtain

$$\left[\frac{\partial^4}{\partial x^4} + \frac{\rho A}{EI}\frac{\partial^2}{\partial t^2}\left(1 - \xi\frac{\partial^2}{\partial x^2}\right)\right]w - \frac{1}{EI}\left(1 - \xi\frac{\partial^2}{\partial x^2}\right)q + \frac{24\alpha_T}{\pi^2 h}\frac{\partial^2\Psi}{\partial x^2} = 0 \tag{21}$$

$$M = \xi\left(\rho A \frac{\partial^2 w}{\partial t^2} - q\right) - EI\left(\frac{\partial^2 w}{\partial x^2} + \frac{24 T_0 \alpha_T}{\pi^2 h}\Theta\right) \quad (22)$$

$$\left(1+\tau_\theta \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 \Psi}{\partial x^2} - \frac{\pi^2}{h^2}\Psi\right) = \left(1+\tau_q \frac{\partial}{\partial t} + \frac{1}{2}\tau_q^2 \frac{\partial^2}{\partial t^2}\right)\left[\frac{\rho C_E}{K}\frac{\partial \Psi}{\partial t} - \frac{\gamma T_0 \pi^2 h}{24K}\frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial x^2}\right)\right] \quad (23)$$

We will present the following dimensionless variables

$$\begin{gathered}\{x', z', u', w', L', b', h'\} = \eta c\{x, z, u, w, L, b, h\},\\ \{t', t_0', \tau_q', \tau_\theta'\} = \eta c^2\{t, t_0, \tau_q, \tau_\theta\},\\ \xi' = \eta^2 c^2 \xi, \quad \Theta' = \frac{\Theta}{T_0}, \quad \Psi' = \frac{\Psi}{T_0}, \quad q' = \frac{Aq}{EI}, \\ M' = \frac{M}{\eta c EI}, \quad c = \sqrt{\frac{E}{\rho}}, \quad \eta = \frac{\rho C_E}{K}\end{gathered} \quad (24)$$

Upon introducing the dimensionless quantities (24) in the governing Eqs. (21)-(23), we can obtain (dropping the primes for convenience)

$$\left[\frac{\partial^4}{\partial x^4} + \frac{12}{h^2}\frac{\partial^2}{\partial t^2}\left(1 - \xi\frac{\partial^2}{\partial x^2}\right)\right]w - \left(1 - \xi\frac{\partial^2}{\partial x^2}\right)q + \frac{24 T_0 \alpha_T}{\pi^2 h}\frac{\partial^2 \Psi}{\partial x^2} = 0 \quad (25)$$

$$\left(1+\tau_\theta \frac{\partial}{\partial t}\right)\left(\frac{\partial^2 \Psi}{\partial x^2} - \frac{\pi^2}{h^2}\Psi\right) = \left(1+\tau_q \frac{\partial}{\partial t} + \frac{1}{2}\tau_q^2 \frac{\partial^2}{\partial t^2}\right)\left[\frac{\partial \Psi}{\partial t} - \frac{\gamma \pi^2 h}{24K\eta}\frac{\partial}{\partial t}\left(\frac{\partial^2 w}{\partial x^2}\right)\right] \quad (26)$$

$$M = \xi\left(\frac{12}{h^2}\frac{\partial^2 w}{\partial t^2} - q\right) - \frac{\partial^2 w}{\partial x^2} - \frac{24 T_0 \alpha_T}{\pi^2 h}\Theta \quad (27)$$

Now, special form of external transverse load is considered. We consider exponential decaying time varying load as

$$q = -q_0\left(1 - \delta e^{-\Omega t}\right) \quad (28)$$

The case of $\delta = 0$ gives the uniformly distributed load.

## 6. Solution in Laplace transforms space

The initial conditions in their dimensionless form are taken as

$$\begin{gathered} w(x,t)|_{t=0} = \left.\frac{\partial w(x,t)}{\partial t}\right|_{t=0} = 0, \\ \Psi(x,t)|_{t=0} = \left.\frac{\partial \Psi(x,t)}{\partial t}\right|_{t=0} = 0 \end{gathered} \quad (29)$$

The nanobeam is assumed to be clamped-clamped at its axial ends, therefore it satisfies the following non-dimensional boundary conditions

$$w(0,t) = w(L,t), \quad \left.\frac{\partial w(x,t)}{\partial x}\right|_{x=0,L} = 0 \quad (30)$$

In addition, let us consider the boundary $x = 0$ of nanobeam is thermally loaded by harmonically varying incidents,

$$\Theta(0,t) = \Theta_0 \cos(\omega t), \quad \omega > 0 \quad (31)$$

The case of $\omega = 0$ for a thermal shock problem. Using Eqs. (15) and (20), then one gets

$$\Psi(0,t) = \Theta_0 \cos(\omega t) + \frac{1}{2}K_1 \Theta_0^2 \cos^2(\omega t) \quad (32)$$

Also, the temperature at $x = L$ should satisfy the relation

$$\frac{\partial \Psi}{\partial x} = 0 \quad (33)$$

If we apply Laplace transform method to both sides of Eqs. (25)-(27), we get

$$\left(\frac{d^4}{dx^4} - A_3 s^2 \frac{d^2}{dx^2} + A_1 s^2\right)\bar{w} = -A_2 \frac{d^2 \bar{\Psi}}{dx^2} - \bar{g}(s) \quad (34)$$

$$\left(\frac{d^2}{dx^2} - B_1\right)\bar{\Psi} = -B_2 \frac{d^2 \bar{w}}{dx^2} \quad (35)$$

$$\bar{M}(x,s) = -\left(\frac{d^2}{dx^2} - A_3 s^2\right)\bar{w} - A_2 \bar{\Theta} + \xi \bar{g}(s) \quad (36)$$

where

$$\begin{gathered} B_1 = A_4 + \frac{s(1+\tau_q s)}{1+\tau_\theta s}, \quad B_2 = \frac{s(1+\tau_q s)}{1+\tau_\theta s}A_5, \\ \bar{g}(s) = q_0\left(\frac{1}{s} - \frac{\delta}{\Omega + s}\right), \\ A_1 = \frac{12}{h^2}, \quad A_2 = \frac{24 T_0 \alpha_T}{\pi^2 h}, \quad A_3 = \frac{12\xi}{h^2}, \\ A_4 = \frac{\pi^2}{h^2}, \quad A_5 = \frac{\gamma \pi^2 h}{24K\eta} \end{gathered} \quad (37)$$

Eliminating $\bar{\Psi}$ from Eqs. (34) and (35), we get the differential equation for $\bar{w}$ as

$$\left(\frac{d^6}{dx^6} - A\frac{d^4}{dx^4} + B\frac{d^2}{dx^2} - C\right)\bar{w} = B_1 \bar{g}(s) \quad (38)$$

where

$$A = B_1 + A_2 B_2 + A_3 s^2,\quad B = s^2(A_1 + A_3 B_1),\quad C = A_1 B_1 s^2 \tag{39}$$

The general solutions of $\bar{w}$ can be obtained from Eq. (38) as follows

$$\bar{w} = \sum_{j=1}^{3}\left(C_j \mathrm{e}^{-m_j x} + C_{j+3}\mathrm{e}^{m_j x}\right) - \eta_1 \tag{40}$$

where $C_j$ and $C_{j+3}$ are integration constants, $\eta_1 = B_1\bar{g}(s)/C$ and $m_1^2$, $m_2^2$ and $m_3^2$ are roots of the characteristic equation

$$m^6 - Am^4 + Bm^2 - C = 0 \tag{41}$$

Substituting Eq. (34) into Eq. (35), leads to

$$\bar{\Psi} = -\frac{1}{A_2 B_1}\left[\frac{\mathrm{d}^4\bar{w}}{\mathrm{d}x^4} - (A_2 B_2 + A_3 s^2)\frac{\mathrm{d}^2\bar{w}}{\mathrm{d}x^2} + A_1 s^2\bar{w} + \bar{g}(s)\right] \tag{42}$$

The general solutions appeared in Eq. (42) with the help of Eq. (40) can be simplified as

$$\bar{\Psi} = \sum_{j=1}^{3} H_j\left(C_j \mathrm{e}^{-m_j x} + C_{j+3}\mathrm{e}^{m_j x}\right) - H_4 \tag{43}$$

$$H_j = -\frac{1}{A_2 B_1}\left[m_j^4 - (A_2 B_2 + A_3 s^2)m_j^2 + A_1 s^2\right],\quad j = 1,2,3,\quad H_4 = \frac{\bar{g}(s) - A_1 s^2 \eta_1}{A_2 B_1} \tag{44}$$

Using Eq. (43) into Eq. (20), we get

$$\bar{\psi} = \left[\sum_{j=1}^{3} H_j\left(C_j \mathrm{e}^{-m_j x} + C_{j+3}\mathrm{e}^{m_j x}\right) - H_4\right]\sin\left(\frac{\pi z}{h}\right) \tag{45}$$

Solving Eq. (15) in Laplace domain and using Eq. (20) gives the temperature $\bar{\theta}$ as

$$\bar{\theta}(x,z,s) = \bar{\Theta}(x,s)\sin\left(\frac{\pi z}{h}\right) = \frac{\sqrt{1 + 2K_1\bar{\psi}} - 1}{K_1} \tag{46}$$

The expressions of $\bar{w}$ and $\bar{\Theta}$ from Eqs. (40) and (46) are used into Eq. (36) to get the solution for $\bar{M}$ in Laplace domain as follows

$$\bar{M} = \sum_{j=1}^{3}(A_3 s^2 - m_j^2)\left(C_j \mathrm{e}^{-m_j x} + C_{j+3}\mathrm{e}^{m_j x}\right) - \frac{A_2\left(\sqrt{1 + 2K_1\bar{\psi}} - 1\right)}{K_1 \sin\left(\frac{\pi z}{h}\right)} + \eta_2 \tag{47}$$

where $\eta_2 = \xi\bar{g}(s) - A_3 s^2 \eta_1$. In addition, axial displacement after employing Eq. (38) becomes

$$\bar{u} = -z\frac{\mathrm{d}\bar{w}}{\mathrm{d}x} = z\sum_{j=1}^{3} m_j\left(C_j \mathrm{e}^{-m_j x} - C_{j+3}\mathrm{e}^{m_j x}\right) \tag{48}$$

Boundary conditions (30)-(33) in Laplace transform domain reduces to

$$\bar{w}(0,s) = \bar{w}(L,s) = 0,\quad \left.\frac{\mathrm{d}\bar{w}(x,s)}{\mathrm{d}x}\right|_{x=0,L} = 0, \tag{49}$$

$$\left.\frac{\mathrm{d}\bar{\Psi}(x,s)}{\mathrm{d}x}\right|_{x=L} = 0,$$

$$\bar{\Psi}(0,s) = \theta_0\left[\frac{s}{s^2 + \omega^2} + \frac{K_1(s^2 + 2\omega^2)}{2s(s^2 + 4\omega^2)}\right] = \bar{G}(s)$$

Substituting Eqs. (40) and (43) into above boundary conditions, one gets six linear equations in the form

$$\sum_{j=1}^{3}\left(C_j + C_{j+3}\right) = \eta_1,\quad \sum_{j=1}^{3}\left(C_j \mathrm{e}^{-m_j L} + C_{j+3}\mathrm{e}^{m_j L}\right) = \eta_1, \tag{50}$$

$$\sum_{j=1}^{3} m_j\left(C_j - C_{j+3}\right) = 0,\quad \sum_{j=1}^{3} m_j\left(C_j \mathrm{e}^{-m_j L} - C_{j+3}\mathrm{e}^{m_j L}\right) = 0,$$

$$\sum_{j=1}^{3} H_j\left(C_j + C_{j+3}\right) = \bar{G}(s)H_4,\quad \sum_{j=1}^{3} H_j m_j\left(C_j \mathrm{e}^{-m_j L} - C_{j+3}\mathrm{e}^{m_j L}\right) = 0$$

It is easy to get $C_j$ and $C_{j+3}$ by solving the above system of six linear equations which complets the solutions of the problem in Laplace transform domain.

## 7. Numerical results

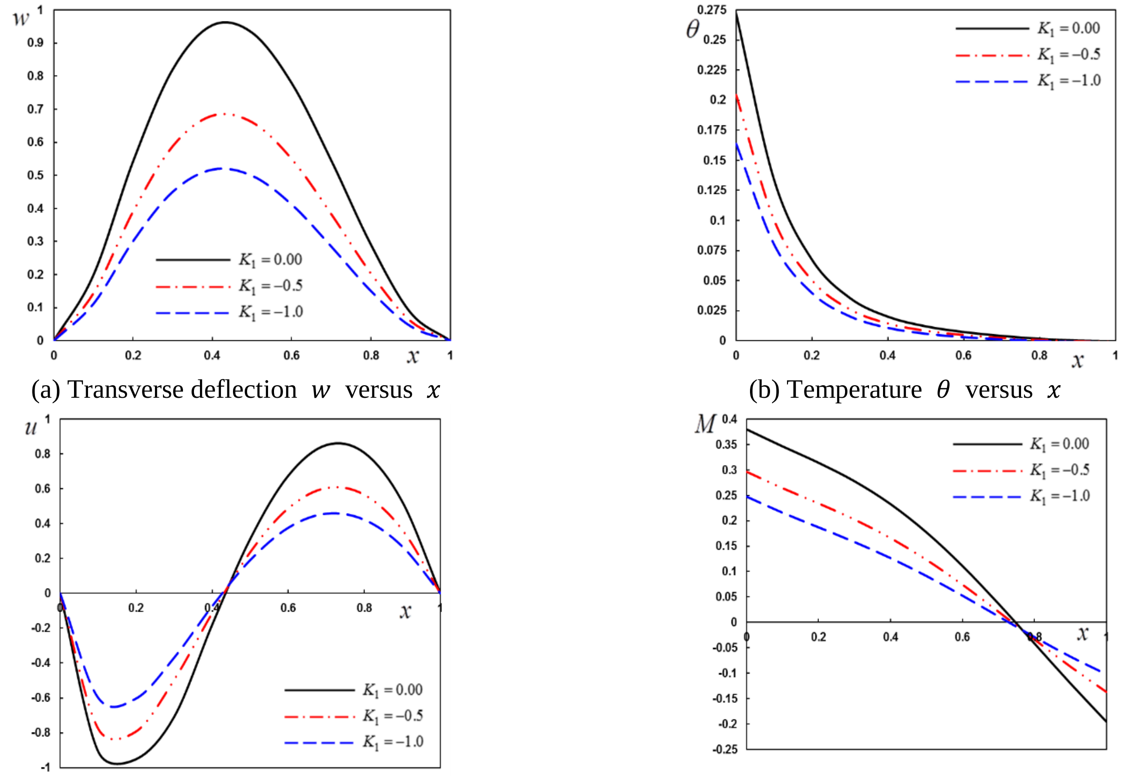
The Riemann-sum approximation method is used to obtain the numerical results for lateral vibration, thermal temperature, displacement, and stress distributions in the time domain. In this method, any function $\bar{f}(x,s)$ in the Laplace domain can be inverted to the time domain $f(x,t)$ as

$$f(x,t) = \frac{e^{\zeta t}}{t}\left[\frac{1}{2}\mathrm{Re}\{\bar{f}(x,\zeta)\} + \mathrm{Re}\left\{\sum_{n=0}^{N}(-1)^n \bar{f}\left(x, \zeta + \frac{\mathrm{i}n\pi}{t}\right)\right\}\right] \tag{51}$$

where $\zeta$ is an arbitrary real number (experimentally, $\zeta \approx 4.7/t$) (Tzou 1996), Re is the real part and $\mathrm{i} = \sqrt{-1}$.

In this current section, some numerical results and graphs are presented to illustrate thermoelastic behavior of nanobeams based on nonlocal generalized thermoelasticity. In addition, results are obtained to establish the effect of variability thermal conductivity, the point load, nonlocal parameter and angular frequency of thermal vibration on the behavior of nanobeam. The following physical parameters for silicon (Si) at $T_0 = 293$ K are employed in calculating the numerical results:

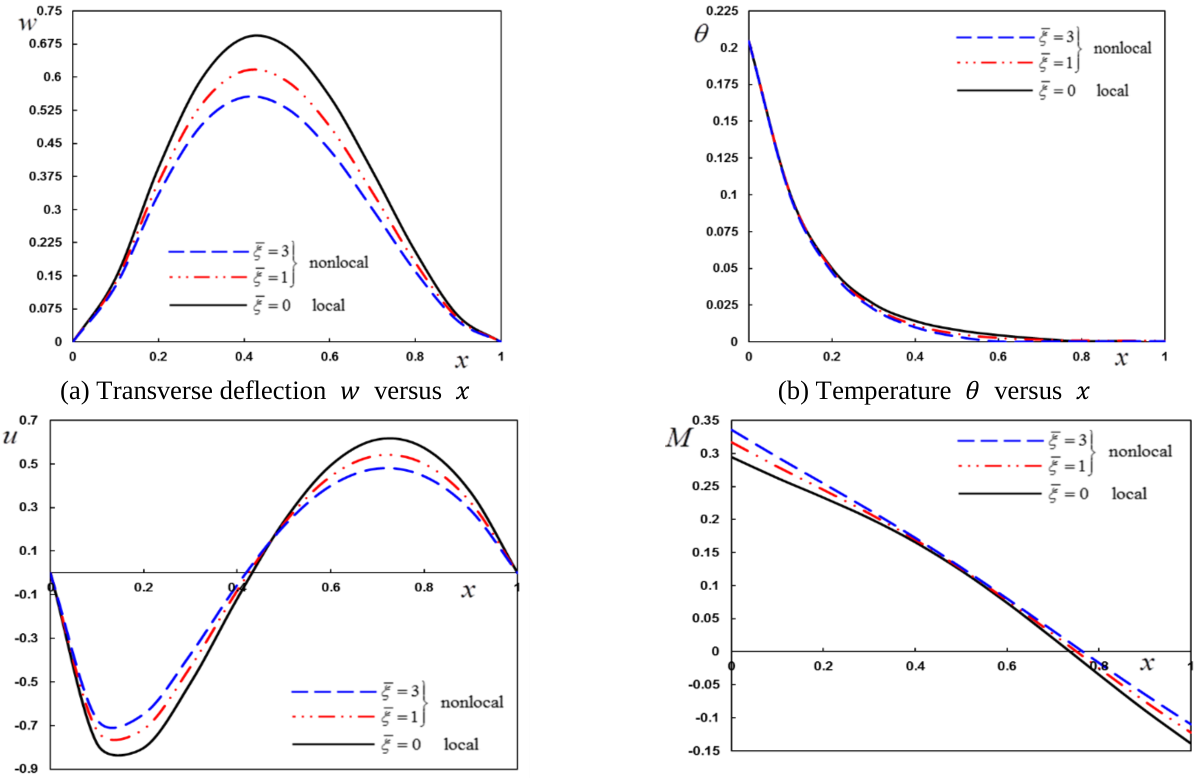
$$E = 169\ \text{GPa},\quad \nu = 0.22,\quad \rho = 2330\ \text{kg/m}^3,\quad K = 156\ W/(m\text{K}),\quad C_E = 713\ \text{J/(kg K)}, C_E = 2.59 \times 10^{-6}\ /\text{(K)} \tag{52}$$




(a) Transverse deflection $w$ versus $x$ (b) Temperature $\theta$ versus $x$

(c) Displacement $u$ versus $x$ (d) bending moment $M$ versus $x$

Fig. 2 The transverse deflection, temperature, displacement and thermal stress distributions of the nanobeam for different values of the variability thermal conductivity parameter $K_1$.




(a) Transverse deflection $w$ versus $x$ (b) Temperature $\theta$ versus $x$

(c) Displacement $u$ versus $x$ (d) Bending moment $M$ versus $x$

Fig. 3 The transverse deflection, temperature, displacement and thermal stress distributions of the nanobeam for different values of the nonlocal thermoelastic parameters $\bar{\xi}$

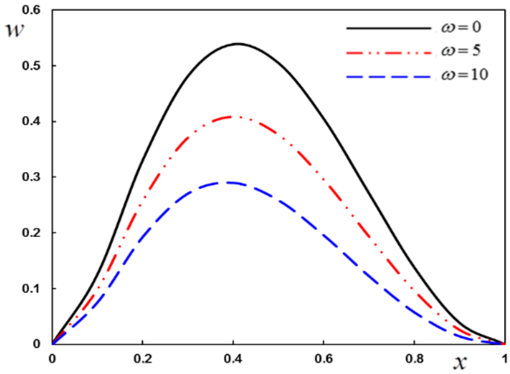
The magnitude $q_0$ has been taken as $(1 \times 10^{-8})$ and the nonlocal parameter $\bar{\xi}$ ($\bar{\xi} = 10^6 \xi$) is also considered. In addition, the following parameters are fixed here: $t = 0.1$, $\Omega = 0.1$, $L/h = 10$, $b/h = 0.5$, $L = 1$, $z = h/3$. To get the solutions lateral vibration, temperature, displacement and bending moment in physical domain, we have applied Laplace inversion formula appeared in Eq. (51) to Eqs. (40), (45)-(48). Numerical code has been obtained using Mathematica programming language. Numerical results are illustrated for some cases as:

In Figs. 2(a)-2(d), the dimensionless lateral vibration $w$, temperature $\theta$, displacement $u$ and bending moment $M$ is plotted against variability thermal conductivity parameter $K_1$ varying from $-1$ to $0$, when $\omega = 5$, $\tau_q = 0.02$ and $\tau_\theta = 0.01$. The thermal conductivity $K$ is temperature-independent when we take $K_1 = 0$. It is also clear from the plotted figures that the parameter $K_1$ acts to decrease the magnitude of the vertical deflection $w$ and attains its maximum value at $x = 0.5$. This variation is shown in Fig. 2(a). The non-dimensional the temperature $\theta$ is plotted in Fig. 2(b) against distance $x$ varying, when $K_1 = 0, -0.5, -1$. Fig. 2(b) shows the dimensionless temperature $\theta$ decreases sharply with the increase in $x$ and attains its maximum value at the boundary $x = 0$ and finally vanishes. It is also seen from Figure 2b that $\theta$ in case temperature-independent ($K_1 = 0$) is larger than the case of variability thermal conductivity ($K_1 = -0.5, -1$). From Fig. 2(c), it can be noticed that magnitude of $u$ decreases as $K_1$ decreases. This figure shows negative values for displacement in the range $0 \le x \le 0.45$ and positive values for the displacement in the range $0.45 \le x \le 1$.

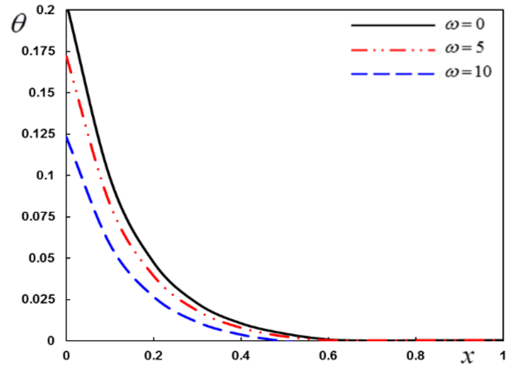
Fig. 2(d) depicts the variation of the bending moment $M$ versus distance $x$ for $K_1 = 0, -0.5, -1$. It is observed from this figure that the parameter $K_1$ acts to decrease the magnitude of the bending moment distribution. From Fig. 2 we have noticed that, $K_1$ has a significant effect on all fields, which add an importance to our consideration about the thermal conductivity to be variable.

Figs. 3(a)-3(d) shows the dimensionless field quantities $w$, $\theta$, $u$ and $M$ against distance $x$ for various values of the angular frequency of thermal vibration coefficient $\omega$ in the case of $K_1$ is fixed to $-0.5$ and also the parameters $\tau_q$ and $\tau_\theta$ remain constants ($\tau_q = 0.2$, $\tau_\theta = 0.1$). Putting $\omega = 0$, for thermal shock problem, and setting $\omega = 5$, $10$ for harmonically varying heat. It is observed from the plotted results 3(a)-(d) that the coefficient $\omega$ has a great influence on all fields.

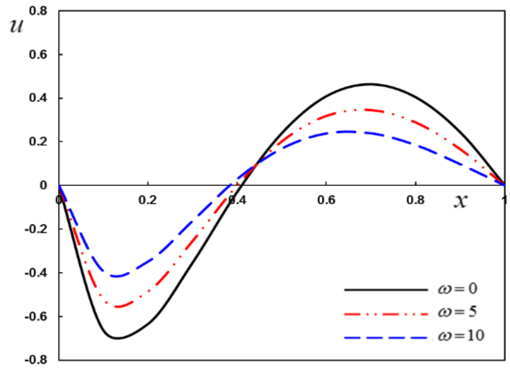
Figs. 4(a)-4(d) are plotted to compare between the results of the nonlocal thermoelastic model (Eringen's theory) ($\bar{\xi} = 1,3$) theory and local thermoelastic theory ($\bar{\xi} = 0$) when $\omega = 5$ and $K_1 = -0.5$. In both of the cases temperature distribution decreases with distance and finally goes to zero but rate of decay for $\bar{\xi} = 1,3$ is slower than the rate of decay for $\bar{\xi} = 0$. From the figures we can see that the amplitudes of lateral vibration $w$ temperature, displacement, and moment decrease as nonlocal parameter $\bar{\xi}$ increase. The figures show that this parameter has significant effect on all fields indicating the difference between the generalized local and nonlocal thermoelasticity theories.



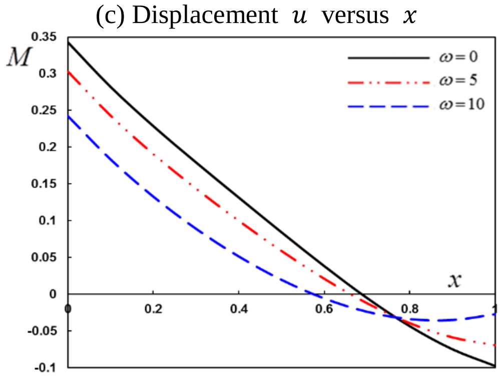

(a) Transverse deflection $w$ versus $x$




(b) Temperature $\theta$ versus $x$

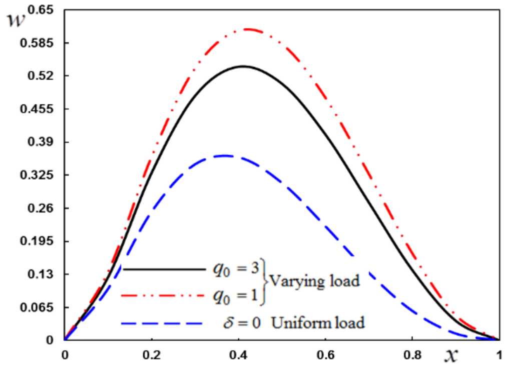



(c) Displacement $u$ versus $x$

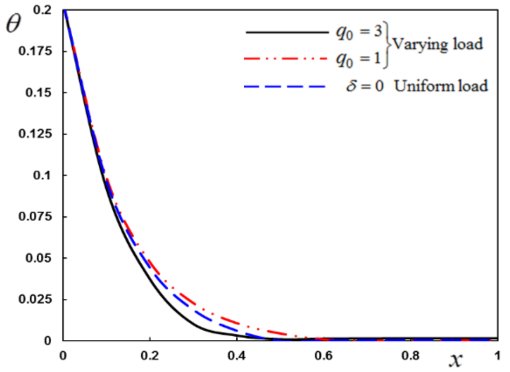



(d) Bending moment $M$ versus $x$

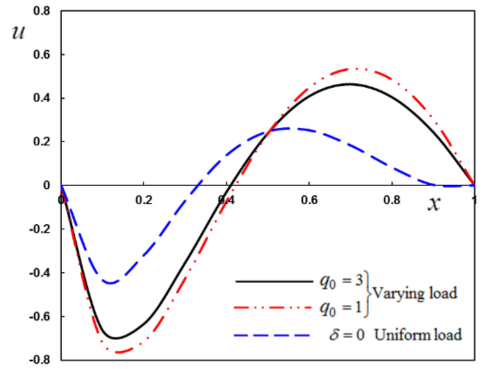
Fig. 4 The transverse deflection, temperature, displacement and thermal stress distributions of the nanobeam for different values of the angular frequency of thermal vibration $\omega$



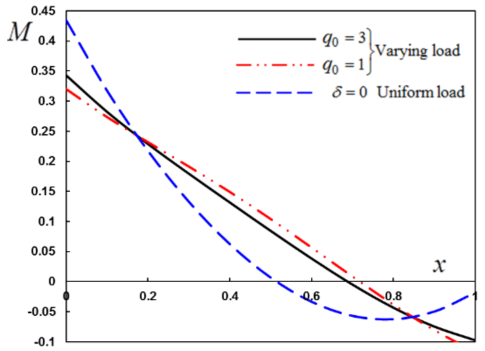

(a) Transverse deflection $w$ versus $x$




(b) Temperature $\theta$ versus $x$




(c) Displacement $u$ versus $x$




(d) Bending moment $M$ versus $x$

Fig. 5 The transverse deflection, temperature, displacement and thermal stress distributions of the nanobeam for different values of the the point load $q_0$

In last case, three different values of the dimensionless magnitude of the point load $q_0$ are considered. We put $\delta = 0$ for the uniformly distributed load and for exponential decaying time varying load, we take $\delta = 1$. For a comparison purpose, lateral vibration, temperature, displacement, and bending moment of nanobeam are shown in Figs. 5(a)-5(d). The effect of point load $q_0$ (harmonic and uniform) plays significant role on lateral vibration, temperature, and displacement fields.

## 8. Conclusions

The dynamic response of an isotropic thermoelastic nanobeams with finite length subjected to harmonically varying heat and transverse loads is investigated in context of nonlocal theory of thermoelasticity. The effects of the dynamic loads $q_0$, the nonlocal parameter $\bar{\xi}$, the variability thermal conductivity parameter $K_1$ and angular frequency of thermal vibration $\omega$ on the field variables are investigated and presented graphically. From the above section, we can arrive at the following conclusions.

- The study can discover applications in design and development of resonator devices under loading environment.
- Non-dimensional studied fields depend on various material parameters.
- Thermoelastic stress, displacement and temperature have a strong dependency on variability thermal conductivity parameter.
- The nonlocal parameter $\bar{\xi}$ has significant effects on all studied fields.
- The effects of dynamic loads on all studied fields are very significant.
- Significant differences in physical quantities are observed between exponential decaying time varying load and the uniformly distributed load.
- The effects of angular frequency of thermal vibration parameter on all studied fields are very significant.
- The vibration of nanotubes is important subject in study of nanotechnology since it relates to the electronic and optical properties of multi-walled carbon nanotubes.
- This study may find requests and applications in design and development of resonator devices under loading environment.

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## Nomenclature

| | |
|---|---|
| $a$ | internal characteristic length |
| $A = bh$ | area of beam cross-section |
| $b$ | width of nanobeam ($-b/2 \leq y \leq b/2$) |
| $C_E$ | specific heat at constant strain |
| $E$ | Young's modulus |
| $e = \mathrm{div}\,\vec{u}$ | volumetric strain |
| $e_0$ | a constant appropriate to each material in nonlocal theory |
| $EI$ | flexural rigidity |
| $h$ | thickness of nanobeam ($-h/2 \leq z \leq h/2$) |
| $I = bh^3/12$ | inertia moment of beam cross-section |
| $K$ | thermal conductivity |
| $k = K/\rho C_E$ | thermal diffusivity |
| $K_0$ | thermal conductivity at reference temperature $T_0$ |
| $K_1$ | slope of thermal conductivity-temperature curve divided by $K_0$ |
| $l$ | external characteristic length |
| $L$ | length of nanobeam ($0 \leq x \leq L$) |
| $M$ | flexural moment |
| $M_T$ | thermal moment of beam |
| $s$ | Laplace's variable |
| $t$ | time |
| $T$ | Temperature field |
| $T_0$ | environment temperature |
| $Q$ | heat source |
| $q(x,t)$ | external load |
| $q_0$ | dimensionless magnitude of point load |
| $\vec{u}$ | displacement vector |
| $u$ | axial displacement |
| $w$ | lateral deflection |
| $x, z$ | axial and normal coordinates |
| $\alpha_T$ | thermal expansion |
| $\psi$ | new function expressing heat conduction |
| $\Psi$ | heat conduction function in terms of $x$ and $t$ |
| $\xi = (e_0 a)^2$ | nonlocal parameter |
| $\delta$ | parameter appeared in Eq. (28) |
| $\delta_{ij}$ | Kronecker's delta function |
| $\gamma = E\alpha_T/(1-2\nu)$ | stress-temperature modulus |
| $\lambda$ and $\mu$ | Lamé's constants |
| $\nu$ | Poisson's ratio |
| $\nabla^2$ | Laplacian operator |
| $\theta = T - T_0$ | thermodynamical temperature |
| $\rho$ | material density |
| $\sigma_{kl}$ | classical (Cauchy) or local stress tensor |
| $\sigma_x$ | nonlocal normal stress |
| $\tau_{kl}$ | nonlocal stress tensor |
| $\tau_q$ | phase-lag of gradient of temperature |
| $\tau_\theta$ | phase-lag of heat flux |
| $\Theta$ | thermal function in terms of $x$ and $t$ |
| $\Theta_0$ | thermal constant |
| $\Omega$ | dimensionless frequency of applied load |
| $\omega$ | angular frequency of thermal vibration |