Smart Structures and Systems, Vol. 2, No. 4 (2006) 313-320
DOI: http://dx.doi.org/10.12989/sss.2006.2.4.313

A reliability-based criterion of structural performance for structures with linear damping

Agnessa Kovaleva†

Space Research Institute, Russian Academy of Sciences, Moscow 117997, Russia

(Received January 1, 2006, Accepted, July 11, 2006)

Abstract. The reliability analysis of structures subjected to stochastic loading involves evaluation of time and probability of the system's residence in a reference domain. In this paper, we derive an asymptotic estimate of exit time for multi-degrees-of-freedom structural systems. The system's dynamics is governed by the Lagrangian equations with linear dissipation and fast additive noise. The logarithmic asymptotic of exit time is found explicitly as a sum of two terms dependent on kinetic and potential energy of the system, respectively. As an example, we estimate exit time and an associated structural performance for a rocking structure.

Keywords: first-passage time problem; reliability; large deviations analysis.

1. Introduction

In problems that cover a wide range of engineering applications, there is a stochastic process model for a system, and the goal is to keep the system within a prescribed admissible domain. Exit from this domain can be in some sense catastrophic. Examples are the loss of integrity and the destruction of structures, the loss of stability, the disruption of normal operation, etc.

There are two parameters, closely related to the system's lifetime and reliability, probability $P(t)$ of exit from an admissible domain over a given time interval $[0, t)$ and the mean exit time $\mathbf{E}\tau$, where τ is the first moment the system leaves the prescribed domain. If exits are rare independent events, exit probability $P_e(t)$ can be approximated by the Poisson law $P_e(t) = 1-e^{-\lambda t}$, where $\lambda = 1/(\mathbf{E}\tau)$ (Bolotin 1984). This makes the mean exit time a parameter of a high priority.

In general, the analysis of the non-stationary exit processes for a mdf nonlinear system with non-white noise perturbation is prohibitively difficult. The appropriate mathematical basis to examine the exit processes is large deviations theory (Kushner 1984a,b, Freidlin and Wentzell 1998). This theory provides an alternative approach to the analysis of the perturbed dynamics. Essentially, the estimates of the mean exit time and related quantities are obtained as the solution of a deterministic variational problem. The large deviations principle considers the cost (action) functional that must be minimized by the "most likely" exit path. The solution the Hamilton-Jacobi equation associated with minimization of the action functional corresponds to the logarithmic asymptotics of the mean exit time.

Large deviations in a multi-degrees-of-freedom Lagrangian diffusion system has been discussed (Kovaleva 2006a,b). In this paper, we extend the large deviations principle to the mdf Lagrangian

†E-mail: a.kovaleva@ru.net

systems with coloured noise. Following (Kushner 1984a,b), we construct the Hamilton-Jacobi equation for the logarithmic asymptotics of the mean exit time. We show that, under some non-restrictive assumptions, this equation has a closed-form solution.

The paper is organized as follows. Section 2 reminds some requisite issues of large deviations theory and introduces the Hamilton-Jacobi equation for the logarithmic asymptotic of exit time.

In Section 3 we show that the solution of this equation can be constructed as a sum of two terms. The first term takes a form of kinetic energy with a linearly transformed matrix. The second term depends on potential energy and satisfies a linear PDE of a simple structure. Under some non-restrictive assumptions, we find the explicit solution of this equation.

Section 4 demonstrates an application of the theory to a problem of the controlled motion of a rocking block. The control task is to prevent overturning during the requisite time interval. The logarithmic asymptotics of the mean exit time from the reference domain is found as an explicit function of the system and noise parameters. This allows design of a regulator effectively counteracting the noise-induced deviations.

2. Basic methodology

The equations of motion take the form

$$\frac{d}{dt}\frac{\partial L(q,\dot q)}{\partial \dot q} - \frac{\partial L}{\partial q} + B\dot q = \Delta\xi^{\varepsilon}(t) \tag{1}$$

where $q \in R^n$, $\xi^{\varepsilon}(t) = \xi(t/\varepsilon)$ is a zero-mean stationary Gaussian process in R^r, Δ is a $n\times r$ - matrix. Lagrangian $L(q, \dot q) = T(q, \dot q) - \Pi(q)$, where $T(q, \dot q) = (\dot q, M\dot q)/2$ and $\Pi(q)$ are kinetic and potential energy of the system, respectively. The matrices M and B are positive definite square matrix. The point $q = 0$ is assumed to be a unique asymptotically stable equilibrium of the unperturbed system in the reference domain.

Define the impulse $p = \partial L(q, \dot q)/\partial q$ and construct the function

$$H(q, p) = (\dot q, p) - L(q, \dot q) = T(\dot q) + \Pi(q) = \tilde T(p) + \Pi(q) \tag{2}$$

in which

$$\dot q(q, p) = M^{-1}p,\ \tilde T(p) = T(M^{-1}p) = \frac{1}{2}(p, M^{-1}p)$$

Substitution of function (2) into Eq. (1) yields

$$\dot q = \frac{\partial H}{\partial p}$$

$$\dot p = -\frac{\partial H}{\partial q} - B\frac{\partial H}{\partial p} + \Delta\xi^{\varepsilon}(t) \tag{3}$$

Let the admissible domain G: $\{(q, p)\in G \subset \mathbf{R}^{2n},\ t < \tau^{\varepsilon}\}$ be an open bounded set with smooth boundary Γ and closure $\overline{G}$. The unperturbed counterpart of system, Eq. (3) ($\Delta = 0$) is assumed to have a single asymptotically stable point O: $\{q = p = 0\} \in \overline{G}^{\,\varepsilon}$, and all trajectories of the unperturbed system originating in $\overline{G}$ tend to O. The task is to estimate the mean time $\mathbf{E}\tau^{\varepsilon}$, where $\tau^{\varepsilon} = \inf\{t: (q(t), p(t)) \notin G\}$ is the first moment the system leaves G.

The convergence of the unperturbed orbits from G to the asymptotically stable origin O entails the convergence of the perturbed orbits originating in G to a small neighbourhood of O. Motion evolves within this neighbourhood over the exponentially large time interval until the burst-like escape from the domain G (Freidlin and Wentzell 1998, Kushner 1984a,b). This implies that escapes from a given domain are rare events, and large deviation theory is useful for the asymptotic analysis. Omitting the detailed analysis, we refer to the main result of large deviation theory concerning the exit time estimation. Under some non-restrictive assumptions, the mean time $\mathbf{E}\tau^{\varepsilon}$ obeys the estimate (Kushner 1984a,b)

$$\lim_{\varepsilon \to 0} \varepsilon \ln(\mathbf{E}\tau^{\varepsilon}) = \inf_{q, p \in \Gamma} S(q, p) = S_0 \tag{4}$$

where $S(q, p)$ is the solution of the Hamilton-Jacobi equation (Kovaleva 2006a,b)

$$[H, S] - \left(B\frac{\partial H}{\partial p}, \frac{\partial S}{\partial q}\right) + \frac{1}{2}\left(\frac{\partial S}{\partial p}, A\frac{\partial S}{\partial p}\right) = 0, S(0, 0) = 0 \tag{5}$$

in which $[H, S]$ is the Poisson bracket

$$[H, S] = \left(\frac{\partial H}{\partial p}, \frac{\partial S}{\partial q}\right) - \left(\frac{\partial H}{\partial q}, \frac{\partial S}{\partial p}\right)$$

A is the positive definite matrix defined as

$$A = \Delta \Psi_0 \Delta', \Psi_0 = \int_{-\infty}^{\infty} \mathbf{E}\xi(t)\xi'(0)\mathrm{d}t \tag{6}$$

Here and below the prime denotes transposition of a vector or a matrix.

3. Calculation of S_0

It is easy to prove that Eq. (5) is satisfied if

$$\frac{\partial S}{\partial p} = K\frac{\partial H}{\partial p}, \frac{\partial S}{\partial q} = K'\frac{\partial H}{\partial q}, K = 2A^{-1}B \tag{7}$$

or, by Eq. (2)

$$\frac{\partial S}{\partial p} = K\frac{\partial \tilde{T}}{\partial p} = KM^{-1}p, \frac{\partial S}{\partial q} = K'\frac{\partial \Pi}{\partial q} \tag{8}$$

that is

$$S(q, p) = \frac{1}{2}(p, KM^{-1}p) + \Theta(q) \tag{9}$$

where the function Θ satisfies the equation

$$\frac{\partial \Theta}{\partial q} = K'\frac{\partial \Pi}{\partial q}, \Theta(0) = 0 \tag{10}$$

Eqs. (9) and (10) define the solution of Eq. (5) and estimate (4). We note that the term $(p, KM^{-1}p)/2$ can be interpreted as kinetic energy of a system with the mass matrix $M^* = MK^{-1}$.

Now we consider some special cases allowing the precise solution of Eq. (10). Suppose that

1. The matrix $K = kI_n$, where I_n is the *n-th* order identity matrix, k is a scalar. We obtain from Eqs. (7)

$$S(q,p) = kH(q,p) \tag{11}$$

2. The matrix $K = \text{diag}\,\{k_1,\ldots, k_n\}$. Potential energy $\Pi(q)$ is in the form

$$\Pi(q) = \sum_{i=1}^{n} f_i(q_i),\ f_i(0) = 0 \tag{12}$$

Thus we have

$$\Theta\,(q) = \sum_{i=1}^{n} k_i f_i(q_i) = (k, f(q)) \tag{13}$$

where the vectors $k = (k_1,\ldots, k_n)$, $f = (f_1,\ldots, f_n)$. Formulas (9), (10) and (13) yield

$$S(q,p) = \frac{1}{2}(p, KM^{-1}p) + (k, f(q)) \tag{14}$$

3. The system is linear; the equations of motion are written as

$$M\ddot{q} + B\dot{q} + Cq = \Delta\xi^{\varepsilon}(t) \tag{15}$$

where C is positive definite matrix. Potential energy of the system is $\Pi(q) = (q, Cq)/2$. In this case we find from Eqs. (9), (10)

$$S(q,p) = \frac{1}{2}[(p, KM^{-1}p) + (q, K'Cq)] \tag{16}$$

3. Definition of the exit point and the terminal state

Exit location and the state of the system at the moment τ^{ε} are defined by condition (4). Let the minimum of the quasi-potential $S(q,p)$ be achieved at a point $(q^*, p^*) \in int\ G$. In this case, the terminal state of the system converges in probability to (q^*, p^*) as $\varepsilon \to 0$ (Freidlin and Wentzell 1998). If all entries of the vectors $q = q^*$ and $p = p^*$ are fixed on the boundary ∂G, then, by (4), $S_0 = S(q^*, p^*)$.

In general, the boundary is defined as a set of equalities

$$\Phi_i(q,p) = 0\ (i = 1,\ldots r),\ (q,p) \in \Gamma \tag{17}$$

or, in the vector form, $\Phi\,(q,p) = 0$, where Φ is the vector with entries Φ_i $(i = 1,\ldots, r)$, Following the standard minimization procedure (Fiacco and McCormick 1990), we introduce the extended function $F(q, p, \lambda) = S(q,p) + (\lambda, \Phi(q,p))$, where entries of the vector λ are the Lagrange multipliers λ_i $(i = 1,\ldots, r)$. Under the regularity assumption (Fiacco and McCormick 1990), the minimum of $F(q, p, \lambda)$ is defined by the equations $\partial F/\partial q = \partial F/\partial p = 0$ and conditions (17).

In a number of applications, the admissible domain G_q is identified by the variable q, that is G_q: $\{\Phi_i(q) < 0\}$, Γ_q: $\{\Phi_i(q) = 0\}$ $(i = 1,\ldots,r)\}$. Formally, there are no particular restrictions to the impulse p. In this case $F(q, p, \lambda) = S(q, p) + (\lambda, \Phi(q))$, that is

$$\frac{\partial F}{\partial p} = \frac{\partial S}{\partial p} = KM^{-1}p = 0\,,\; p^* = 0. \tag{18}$$

Therefore, if the boundary value of the impulse p is not prescribed, then $p^* = p(\tau^\varepsilon) = 0$ at the moment "the most likely" exit path reaches the boundary Γ_q. Then by (4), (18)

$$\lim_{\varepsilon \to 0} \varepsilon \ln(\mathbf{E}\,\tau^\varepsilon) = \inf_{\Gamma} S(q, p) = \inf_{\Gamma_q} \Theta(q) \tag{19}$$

5. Example

As an example, we analyse rocking motion of a structure standing free on a moving foundation (Fig. 1). Performance criterion is toppling probability, which is closely associated with exit time. The task is to choose the control parameters maximizing the mean exit time.

During the earthquakes, many of the buildings and elevated highways move like rigid blocks and end up leaning or fallen to one side. A detailed analysis (Kovaleva 2005) has proved that the effect of structural flexibility on the rigid motion of a structure as a whole is negligibly small if the height of the structure significantly exceeds its width. This makes the rigid block motion a credible dynamic model in analysis of rocking response of a slender structure.

We consider the simplest case of a planar response of a slender, rigid, uniform, rectangular block. Assuming rigid foundation, large friction to prevent sliding, and the Newton restitution law at impact, the only possible response mechanism under ground excitation is rocking about the corners of the block (Fig. 1). Rocking oscillations are excited by random ground acceleration resulting in overturning with a non-zero probability. The control U is implemented to minimize the toppling probability.

Let M be the mass of a rectangular block of width $2h$ and height $L >> h$; $K = M\rho^2$ and ρ be the mass moment of inertia and the radius of inertia about the axes of rotation O_R or O_L. The position of the centre of gravity C on the axe of symmetry is determined by the angle $\alpha = \arcsin(h/l) << 1$; the distance from the bottom edges to the center of gravity is $O_R C = O_L C = l$. The angle of rotation about the corner

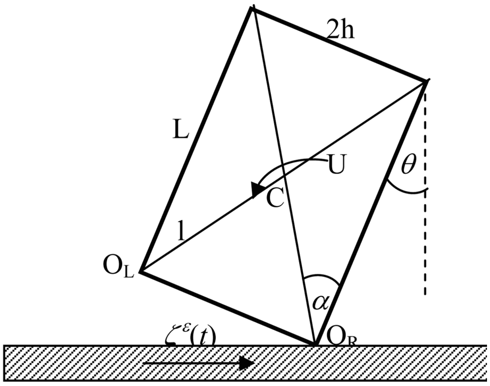

Fig. 1 Block rocking about the bottom of the corner

of the block is $q << 1$ (Fig. 1).

The horizontal ground motion induces rocking; the control torque U counteracts rotation. The ground acceleration $\zeta^{\varepsilon}(t) = \zeta(t/\varepsilon)$ is assumed to be zero-mean stationary Gaussian process, $0 < \varepsilon \le 1$.

The equation of rocking motion is written as

$$K\ddot{q} + Mgl\sin(\alpha - |q|) = -Ml\zeta(t)\cos(\alpha - |q|) - U \tag{20}$$

Whenever $q = 0$, an impact rule is applied: $t = t_s$, $q(t_s) = 0$, $\ddot{q}_+(t_s) = \rho\ddot{q}_-(t_s)$, in which $\rho\in(0, 1]$ is the restitution coefficient, $\dot{q}_{\pm}(t_s) = \dot{q}(t_s \pm 0)$. For simplicity, we let $\rho = 1$.

As shown in Kovaleva (2005), two parameters affect the structural stability, the position of the centre of gravity and dissipation in the system and at impact. Since the position of centre of gravity is unchangeable, the control torque is chosen as $U = \beta\dot{q}$. The coefficient b can be interpreted as the gain factor of the velocity feedback.

Linearisation with respect to the small angles q and α yields the equation

$$\ddot{q} + b\dot{q} - n^2(q - \alpha\,\mathrm{sign}q) = \sigma\zeta^{\varepsilon}(t) \tag{21}$$

where $n^2 = Mgl/K$, $\sigma = Ml/K = n^2/g$, $b = \beta/K$. Considering Eq. (21) as the Lagrange equation, we define kinetic, potential and total energy as

$$T(p) = \frac{1}{2}\dot{q}^2 = \frac{1}{2}p^2,\ \Pi(q) = \frac{n^2}{2}[\alpha^2 - (q - \alpha\,\mathrm{sgn}q)^2]$$

$$H(q, p) = \frac{1}{2}\{p^2 + n^2[\alpha^2 - (q - \alpha\,\mathrm{sgn}q)^2]\} \tag{22}$$

The potential function $\Pi(q)$ is presented in Fig. 2. The dashed line (-A, A) is tangent to the maxima of the potential function $\Pi(\alpha) = \Pi(-\alpha) = (n\alpha)^2/2$. The curvilinear triangle (-A, 0, A) represents the domain of rocking oscillations, exit from this domain entails overturning. In the phase plane, the admissible domain is restricted with the diamond-shaped separatrix Γ (Kovaleva 2005).

$$\Gamma:\ \{|p| = n(|q| - \alpha)\} \tag{23}$$

By Eqs. (4), (11), the mean exit time is calculated as

$$\lim_{\varepsilon\to 0}\varepsilon\ln(\mathbf{E}\,\tau^{\varepsilon}) = \inf_{q,p\,\in\,\Gamma}[kH(q, p)] = S_0 \tag{24}$$

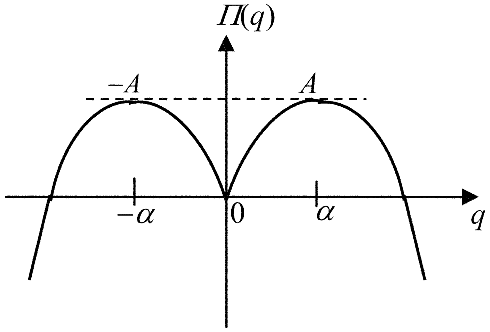

Fig. 2 The potential of the rocking oscillator

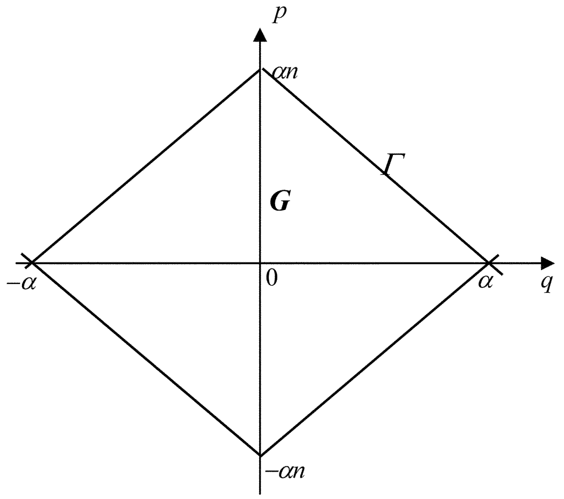

Fig. 3 The conservative phase portrait of the rocking oscillator

where, by Eqs. (6), (7), $k = 2b/(\sigma^2 S(0))$, $S(\omega)$ is spectral density of the process $\zeta(t)$. In order to minimize $H(q, p)$ at the boundary Γ, we rewrite Eq. (23) as

$$\Gamma : \{p^2 = (q - \alpha \operatorname{sgn} q)^2\} \tag{25}$$

Substituting Eq. (25) into Eq. (24), we obtain

$$\lim_{\varepsilon \to 0} \varepsilon \ln(\mathbf{E}\tau^{\varepsilon}) = \frac{b(n\alpha)^2}{\sigma^2 S(0)} = \frac{bg\rho^2\alpha^2}{lS(0)} \tag{26}$$

Formula (6) and (26) imply that an increase of the spectral density $S(0)$ reduces $\mathbf{E}\tau^{\varepsilon}$. Physically, this stems from the fact that free motion of the system is a slow process compared with noise, and the effect of the low-frequency portion of the noise spectrum is similar to the effect of the resonance excitation. Then the value $\mathbf{E}\tau^{\varepsilon}$ depends on both the geometric parameters of the block and the gain factor b but it is independent of the mass. If the geometric parameters are unchangeable, an increase of the parameter b improves structural performance.

5. Conclusions

The paper has developed an algorithm for calculating a reliability-based performance criterion, associated with the first-passage problem. It has been shown that the mean time until escape from an admissible domain of structural integrity can be calculated in the closed form. The approach employed is based on the solution of the Hamilton-Jacobi equation for the logarithmic asymptotics of the mean exit time. The solution of this equation explicitly depends on kinetic and potential energy of the system and can be found for both non-linear and linear systems.

This approach is of potential use in the dynamics and control problems for a wide class of mechanical and physical systems. As an illustration, the time until overturning for a controlled rocking structure is estimated. The solution is found as an explicit function of the block's geometric parameters, the regulator's gain factor, and the noise spectrum.

Acknowledgements

Partial financial support of this research by National Institute of Standards and Technology (Gaithersburg, USA), INTAS (grant 03-51-5547) and RFBR (grant 05-01-00225) is acknowledged with thanks.

References

Bolotin, V. V. (1984), *Random Vibrations of Elastic Systems*, Martinus Nijhoff Publishers, The Hague.

Fiacco, A. V. and McCormick, G. P. (1990), *Nonlinear Programming: Sequential Unconstrained Minimization Techniques*, SIAM Classics in Applied Mathematics 4, SIAM, Philadelphia, PA.

Freidlin, M. I. and Wentzell, A. D. (1998), *Random Perturbations of Dynamical Systems*, 2nd Ed., Springer-Verlag, New York.

Kovaleva A. (2005), Control of random rocking dynamics of block-like structures. *Proceedings of the 3rd European Conference on Structural Control, July 2004*. TU Wien, *Vienna, Austria.*

Kovaleva A. (2006a), "Large deviations estimates of escape time for Lagrangian systems", *Proceedings of 44th IEEE Conference on Decision and Control*, Seville, Spain.

Kovaleva A. (2006b), "Solution of the exit time problem for mechanical systems with fast noise", *Probabilistic Engineering Mechanics* (*in press*).

Kushner, H. J. (1984a), *Approximation and Weak Convergence Methods for Random Processes, with Applications to Stochastic System Theory*, The MIT Press, Cambridge, MA.

Kushner, H. J. (1984b), "Robustness and approximation of escape time and large deviations estimates for systems with small noise effect", *SIAM J. Appl. Math.*, **44**, 160-182.

FC