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Modeling of fractional magneto-thermoelasticity for a perfect conducting materials

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Abstract. A unified mathematical model of the equations of generalized magneto-thermoelasticty based on fractional derivative heat transfer for isotropic perfect conducting media is given. Some essential theorems on the linear coupled and generalized theories of thermoelasticity e.g., the Lord- Shulman (LS) theory, Green-Lindsay (GL) theory and the coupled theory (CTE) as well as dual-phase-lag (DPL) heat conduction law are established. Laplace transform techniques are used. The method of the matrix exponential which constitutes the basis of the state-space approach of modern theory is applied to the non-dimensional equations. The resulting formulation is applied to a variety of one-dimensional problems. The solutions to a thermal shock problem and to a problem of a layer media are obtained in the present of a transverse uniform magnetic field. According to the numerical results and its graphs, conclusion about the new model has been constructed. The effects of the fractional derivative parameter on thermoelastic fields for different theories are discussed.

Keywords: generalized magneto-thermoelasticity; fractional calculus; laplace transforms; state space approach; numerical results

1. Introduction

Mathematical modeling is the process of constructing mathematical objects whose behaviors or properties correspond in some way to a particular real-world system. The term real-world system could refer to a physical system, a financial system, a social system, an ecological system, or essentially any other system whose behaviors can be observed. In this description, a mathematical object could be a system of equations, a stochastic process, a geometric or algebraic structure, an algorithm or any other mathematical apparatus like a fractional derivative, integral or fractional system of equations. The fractional calculus and the fractional differential equations are served as mathematical objects describing many real-world systems.

According to Duhamel (1937), the classical uncoupled theory of thermoelasticity predicts two phenomena not compatible with physical observations. First, the equation of heat conduction of this theory does not contain any elastic terms; second, the heat equation is of a parabolic type, predicting infinite speeds of propagation for heat waves. Biot (1956) proposed the coupled theory

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of thermoelasticity to overcome the first shortcoming in the uncoupled theory. The equations of elasticity and of heat conduction for this theory are coupled, eliminating the first paradox of the classical uncoupled theory. However, both theories share the second shortcoming since the heat equation for the coupled theory is also parabolic.

Cattaneo’s theory (1958) allows for the existence of thermal waves, which propagate at finite speeds. Starting from Maxwell’s idea (Truesdell and Muncaster 1980) and from the paper (Cattaneory 1958), an extensive amount of literature (Glass and Vick 1985, Joseph and Preziosi 1989, Dreyer and Struchtrup 1993) has contributed to the elimination of the paradox of instantaneous propagation of thermal disturbances. The approach used is known as extended irreversible thermodynamics, which introduces time derivative of the heat flux vector, Cauchy stress tensor and its trace into the classical Fourier law by preserving the entropy principle. A history of heat conduction also appears in the review article (Lord and Shulman 1967). They discussed low temperature heat propagation in dielectric solids where second sound effects are present.

Several generalizations to the coupled theory of thermoelasticity are introduced. The mathematical aspects of Lord and Shulman (1967) theory are explained and illustrated in detail in the works of Ignaczak and Ostoja-starzeweski (2009) and Joseph and Preziosi (1990) state that the Cattaneo (1958) heat conduction law is the most obvious and simple generalization of the Fourier law that gives rise to a finite propagation speed. One can refer to Ignaczak (1989) and to Chandrasekharaiah (1998) for a review, presentation of generalized theories. Hetnarski and Ignaczak (1999) in their survey article examined five generalizations to the coupled theory and obtained a number of important analytical results. Hetnarski and Eslami (2009) introduced a unified generalized thermoelasticity theory and present advanced theory and applications of classical thermoelasticity, generalized thermoelasticity, and mathematical and mechanical background of thermodynamics and theory of elasticity as well. The uniqueness theorem for generalized thermo-viscoelasticity with one relaxation time under different conditions is proved by Ezzat an El-Karamany (2002a, 2003a). Sherief (1986) obtained the fundamental solution of this theory. El-Karamany and Ezzat (2002) introduced a formulation of the boundary integral equation method for generalized thermo-viscoelasticity with one relaxation time. El-Karamany and Ezzat (2004_a, 2005) and Ezzat and El-Karamany (2006) investigated the propagation of discontinuities of solutions in this theory.

Generalizations of thermoelasticity theory with two relaxation times was made by Green and Lindsay (1972) who obtained an explicit version of the constitutive equations and Șuhubi (1975) obtained independently these equations. The fundamental solutions for this theory are obtained by Sherief (1992), while the generalizations of thermo-viscoelasticity with two relaxation times have been made: Ezzat and El-Karamany (2002b) proved the uniqueness and reciprocity theorems for anisotropic media, El-Karamany and Ezzat (2004b) introduced a formulation of the boundary integral equation method for generalized thermoviscoelasticity with two relaxation times , Ezzat *et al*. (2002) introduced the model of the two-dimensional equations of generalized thermo-viscoelasticity with two relaxation times and solved some problems by using state space approach (Ezzat and El-Karamany 2012, Ezzat *et al*. 2004). Othman *et al*. (2002) used normal mode analysis to solve three different concrete problems in this theory. Ezzat *et al*. (2003) introduced the model of the equations of generalized thermo-viscoelasticity with one, when the relaxation effects of the volume properties of the material are taken into account, respectively.

Tzou (1995) proposed a dual-phase-lag heat conduction law in which two different phase-lags: one for the heat flux vector and other for the temperature gradient have been introduced in the

Fourier law to capture the micro-structural effects for heat transport mechanism into the delayed response in time in the macroscopic formulation. For more on the dual-phase-lag model are found in references Quintanilla and Racke (2006), Horgan and Quintanilla (2005), Jou and Criado-Sancho (1998), and El-Karamny and Ezzat (2014, 2004c).

The foundation of magnetoelasticitywas presented by Knopoff (1955) and Chadwick (1960) and developed by Kaliski and Petykiewicz (1959). Increasing attention is being devoted to the interaction between magnetic field and strain field in a thermoelastic solid due to its many applications in the fields of geophysics, plasma physics and related topics. In the preceding references, it was assumed that the interactions between the two fields take place by means of the Lorentz forces appearing in the equations of motion and by means of a term entering Ohm's law and describing the electric field produced by velocity of a material charge, moving in a magnetic field. Increasing attention is being devoted to the interaction between magnetic fields and strain in a thermoelastic solid due to its many applications in the fields of geophysics, plasma physics, and related topics. In the nuclear field, the extremely high temperatures and temperature gradients as well as the magnetic fields originating inside nuclear reactors influence their design and operation (Nowinski 1978). This is the domain of the theory of magneto-thermoelasticity. It is the combination of two different disciplines: those of the theories of electromagnetism and thermoelasticity. Among the authors who considered the generalized magneto-thermoelasticity equations are Nayfeh and Nasser (1973) who studied the propagation of plane waves in a solid under the influence of an electromagnetic field. Choudhuri (1984) extend these results to rotating media. El Karamany and Ezzat (2009) and Ezzat and Awad (2010) proved the uniqueness and reciprocal theorems in linear micropolar electro-magnetic thermoelasticity. Sherief and Ezzat (1998) solved a problem for an infinitely long annular cylinder, while Ezzat (1997, 2001, 2006), Ezzat and Othman (2002) and Ezzat and El Karamany (2003_b, 2006) solved some problems for a perfect conducting media. Zenkour and Abbas (2015) introduced electro-magneto-thermo-elastic analysis problem of an infinite functionally graded hollow cylinder is studied in the context of Green–Naghdi's generalized thermoelasticity theory (without energy dissipation).

Differential equations of fractional order have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering. The most important advantage of using fractional differential equations in these and other applications is their non-local property. It is well known that the integer order differential operator is a local operator but the fractional order differential operator is non-local. This means that the next state of a system depends not only upon its current state but also upon all of its historical states. This is more realistic and it is one reason why fractional calculus has become more and more popular. Fractional calculus has been used successfully to modify many existing models of physical processes. One can state that the whole theory of fractional derivatives and integrals was established in the 2nd half of the 19th century. The first application of fractional derivatives was given by Abel who applied fractional calculus in the solution of an integral equation that arises in the formulation of the tautochrone problem. The generalization of the concept of derivative and integral to a non-integer order has been subjected to several approaches and some various alternative definitions of fractional derivatives appeared. In the last few years fractional calculus was applied successfully in various areas to modify many existing models of physical processes, e.g., chemistry, biology, modeling and identification, electronics, wave propagation and viscoelasticity. Caputo (1974) found good agreement with experimental results when using fractional derivatives for description of viscoelastic materials and established the connection between fractional derivatives and the theory of linear viscoelasticity. Adolfsson *et al*.

(2005) constructed a newer fractional order model of viscoelasticity.

Recently, Ezzat (2010, 2011a,b,c, 2012) established a new model of fractional heat conduction equation using the Taylor-Riemann series expansion of time-fractional order. Sherief *et al*. (2010) introduced a fractional formula of heat conduction and proved a uniqueness theorem and derived a reciprocity relation and a variational principle. El.-Karamany and Ezzat (2011a) introduced two general models of fractional heat conduction law for a non-homogeneous anisotropic elastic solid. El.-Karamany and Ezzat (2011b) proved uniqueness and reciprocal theorems and established the convolutional variational principle and used to prove a uniqueness theorem with no restriction on the elasticity or thermal conductivity tensors except symmetry conditions. One can refer to Ezzat *et al*. (2010, 2011a, b, c, 2012) for a survey of applications of fractional calculus. Abbas (2015) considered the problem of fractional order thermoelastic interaction in a material placed in a magnetic field and subjected to a moving plane of heat source.

The purpose of the present article is to introduce a unified mathematical model for the linear theory of thermoelasticity by using the methodology of fractional calculus theory based on the generalized theories. For this model we shall formulate the state space approach developed in Refs. (Ezzat *et al*. 1999, Ezzat and Youssef 2010). The resulting formulation is applied to specific one-dimensional problems for a perfect electrically conducting medium in the presence of a constant magnetic field. Laplace-transform technique is used throughout. The inversion of the transforms is carried out using a numerical inversion technique proposed by Honig and Hirdes (1984). Numerical results for the temperature; the stress and displacement distributions are given and illustrated graphically for given problems. Comparisons are made with the results predicted by the four theories and the unified model.

2. The mathematical model

We shall consider a prefect conducting thermoelastic medium permeated by an initial magnetic field $\boldsymbol{H}$. This produces an induced magnetic field $\boldsymbol{h}$ and induced electric field $\boldsymbol{E}$, which satisfy the linearized equations of electromagnetism and are valid for slowly moving media (Ezzat 1997):

The first set of equations constitutes the equations of electrodynamics of slowly moving bodies

$$\mathrm{curl}\,\boldsymbol{h} = \boldsymbol{J} + \varepsilon_o \frac{\partial \boldsymbol{E}}{\partial t} \tag{1}$$

$$\mathrm{curl}\,\boldsymbol{E} = -\mu_o \frac{\partial \boldsymbol{h}}{\partial t} \tag{2}$$

$$\boldsymbol{E} = -\mu_o \left(\frac{\partial \boldsymbol{u}}{\partial t} \wedge \boldsymbol{H}\right) \tag{3}$$

$$\mathrm{div}\,\boldsymbol{h} = 0 \tag{4}$$

Here the vectors $\boldsymbol{h}$ and $\boldsymbol{E}$ denote perturbations of the magnetic and electric fields, respectively, $\boldsymbol{J}$ is the electric current density vector, $\boldsymbol{H}$ the initial constant magnetic field, $\boldsymbol{u}$ the displacement vector.

The second group of equations is the equations of motion

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \sigma_{ij,j} + T_{ij,j} + X_i \tag{5}$$

where σ_{ij} is the stress tensor represents the fractional constitutive equation (Hamza *et al*. 2014)

$$\sigma_{ij} = \lambda e_{kk}\, \delta_{ij} + 2\mu e_{ij} - \gamma\left(\Theta + \frac{\upsilon^\alpha}{\alpha!}\frac{\partial^\alpha \Theta}{\partial t^\alpha}\right)\delta_{ij}, \qquad 0 < \alpha \le 1 \tag{6}$$

and T_{ij} the Maxwell electromagnetic stress tensor related to the quantity h in the following manner (Ezzat 2006)

$$T_{ij} = \mu_o\left[H_i\, h_j + H_j\, h_i - \delta_{ij}\,(h_k\, H_k)\right] \tag{7}$$

so that the quantity $T_{ij,j} = \mu_o \in_{ijk} J_j H_k$ is the $i-$ component of the Lorentz force,
The above equations should be supplemented by the relations between strain and displacements

$$e_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{8}$$

and the generalized fractional heat conduction equation

$$\begin{aligned} k\left(1 + \frac{\tau_\theta^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)\Theta_{,ii} &= \rho C_E \frac{\partial}{\partial t}\left(1 + \frac{\tau^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha}}{2\alpha!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\Theta \\ &+ \gamma T_o \frac{\partial}{\partial t}\left(1 + n\frac{\tau^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha}}{2\alpha!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)e, \qquad 0 < \alpha \le 1 \end{aligned} \tag{9}$$

In the above equations a comma denotes material derivatives and the summation convention are used.

The previous equations constitute a complete system of unified generalized magneto-thermoelasticity with fractional derivative heat transfer for a perfect conducting medium.

2.1 Limiting cases

1- Coupled thermoelasticity theory (CTE)

The model of Eqs. (6)-(9) in the limiting case, $\tau_\theta = \tau_q = \tau = \upsilon = n = 0, \alpha = 0$, transforms to the work of Biot (1956).

2- Generalized thermoelasticity theory (LS Theory)

The model of Eqs. (6)-(9) in the limiting case, $\alpha \to 1, \tau_\theta = \tau_q = \upsilon = 0,\ \tau > 0, n = 1,$, transforms to the works of Lord and Shulman (1967), Glass and Vick (1985), Joseph and Preziosi (1990), Ignaczak (1989), and Sherief (1986) in thermoelasticity with one relaxation time.

3- Generalized thermoelasticity theory (GL Theory)

The model of Eqs. (6)-(9) in the limiting case, $\alpha \to 1, \tau_q = \tau_\theta = n = 0, \upsilon \ge \tau > 0,$, transforms to the works of Green and Lindsay (1972), Șuhubi (1975) and Shereif (1992) in thermoelasticity with two relaxation times.

4- Generalized thermoelasticity theory (DFL Theory)

The model of Eqs. (6)-(9) in the limiting case $\alpha \to 1, \tau_\theta > \tau_q > 0, \tau = \tau_q, \upsilon = 0, n = 1,$, transforms to the work of Tzou (1995), Dreyer and Struchtrup (1993), Quintanilla and Racke (2006) and El-Karamany and Ezzat (2014) in dual-phase-lag thermoelasticity.

5- Generalized fractional thermoelasticity based on LS theory

The model of Eqs. (6)-(9) in the limiting case $\tau_\theta = \upsilon = 0, n = 1, \tau_q = \tau > 0, 1 > \alpha > 0,$ transforms to the works of Ezzat (2010, 2011a,b,c, 2012), El-Karamany and Ezzat (2011a,b), Ezzat and El-Karamany (2011a,b,c), Sherief *et al*. (2010), Ezzat and El-Bary (2012).

6- Generalized fractional thermoelasticity based on GL theory

The model of Eqs. (6)-(9) in the limiting case $\tau_\theta = \tau_q = 0, n = 0, \ \upsilon \geq \tau > 0, \ 1 > \alpha > 0,$ transforms to the work of Hamza *et al*. (2014).

7- Generalized fractional thermoelasticity based on DFL theory

The model of Eqs. (6)-(9) in the limiting case $\tau_\theta > \tau_q > 0, \tau = \tau_q, \upsilon = 0, \ n = 1, \ 1 > \alpha > 0,$ transforms to the works of Ezzat *et al*. (2012).

3. Physical problem

Now, we shall consider an infinite homogeneous isotropic perfect conducting thermo-viscoelastic medium permeated by an initial magnetic field $\boldsymbol{H} \equiv (0, 0, H_o)$ occupying the region $x \geq 0$, which is initially quiescent and where all the state functions depend only on the dimension x and the time t. The x-axis is taken perpendicular to the bounding plane pointing inwards. Due to the effect of this magnetic field there arises in the conducting medium an induced magnetic field $\boldsymbol{h} \equiv (0, 0, h)$ and induced electric field $\boldsymbol{E} \equiv (0, E, 0)$. Also, there arises a force $\boldsymbol{F}$ (the Lorentz Force). Due to the effect of the force, points of the medium undergo a displacement $\boldsymbol{u} \equiv (u, 0, 0)$, which gives rise to a temperature. The system of fractional magneto-thermoelasticity for a medium with a perfect electric conductivity can be written as:

The displacement vector has components

$$u_x = u(x,t), \quad u_y = u_z = 0 \tag{10}$$

The strain component takes the form

$$e = \varepsilon_{xx} = \frac{\partial u}{\partial x} \tag{11}$$

The linearzed equations of electromagnetism for a perfect conducting medium

$$J = -\left(\frac{\partial h}{\partial x} + \varepsilon_o \mu_o H_o \frac{\partial^2 u}{\partial t^2} \right) \tag{12}$$

$$h = -H_o \frac{\partial u}{\partial x} \tag{13}$$

$$E = \mu_o H_o \frac{\partial u}{\partial t} \tag{14}$$

The equation of motion takes the form

$$(\rho + \varepsilon_o \mu_o^2 H_o^2)\frac{\partial^2 u}{\partial t^2} = (\lambda + 2\mu)\frac{\partial^2 u}{\partial x^2} + \mu_o H_o^2 \frac{\partial^2 u}{\partial x^2} - \gamma \frac{\partial}{\partial x}\left(1 + \frac{\upsilon^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)\Theta, \quad 0 < \alpha \le 1 \tag{15}$$

The fractional constitutive equation yield

$$\sigma_{xx} = (\lambda + 2\mu)\frac{\partial u}{\partial x} - \gamma\left(\Theta + \frac{\upsilon^\alpha}{\alpha!}\frac{\partial^\alpha \Theta}{\partial t^\alpha}\right), \qquad 0 < \alpha \le 1 \tag{16}$$

The generalized fractional heat equation is given by

$$\begin{aligned} k\left(1 + \frac{\tau_\theta^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial^2 \Theta}{\partial x^2} &= \rho C_E \frac{\partial}{\partial t}\left(1 + \frac{\tau^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha}}{2\alpha!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\Theta \\ &+ \gamma T_o \frac{\partial}{\partial t}\left(1 + n\frac{\tau^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha}}{2\alpha!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)e, \qquad 0 < \alpha \le 1 \end{aligned} \tag{17}$$

Let us introduce the following non-dimensional variables

$$x^* = C_o \eta_o x, \; u^* = C_o \eta_o u, \; t^* = C_o^2 \eta_o t, \; \tau^* = C_o^2 \eta_o \tau, \; \upsilon^* = C_o^2 \eta_o \upsilon, \; \Theta^* = \frac{\gamma \Theta}{\rho C_o^2},$$

$$\varepsilon = \frac{\delta_o \gamma}{\rho C_E}, \; \sigma^* = \frac{\sigma}{\lambda + 2\mu}, \; h^* = \frac{h}{H_o}, \; E^* = \frac{E}{\mu_o H_o C_o}.$$

In terms of these non-dimensional variables, we have (dropping asterisks for convenience)

$$h = -\frac{\partial u}{\partial x} \tag{18}$$

$$E = \frac{\partial u}{\partial t} \tag{19}$$

$$\begin{aligned} \left(1 + \frac{\tau_\theta^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)\frac{\partial^2 \Theta}{\partial x^2} &= \frac{\partial}{\partial t}\left(1 + \frac{\tau^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha}}{2\alpha!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)\Theta \\ &+ \varepsilon \frac{\partial^2}{\partial x \partial t}\left(1 + n\frac{\tau^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha}}{2\alpha!}\frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right)u, \qquad 0 < \alpha \le 1 \end{aligned} \tag{20}$$

$$\left(1+\frac{\alpha_o^2}{c_o^2}\right)\frac{\partial^2 u}{\partial t^2} = \left(1+\alpha_o^2\right)\frac{\partial^2 u}{\partial x^2} - \frac{\partial}{\partial x}\left(1 + \frac{\upsilon^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)\Theta, \quad 0<\alpha\le 1 \tag{21}$$

$$\sigma_{xx} = \frac{\partial u}{\partial x} - \left(1 + \frac{\upsilon^\alpha}{\alpha!}\frac{\partial^\alpha}{\partial t^\alpha}\right)\Theta, \quad 0<\alpha\le 1 \tag{22}$$

where $c_o = 1/\sqrt{\varepsilon_o \mu_o}$ is light speed.

Taking Laplace transform, defined by the relations

$$\left.\begin{aligned} \mathrm{L}\{g(t)\} &= \bar{\mathrm{g}}(s) = \int_0^\infty \mathrm{e}^{-st}\mathrm{g}(t)\,\mathrm{d}t \\ \mathrm{L}\{D^{\mathrm{n}}\mathrm{g}(t)\} &= s^{\mathrm{n}}\,\mathrm{L}\{\mathrm{g}(t)\} \end{aligned}\right\}, \quad s>0$$

of both sides Eqs. (18)-(22), with quiescent initial conditions, we obtain

$$\bar{h} = -D\bar{u} \tag{23}$$

$$\bar{E} = s\bar{u} \tag{24}$$

$$\left(\frac{\partial^2}{\partial x^2} - a\right)\bar{\Theta} = b\,\varepsilon\frac{\partial \bar{u}}{\partial x} \tag{25}$$

$$\left(\frac{\partial^2}{\partial x^2} - cs^2\right)\bar{u} = d\,\frac{\partial \bar{\Theta}}{\partial x} \tag{26}$$

$$\bar{\sigma}_{xx} = \frac{\partial \bar{u}}{\partial x} - \left(1+\frac{\upsilon^\alpha}{\alpha!}s^\alpha\right)\bar{\Theta} \tag{27}$$

where

$$a = s\left(1+\frac{\tau^\alpha}{\alpha!}s^\alpha + \frac{\tau_q^{2\alpha}}{2\alpha!}s^{2\alpha}\right)\Big/\left(1+\frac{\tau_\theta^\alpha}{\alpha!}s^\alpha\right),\ b = s\left(1+n\,\frac{\tau^\alpha}{\alpha!}s^\alpha + \frac{\tau_q^{2\alpha}}{2\alpha!}s^{2\alpha}\right)\Big/\left(1+\frac{\tau_\theta^\alpha}{\alpha!}s^\alpha\right),$$

$$c = \left(1+\frac{\alpha_o^2}{c_o^2}\right)\Big/\left(1+\alpha_o^2\right) \text{ and } d = \left(1+\frac{\tau_\upsilon^\alpha}{\alpha!}s^\alpha\right)\Big/\left(1+\alpha_o^2\right)$$

4. State space approach

We shall choose as state variables the temperature increment $\bar{\Theta}(x,s)$ and the displacement component $\bar{u}(x,s)$ in the *x*-direction. Eqs. (25) and (26) can be written in the

$$\frac{\mathrm{d}^2\bar{G}(x,s)}{\mathrm{d}x^2} = A(s)\bar{G}(x,s) \tag{28}$$

where

$$A(s) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ a & 0 & 0 & b\varepsilon \\ 0 & cs^2 & d & 0 \end{bmatrix}, \quad \bar{G}(x,s) = \begin{bmatrix} \bar{\Theta} \\ \bar{u} \\ \bar{\Theta}' \\ \bar{u}' \end{bmatrix}$$

The formal solution of the system (39) can be written as

$$\bar{G}(x,s) = \exp[A(s)]\cdot\bar{G}(0,s) \tag{29}$$

We shall use the well-known Cayley-Hamilton theorem to find the form of the matrix $\exp[A(s)]$. The characteristic equation of the matrix $A(s)$ can be written as

$$k^4 - [\mathrm{c}s^2 + a + bd\,\varepsilon]k^2 + acs^2 = 0 \tag{30}$$

where k_i, $i = 1,2,3,4$ are a characteristic roots. The Cayley-Hamilton theorem states that the matrix A satisfies its own characteristic equation in the matrix sense. Therefore, it follows that

$$A^4 - [\mathrm{c}s^2 + a + bd\,\varepsilon]A^2 + acs^2 I = 0. \tag{31}$$

Eq. (31) shows that A^4 and all higher powers of A can be expressed in terms of I, A, A^2 *and* A^3, the unit matrix of order 4. The matrix exponential can now be written in the form

$$\exp[Ax] = a_o(x,s)I + a_1(x,s)A(s) + a_2(x,s)A^2(s) + a_3(x,s)A^3(s) \tag{32}$$

The scalar coefficients of Eq. (56) are now evaluated by replacing the matrix A by its characteristic roots $\pm k_1$ and $\pm k_2$, which are the roots of the biquadratic Eq. (30), satisfying the relations

$$k_1^2 + k_2^2 = \mathrm{c}s^2 + a + bd\,\varepsilon \tag{33a}$$

$$k_1^2\, k_2^2 = acs^2 \tag{33b}$$

This leads to the system of equations

$$\exp(\pm k_1 x) = a_o \pm a_1 k_1 + a_2 k_1^2 \pm a_3 k_1^3 \tag{34a}$$

$$\exp(\pm k_2 x) = a_o \pm a_1 k_2 + a_2 k_2^2 \pm a_3 k_2^3 \tag{34b}$$

By solving the system of linear Eqs. (34), we can determine $a_o - a_3$ (See Appendix A).

Substituting for the parameters $a_o - a_3$ into Eq. (32), computing A^2 and A^3 and using Eqs. (33(a)) and (33(b)), one can obtain after some lengthy algebraic manipulation

$$\exp\left[A(s).x\right] = L(x,s) = [\ell_{ij}(x,s)], \qquad i,j = 1,2,3,4 \tag{35}$$

where the elements $\ell_{ij}(\mathrm{x,s})$ are given in Appendix B.

In the actual physical problem the space is divided into two regions accordingly as $x \geq 0$ or $x < 0$, inside the region $0 \leq \mathrm{x} < \infty$ the positive exponential terms, not bounded at infinity, must be suppressed. Thus, for $x \geq 0$ we should replace each $\sinh(kx)$ by $-\frac{1}{2}\exp(-kx)$ and each $\cosh(kx)$ by $\frac{1}{2}\exp(-kx)$ and the negative exponentials in the region $x \leq 0$ are suppressed instead.

5. Applications

5.1 Problem I: Prescribed boundary conditions

We shall consider a viscoelastic conducting medium occupying a semi-infinite region $x \geq 0$.
The boundary conditions will be
(i) Thermal boundary condition:
A thermal shock is applied to the boundary plane x = 0 in the form

$$\Theta(0,t) = \Theta_o H(t)\text{, or } \bar{\Theta}(0,\mathrm{s}) = \frac{\Theta_o}{\mathrm{s}} \tag{36}$$

where Θ_o is a constant and $H(t)$ is the Heaviside unit step function.

(i) Mechanical boundary condition:
The bounding plane $x = 0$ is taken to be traction-free, i.e.

$$\sigma(0,\ t) + T_{11}(0,t) - T_{11}^{o}(0,t) = 0 \tag{37}$$

where T_{11}^{o} is the Maxwell stress tensor in a vacuum.

Since the transverse components of the vectors E and h are continuous across the bounding plane, i.e., $E(0, t) = E^{o}(0, t)$ and $h(0, t) = h^{o}(0, t)$, $t > 0$, where E^{o} and h^{o} are the components of the induced electric and magnetic field in free space and the relative permeability is very nearly unity, it follows that $T_{11}(0,t) = T_{11}^{o}(0,t)$ and Eq. (37) reduces to

$$\sigma(0,t) = 0\,, \qquad \text{or} \qquad \bar{\sigma}(0,s) = \bar{\sigma}_0 = 0 \tag{38}$$

and from Eqs. (38), (36) and (27), one can get

$$\bar{u}'(0,s) = \frac{\Theta_o}{s}\left(1 + \frac{\upsilon^{\alpha}}{\alpha!}s^{\alpha}\right) \tag{39}$$

The two remaining components $\bar{\Theta}'(0,\mathrm{s})$ and $\bar{u}(0,s)$ can be obtained from Eq. (29) by substituting $x = 0$ in both sides and performing the necessary matrix operations, we obtain a system of linear algebraic equations, whose solutions are

$$\bar{u}(0,s) = -\frac{\Theta_o(1+\frac{\upsilon^\alpha}{\alpha!}s^\alpha)}{s(k_1+k_2)} \tag{40}$$

$$\bar{\Theta}'(0,s) = -\Theta_o \frac{k_1^2 + k_1k_2 + k_2^2 - cs^2}{s(k_1-k_2)} \tag{41}$$

Inserting the values from (38)-(41) into the right hand side of Eq. (29), we obtain upon using Eqs. (33)

$$\bar{\Theta}(x,s) = \Theta_o \left[\frac{(k_1^2 - cs^2)\,\mathrm{e}^{-k_1x} - (k_2^2 - cs^2)\,\mathrm{e}^{-k_2x}}{s(k_1^2 - k_2^2)} \right] \tag{42}$$

$$\bar{u}(x,s) = -\Theta_o(1+\frac{\upsilon^\alpha}{\alpha!}s^\alpha)\left[\frac{k_1\,\mathrm{e}^{-k_1x} - k_2\,\mathrm{e}^{-k_2x}}{s\left(k_1^2 - k_2^2\right)} \right] \tag{43}$$

$$\bar{\sigma}(x,s) = \Theta_o cs(1+\frac{\upsilon^\alpha}{\alpha!}s^\alpha)\left[\frac{\mathrm{e}^{-k_1x} - \mathrm{e}^{-k_2x}}{k_1^2 - k_2^2} \right] \tag{44}$$

Putting Eq. (43) into Eqs. (23) and (24) the induced magnetic and electric fields take the following forms

$$\bar{h}(x,s) = \Theta_o(1+\frac{\upsilon^\alpha}{\alpha!}s^\alpha)\left[\frac{k_1^2\,\mathrm{e}^{k_1x} - k_2^2\,\mathrm{e}^{k_2x}}{s(k_1^2 - k_2^2)} \right] \tag{45}$$

$$\bar{E} = -\Theta_o(1+\frac{\upsilon^\alpha}{\alpha!}s^\alpha)\left[\frac{k_1\,\mathrm{e}^{k_1x} - k_2\,\mathrm{e}^{k_2x}}{k_1^2 - k_2^2} \right] \tag{46}$$

This completes the solution of the previous problem in the Laplace transform domain.

5.2 Problem II: A problem for a layered medium

We consider, now, a perfectly conducting medium occupying the region $0 \le x \le X$ and resting on non- conducting rigid at a plane and the surface $x = 0$ is taken as traction free, then

$$\sigma(0,t) = 0, \qquad \text{or} \qquad \bar{\sigma}(0,s) = 0 \tag{47}$$

and subjected to a thermal shock

$$\Theta(0,t) = \Theta_o H(t), \qquad \text{or} \qquad \bar{\Theta}(0,\mathrm{s}) = \frac{\Theta_o}{s} \tag{48}$$

At the rigid base $x = X$

$$u(X, t) = 0, \qquad \text{or} \qquad \bar{u}(X, s) = 0 \tag{49}$$

and

$$q(X, t) = 0, \qquad \text{or} \qquad \bar{q}(X, s) = 0 \tag{50}$$

where q denotes the components of the heat flux vector normal to the surface of the layer. Condition Eq. (50) means that the rigid base is thermally insulated. From Fourier law of heat conduction Eq. (50) reduces to

$$\bar{\Theta}'(X, s) = 0 \tag{51}$$

Eqs. (27), (47) and (48) can be combined to give

$$\bar{u}'(0, s) = \frac{\Theta_o}{s}(1 + \frac{\upsilon^\alpha}{\alpha!} s^\alpha) \tag{52}$$

We use Eq. (29) between $x = 0$ and $x = X$ to get

$$\bar{u}(0, s) = -\Theta_o(1 + \frac{\upsilon^\alpha}{\alpha!} s^\alpha)\left[\frac{k_1 \tanh k_1 X - k_2 \tanh k_2 X}{s\left(k_1^2 - k_2^2\right)}\right] \tag{53}$$

$$\bar{\Theta}'(0, s) = -\frac{\Theta_o}{s(k_1 - k_2)}\left[k_1\left(k_1^2 - cs^2\right)\tanh k_1 X - k_2\left(k_2^2 - cs^2\right)\tanh k_2 X\right] \tag{54}$$

Finally, we can find the solutions of the problem as

$$\bar{\Theta}(x, s) = \frac{\Theta_o}{s(k_1^2 - k_2^2)}\left[(k_1^2 - cs^2)\frac{\cosh k_1(X - x)}{\cosh k_1 X} - (k_2^2 - cs^2)\frac{\cosh k_2(X - x)}{\cosh k_2 X}\right] \tag{55}$$

$$\bar{u}(x, s) = -\frac{\Theta_o(1 + \frac{\upsilon^\alpha}{\alpha!} s^\alpha)}{s\left(k_1^2 - k_2^2\right)}\left[k_1 \frac{\sinh k_1(X - x)}{\cosh k_1 X} - k_2 \frac{\sinh k_2(X - x)}{\cosh k_1 X}\right] \tag{56}$$

$$\bar{\sigma}(x, s) = \frac{\Theta_o cs\,(1 + \frac{\upsilon^\alpha}{\alpha!} s^\alpha)}{(k_1^2 - k_2^2)}\left[\frac{\cosh k_1(X - x)}{\cosh k_1 X} - \frac{\cosh k_2(X - x)}{\cosh k_2 X}\right] \tag{57}$$

The induced magnetic and electric fields can be obtained and take the forms

$$\bar{h}(x, s) = \frac{\Theta_o(1 + \frac{\upsilon^\alpha}{\alpha!} s^\alpha)}{s\left(k_1^2 - k_2^2\right)}\left[k_1^2 \frac{\cosh k_1(X - x)}{\cosh k_1 X} - k_2^2 \frac{\cosh k_2(X - x)}{\cosh k_1 X}\right] \tag{58}$$

$$\bar{E}(x,s) = -\frac{\Theta_o\left(1+\frac{\upsilon^{\alpha}}{\alpha!}s^{\alpha}\right)}{k_1^2-k_2^2}\left[k_1\frac{\sinh k_1(X-x)}{\cosh k_1 X} - k_2\frac{\sinh k_2(X-x)}{\cosh k_1 X}\right] \tag{59}$$

This completes the solution of the above problem in the Laplace transform domain.

6. Numerical results and discussions

The method based on a Fourier series expansion proposed by Honig and Hirdes (1984) and is developed in detail in many texts such as Ogata (1967) and Ezzat and Abd Elaal (1997) is adopted to invert the Laplace transform in Eqs. (42)- (46) and (55)- (59).
The values of the material constants required for the calculation are given in Table 1.

The computations were carried out for one value of time, namely t = 0.1 and two different values of relaxation times, namely, τ = 0.02 and υ = 0.04, the orders of the differential fractional order are taken as α = 0.0, 1.0, 0.5. The temperature, stress and displacement distributions are obtained and plotted. The solutions corresponding to *problem I* are shown in Figs. 1-3 while the solutions corresponding to *problem II* are shown in Figs. 4-8. In the first groups of the figures, the solid lines represent the solution obtained in the frame the new unified model of magneto thermoelasticity and other lines represent the different theories.

The effects of the Alfven velocity parameter on all fields are shown in the second groups of the figures. With the help of Mathematica software (Version 6) numerical results have been obtained. Subsequently, a comparative study of analytical and numerical results has been done to analyze the effect of fractional order parameters in details. While doing analysis of analytical and numerical results, we have found following highlighted results:

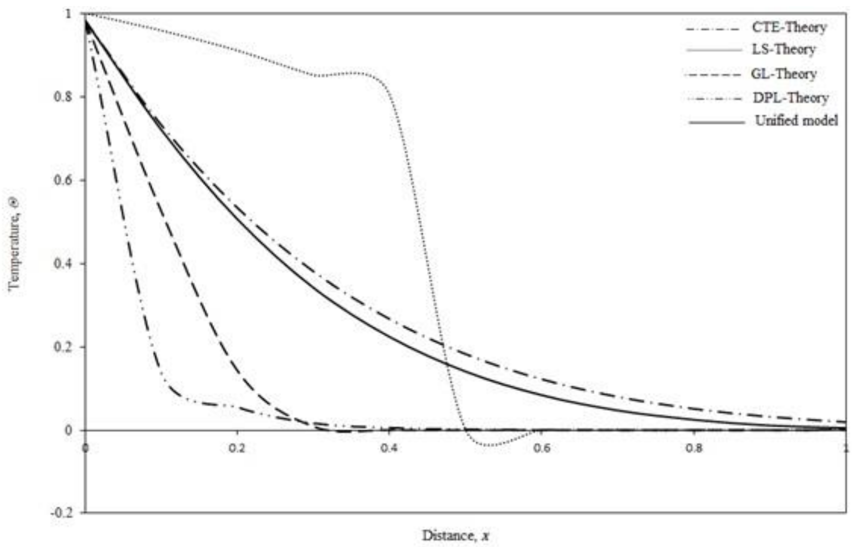

Fig. 1 The variation of temperature for different theories for α = 0.5

Table 1 Material data for copper-like material

$T_o = 293K$	$k = 386\ NK^{-1}s^{-1}$	$\alpha_T = 1.78\times10^{-5}\ K^{-1}$	$C_E = 383.1 m^2 K^{-1}$
$\eta_o = 8886.73 s\ m^{-2}$	$\mu = 3.86\times10^{10} Nm^{-2}$	$\lambda = 7.76\times10^{10} Nm^{-2}$	$\rho = 8954\ kg\ m^{-3}$
$c_o = 4158 ms^{-1}$	$\varepsilon = 0.0168$	$\tau_o = 0.02s$	$B_o = \mu_o H_o = 1T$

In all figures, we notice that

1- The speed of the wave propagation of fractional thermoelastic variable fields according to the new model is finite like the generalized theories and coincides with the physical behaviors of elastic materials.

2- The response to the thermal and mechanical effects does not reach infinity instantaneously but remains in the bounded region of space that expands with the passing of time.

3- It is noticed that the fractional orders α has a significant effect on all fields.

4- In the new unified frame, it is observed that the thermal waves are continuous functions, smooth and reach to steady state depending on the value of α which means that the particles transport the heat to the other particles easily and this makes the decreasing rate of the temperature greater than the other ones.

5- The effects of Alfven velocity on stress, displacement and induced magnetic field as well as induced electric field are discussed in Figs. 5-8. Their effects are more noticeable as shown in these figures. The magnetic field acts to decrease the fields. This is mainly due to the fact that the magnetic field corresponds to a term signifying a positive force that tends to accelerate the charge carriers.

6- In this work, the method of direct integration by means of the matrix exponential, which is a standard approach in modern control theory and developed in many texts (see e.g., Ezzat and El-Karamany 2003), is introduced in the field of electromagneto-thermoelasticity with fractional order heat transfer when the elastic medium is taken as a perfect conductor and applied to one-dimensional problems in which the temperature, displacement and electromagnetic fields are coupled. This method gives exact solution in the Laplace transform domain without any assumed restrictions on either the applied magnetic field or the temperature and displacement fields.

7- The field quantities are sensitive to the variations of different parameters. The method used here may be applicable to a wide range of problems in thermoelasticity and fluids mechanics (Ezzat and Abd Elaal 1997). The numerical results presented here may be considered as more general in the sense that they include the exact analysis Laplace transform domain of different field quantities.

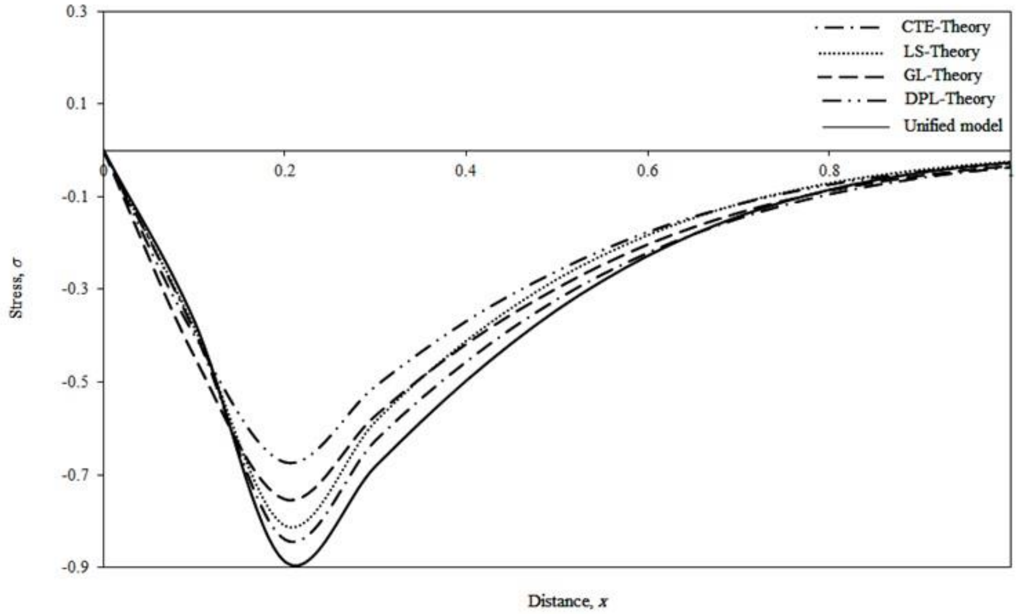

Fig. 2 The variation of stress for different theories for $\alpha = 0.5$

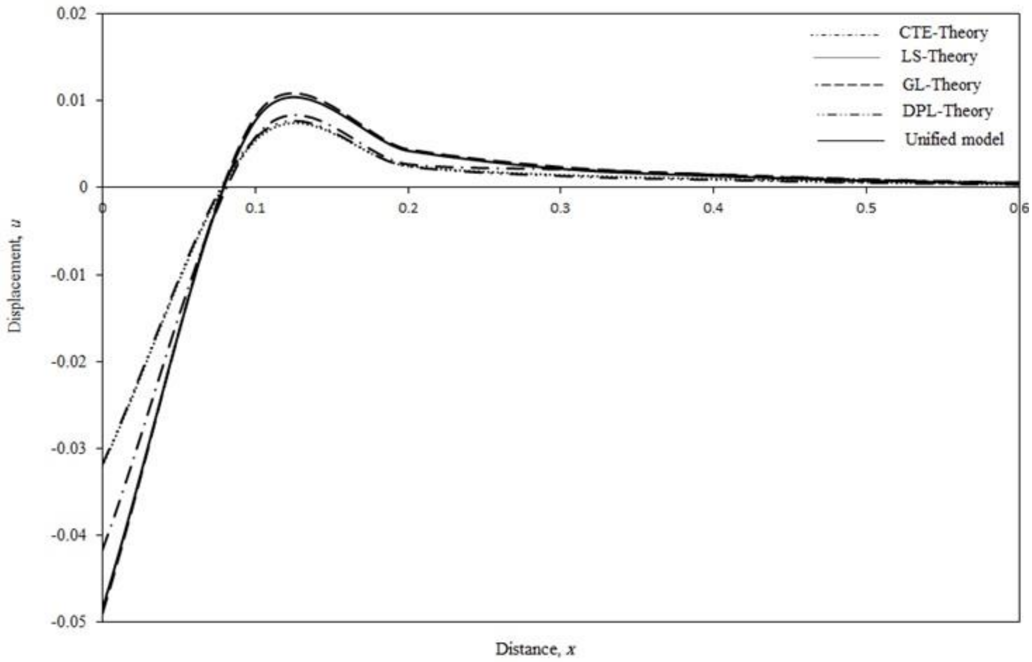

Fig. 3 The variation of displacement for different theories for $\alpha = 0.5$

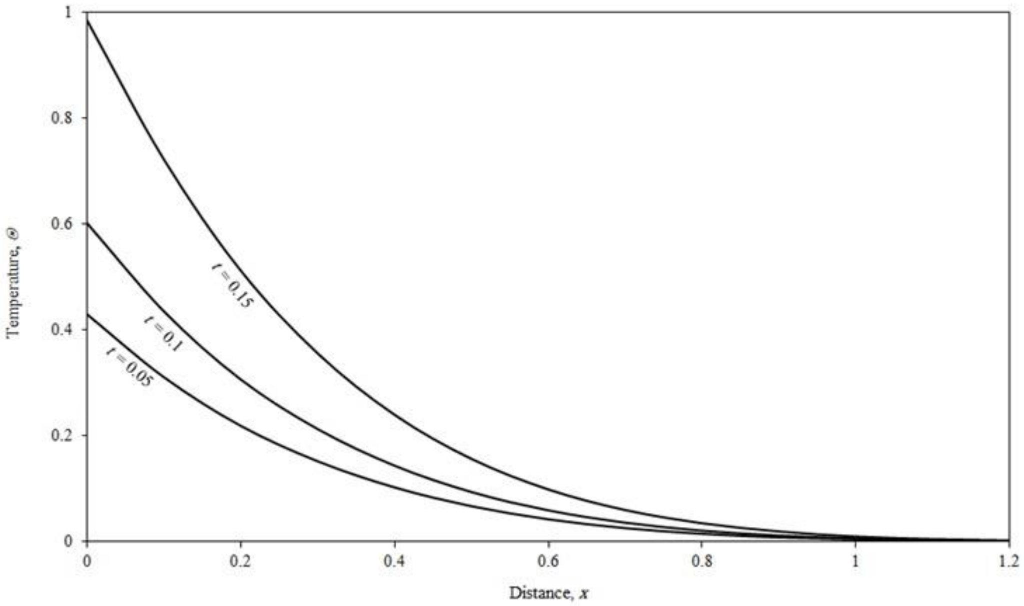

Fig. 4 The variation of temperature for different values of time

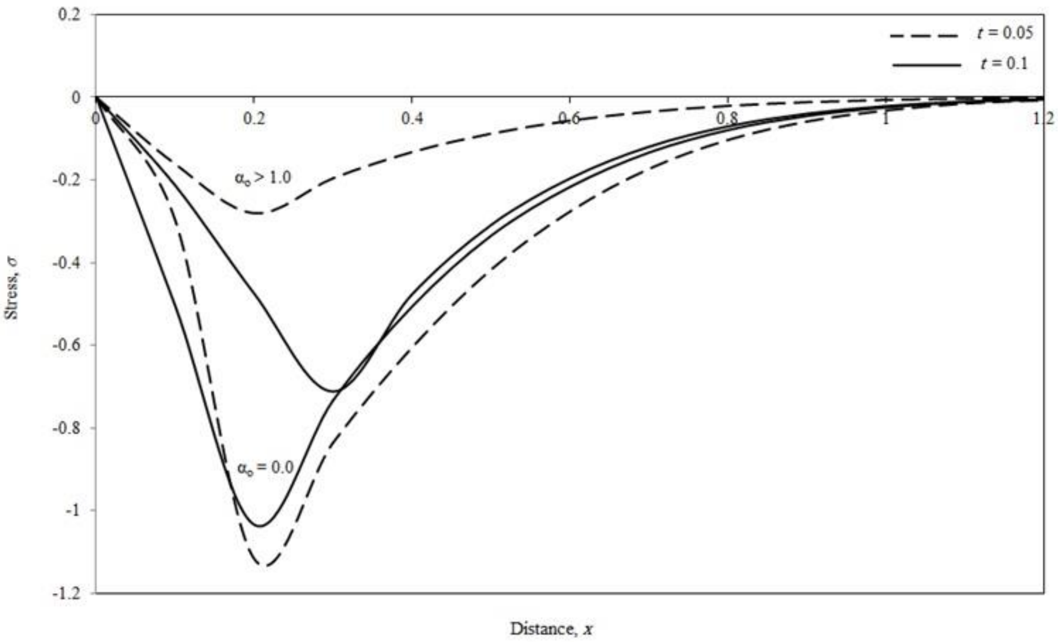

Fig. 5 The variation of stress for different values of Alfven velocity α_0

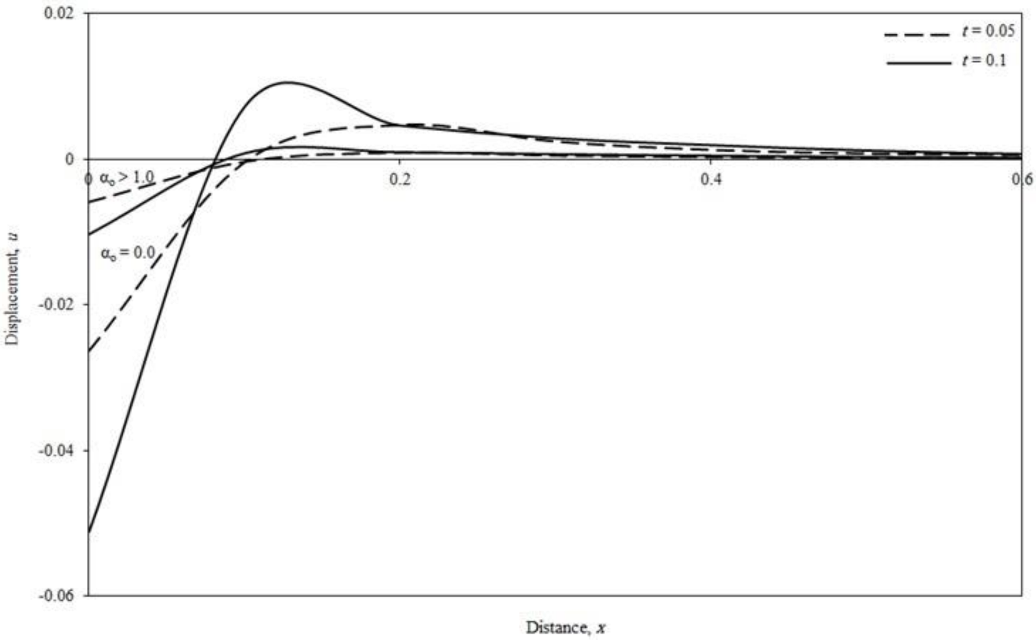

Fig. 6 The variation of stress for different values of Alfven velocity α_0

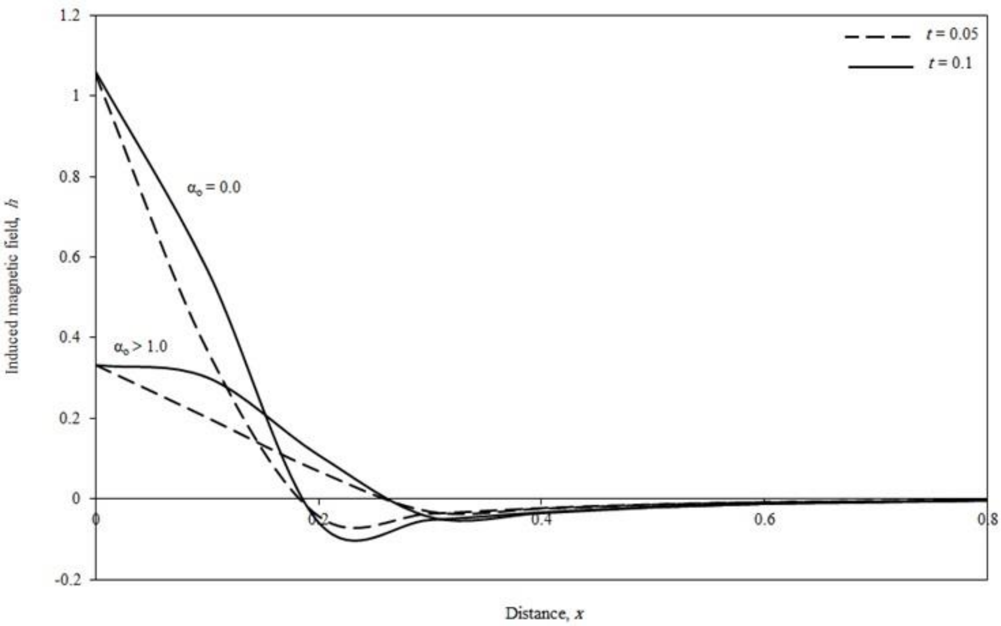

Fig. 7 The variation of induced magnetic field for different values of Alfven velocity α_0

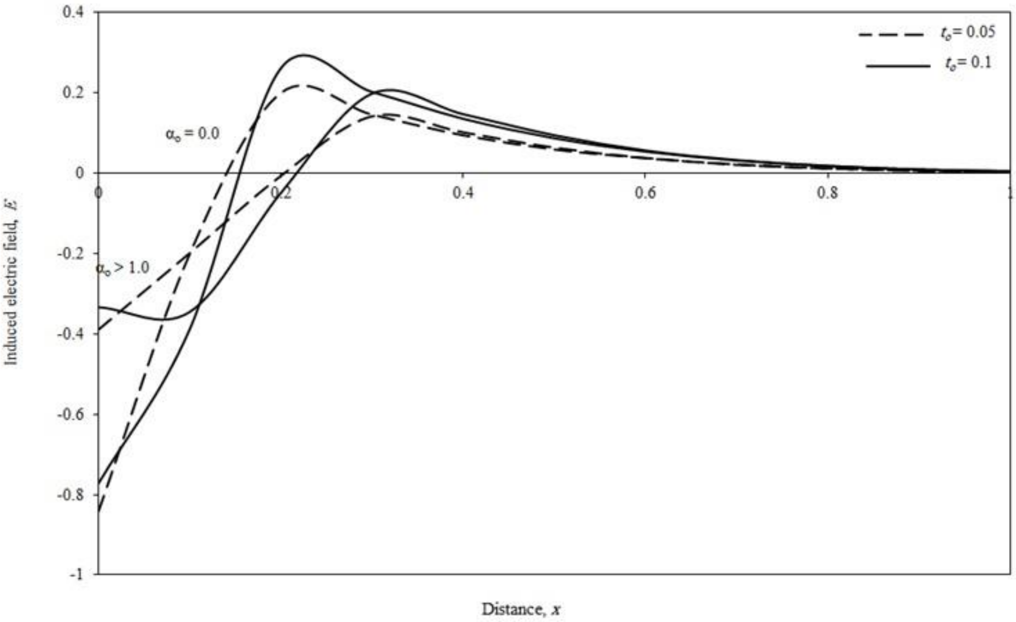

Fig. 8 The variation of induced electric field for different values of Alfven velocity α_0

7. Conclusions

- The main goal of this work is to introduce a unified generalized model for fractional Fourier law of heat conduction.
- From this model we can establish some essential theorems on the linear coupled and generalized theories of thermoelasticity e.g., the Lord- Shulman (LS) theory, Green-Lindsay (GL) theory and the coupled theory (CTE) as well as dual-phase-lag (DPL) heat conduction law and we can compare them.
- The results of all the functions for the new unified model are distinctly different from those obtained for coupled and generalized theories.
- The advantage of the considered unified model consists in:
  i) The discontinuities in temperature distribution disappeared.
  ii) The negative values of temperature that usually appear in the generalized theories of thermoelasticity vanished.
- The method used in the present article is applicable to a wide range of problems in thermodynamics and fluid dynamics when the governing equations are coupled.

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Appendix A

The solution of the system (34) is given by

$$a_0 = \frac{k_1^2 \cosh(K_2 x) - k_2^2 \cosh(K_1 x)}{k_1^2 - k_2^2},$$

$$a_1 = \frac{k_1^3 \sinh(k_1 x) - k_2^3 \sinh(k_2 x)}{k_1 k_2 \left(k_1^2 - k_2^2\right)},$$

$$a_2 = \frac{\cosh(k_2 x) - \cosh(k_1 x)}{k_1^2 - k_2^2},$$

$$a_3 = \frac{k_2 \sinh(k_1 x) - k_1 \sinh(k_2 x)}{k_1 k_2 \left(k_1^2 - k_2^2\right)},$$

Appendix B

The components $[\ell_{ij}\left(x,s\right)]$ are defined as

$$\ell_{11} = \frac{1}{k_1^2 - k_2^2}\left[(k_1^2 - a)\cosh(k_2 x) - (k_2^2 - a)\cosh(k_1 x)\right],$$

$$\ell_{12} = \frac{bc\varepsilon s^2}{k_1^2 - k_2^2}\left[\frac{\sinh(k_1 x)}{k_1} - \frac{\sinh(k_2 x)}{k_2}\right],$$

$$\ell_{13} = \frac{1}{k_1^2 - k_2^2}\left[\frac{(k_1^2 - cs^2)\sinh(k_1 x)}{k_1} - \frac{(k_2^2 - cs^2)\sinh(k_2 x)}{k_2}\right],$$

$$\ell_{14} = \frac{b\varepsilon}{k_1^2 - k_2^2}\left[\cosh(k_1 x) - \cosh(k_2 x)\right],$$

$$\ell_{21} = \frac{ad}{k_1^2 - k_2^2}\left[\frac{\sinh(k_1 x)}{k_1} - \frac{\sinh(k_2 x)}{k_2}\right],$$

$$\ell_{23} = \frac{d}{k_1^2 - k_2^2}\left[\cosh(k_1 x) - \cosh(k_2 x)\right],$$

$$\ell_{24} = \frac{1}{k_1^2 - k_2^2}\left[\frac{(k_1^2 - a)\ \sinh(k_1 x)}{k_1} - \frac{(k_2^2 - a)\ \sinh(k_2 x)}{k_2}\right],$$

$$\ell_{31} = a\ell_{13}\,,\ \ell_{32} = \varepsilon bs\,\ell_{23}\,,$$

$$\ell_{33} = \frac{1}{k_1^2 - k_2^2}\left[(k_1^2 - cs^2)\ \cosh(k_1 x) - (k_2^2 - cs^2)\ \cosh(k_2 x)\right],$$

$$\ell_{34} = \frac{b\varepsilon}{k_1^2 - k_2^2}\left[k_1 \sinh(k_1 x) - k_2 \sinh(k_2 x)\right],$$

$$l_{41} = \frac{w}{\varepsilon} l_{14}\,,\ \ell_{42} = cs^2 \ell_{24}\,,\ \ell_{43} = \frac{d}{b\varepsilon}\ell_{34}\,,$$

$$\ell_{44} = \frac{1}{k_1^2 - k_2^2}\left[(k_1^2 - a)\cosh(k_1 x) - (k_2^2 - a)\cosh(k_2 x)\right].$$

Nomenclature

λ, μ	Lame' constants
t	time
ρ	density
C_E	specific heat at constant strains
C_o^2	$= \dfrac{\lambda + 2\mu}{\rho}$, longitudinal wave speed
T	absolute temperature
u_i	components of displacement vector
ε_{ij}	components of strain tensor
e_{ij}	components of strain deviator tensor
σ_{ij}	components of stress tensor
e	$= \varepsilon_{ii}$, dilatation
k	thermal conductivity
$\boldsymbol{H}$	strength of the applied magnetic field
$\boldsymbol{J}$	electric current density
$\boldsymbol{B}$	magnetic induction vector
$\boldsymbol{h}$	induced magnetic field
$\boldsymbol{E}$	induced electric field
μ_o	magnetic permeability
ε_o	electric permeability
α_o	$= \dfrac{\mu_o H_o^2}{\rho}$, Alfven velocity
X_i	body force
α_T	coefficient of linear thermal expansion
γ	$= (3\lambda + 2\mu)\alpha_T$
δ_{ij}	Kronecker's delta function
T_o	reference temperature
η_o	$= \rho C_E / k$
ε	$= \dfrac{\gamma^2 T_0}{k \eta_o \rho C_0^2}$, thermal coupling parameter
Θ	$= T - T_0$, such that $\lvert \Theta / T_0 \rvert << 1$,
τ, υ	two relaxation times

$\Gamma(.)$	Gamma function
n	constant