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Displacement tracking of pre-deformed smart structures

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Abstract. This paper is concerned with the dynamics of hyperelastic solids and structures. We seek for a smart control actuation that produces a desired (prescribed) displacement field in the presence of transient imposed forces. In the literature, this problem is denoted as displacement tracking, or also as shape morphing problem. One talks about shape control, when the displacements to be tracked do vanish. In the present paper, it is assumed that the control actuation is provided by imposed eigenstrains, e.g., by the electric field in piezoelectric actuators, or by thermal actuators, or via analogous physical effects, such as magneto-striction or pre-stress. Structures with a controlled eigenstrain-type actuation belong to the class of smart structures. The action of the eigenstrains can be conveniently characterized by actuation stresses. Our theoretical derivations are performed in the framework of the theory of small incremental dynamic deformations superimposed upon a statically pre-deformed configuration of a hyperelastic solid or structure. We particularly ask for a distribution of incremental actuation stresses, such that the incremental displacements follow exactly a prescribed trajectory field, despite the imposed incremental forces are present. An exact solution of this problem is presented under the assumption that the actuation stresses can be tailored freely and applied everywhere within the body. Extending a Neumann-type solution strategy, it is shown that the actuation stresses due to the distributed control eigenstrains must satisfy certain quasi-static equilibrium conditions, where auxiliary body-forces and auxiliary surface tractions are to be taken into account. The latter auxiliary loading can be directly computed from the imposed forces and from the desired displacement field to be tracked. Hence, despite the problem is a dynamic one, a straightforward computation of proper actuator distributions can be obtained in the framework of quasi-static equilibrium conditions. Necessary conditions for the functioning of this concept are presented. Particularly, it must be required that the intermediate configuration is infinitesimally superstable. Previous results of our group for the case of shape control and displacement tracking in linear elastic structures are included as special cases. The high potential of the solution is demonstrated via Finite Element computations for an irregularly shaped four-corner plate in a state of plain strain.

Keywords: displacement tracking; shape control; smart structures; piezoelectric actuation; pre-deformed configuration, small superimposed displacements

1. Introduction

In the present paper, we study vibrations of hyperelastic material bodies (solids or structures)

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under the action of imposed forces. An additional control actuation is provided by imposed eigenstrains, e.g., by the electric field acting in piezoelectric bodies, see e.g., Tiersten (1969) and Yang (2005), by the temperature field in a thermoelastic body, see e.g., Parkus (1976), or by analogous types of physical actuation, such as magneto-striction or pre-stress. Structures with a controlled eigenstrain-type actuation belong to the class of smart structures. Our present theoretical derivations are performed in the framework of the theory of small incremental dynamic deformations superimposed upon a statically pre-deformed configuration of hyperelastic bodies, see e.g., Knops and Wilkes (1973) for foundations of the latter theory. The pre-deformed configuration is called the intermediate configuration in the following. We seek for an additional eigenstrain-type control actuation, which produces a desired (prescribed) transient displacement field from the intermediate configuration, despite some transient forces are imposed upon the latter. In the following we talk about incremental forces, when they are applied to the intermediate configuration as functions of space and time. Analogously, we talk about incremental stresses, incremental displacements, incremental eigenstrains and incremental actuation stresses that evolve from the intermediate configuration as functions of space and time. Under the action of the imposed incremental forces alone, incremental displacements would take place, which are generally different from the desired ones. We however wish to produce the desired incremental displacement fields by applying proper incremental eigenstrains. To find proper eigenstrain fields that satisfy this requirement is often denoted as displacement tracking problem, or also as shape morphing problem in the literature. One talks about shape control, when the incremental displacements to be tracked do vanish. The incremental eigenstrains can be characterized by incremental actuation stresses, which represent a linear mapping of the incremental eigenstrains via the fourth order tensor of linearized elastic parameters in the intermediate configuration. Since the intermediate configuration may represent a large deformation from an undeformed and unstressed configuration, these parameters generally will be distributed in an inhomogeneous manner throughout the body. The undeformed and unstressed configuration will be subsequently denoted as natural state or configuration.

In the following, we assume that the incremental actuation stresses can be tailored freely, and that they can be applied everywhere within the body. For some literature reviews on shape control, displacement tracking and various structural applications, see Irschik (2002), Irschik *et al*. (2012), where emphasis has been laid upon the previous work of our own group, and Irschik and Krommer (2013). The latter paper contains references to the work of various other groups, also with respect to automatic or feedback control, a topic that is not addressed in the present contribution. Here, we assume that the imposed incremental forces and the geometric and constitutive parameters under consideration are known, such that feedback control is not needed. For works on shape control of our group, the interested reader is referred e.g. to Krommer and Irschik (2007), Zehetner and Irschik (2008), Schöftner and Irschik (2011), Schöfnter and Buchberger (2013) and Zenz *et al.* (2013).

Subsequently, we present a class of analytic solutions for the displacement tracking problem described above. As a main result of our present considerations, we show that the incremental actuation stresses necessary for displacement tracking must satisfy some auxiliary quasi-static equilibrium conditions only. This important result includes the previous work of Irschik and Pichler (2004) on shape control of displacements from a natural configuration as a special case. Similar to the previous work of Irschik and Pichler (2004), our present formulation can be considered as an extension of the Neumann procedure for proofing the uniqueness of initial boundary value problems of the linear theory of elastodynamics. For Neumann's original procedure, in which eigenstrains, actuation stresses and intermediate configurations of hyperelastic bodies were not considered, see e.g. Chandrasekharaiah and Debnath (1994). In the present context of small incremental deformations superimposed upon an intermediate configuration it turns out that one must also require infinitesimal superstability of the intermediate configuration, see Knops and Wilkes (1973) for the latter notion, and see also Wang and Truesdell (1973). The condition of infinitesimal superstability is trivially satisfied in the context of shape control of displacements from a natural configuration that was considered in Irschik and Pichler (2004). Derivations on displacement tracking from a natural configuration, which are preliminary to the subsequent results, were presented in Irschik and Krommer (2005), where visco-elastic effects were also included, and in Irschik *et al.* (2007). An alternative approach for solving the problem of displacement tracking from a natural configuration was presented in Krommer and Irschik (2007), where a virtual displacement and power balance formulation was used, and where several aspects that are important for feedback-control, such as collocation of actuators and sensors, were addressed in detail.

In the present contribution, we also include a generalized form of boundary conditions, namely the possibility of prescribing the normal component of the boundary displacement and the tangential component of the surface traction, additional to the usual boundary conditions of prescribing displacement vectors or surface tractions in total, see e.g., Gurtin (1972) for details of a corresponding generalized formulation. In extension, this latter formulation of boundary conditions here is applied within the framework of small incremental deformations from an intermediate configuration.

Our subsequent solution for displacement tracking is felt to represent a contribution to the incremental theory of hyperelasticity in its own right. From a practical point of view, it can be used as a proper starting point for the actuator design when restrictions in the actuation with respect to its spatial distribution and/or intensity have to be taken into account, e.g. when the actuation stress is to be provided by discrete networks of actuators. It also can be conveniently used as a preliminary design when methods of automatic control need to be applied.

As an illustrative numeric example, we treat an irregularly shaped four-corner plate in a state of plane strain that was used in Irschik and Pichler (2004) for discussing the problem of shape control from a natural state. In the present contribution, a spatially linear displacement field with a prescribed time-wise evolution is desired to be tracked by proper distributions of actuation stresses. For simplicity and comparison sake, in this example we restrict to the case of incremental displacements evolving from the natural configuration, see Irschik and Pichler (2004) for shape control. The quasi-static boundary value problem for a field of actuation stresses that can achieve the desired goal of displacement tracking is derived and discussed in some detail. Computations are performed using the commercial Finite-Element code ABAQUS together with a self-written code that assigns the actuation stresses to the Finite Elements in the form of anisotropic transient tensors of thermal actuation. The latter eigenstrain-type actuation is applied, and it is demonstrated that the required displacement fields are indeed obtained, despite imposed forces are present.

2. Initial boundary value problem in the presence of actuation stresses

Consider some continuous material body B. Imposed body forces per unit volume are denoted as b. Additional to these forces, a field of actuation stresses S_A is applied within B. On some part

∂B_1 of the boundary ∂B of *B*, the displacement is described as $\hat{u}$, while another part, ∂B_2, is loaded by imposed surface tractions $\hat{s}$ per unit surface area. On the remaining part of ∂B, on ∂B_3, the normal component of the displacement and the tangential component of the surface traction is prescribed. The above quantities, i.e. body forces *b*, actuation stresses S_A, surface displacements $\hat{u}$, and surface tractions $\hat{s}$, or the given components of the latter two, may vary with time, such that the particles of *B* will be accelerated, and time-dependent incremental stresses *S* and displacements *u* will be produced. This motion may be associated with a prescribed non-vanishing initial displacement field u_0, where a non-vanishing initial velocity field v_0 may be also present.

In the present paper, we assume that the body *B* is hyperelastic. We restrict ourselves to the case of small incremental deformations superimposed upon an intermediate configuration, which is assumed to represent a pre-deformed equilibrium state. This static intermediate state might represent a large deformation with respect to an undeformed and unstressed configuration, which we call a natural state or configuration of *B*. In order to enable a proper mathematical description of the small incremental deformations from the intermediate state, a suitable reference configuration of *B* must be selected. Since *B* is taken as hyperelastic, the mathematic relations presented subsequently can be shown to hold formally for two different ways of describing the small incremental motion, compare e.g., Knops and Wilkes (1973). First, the natural configuration may be selected as reference configuration; this case will be denoted as the Lagrange description. Secondly, the intermediate configuration, which in general will be pre-stressed and pre-deformed, can be used as the reference configuration, a formulation, which we call the up-dated Lagrange description. In both descriptions, the subsequent relations have the same mathematical form, but there is a difference in the mechanical meaning of the symbols under consideration. E.g., in the Lagrange description, incremental body forces *b* and surface tractions $\hat{s}$ must be taken per unit volume or per unit surface area of the natural state, respectively. In contrast, unit volume and unit surface area of the intermediate state have to be considered when dealing with the up-dated Lagrange version.

In a first step, we formulate the above described incremental initial boundary-value problem in mathematical terms by starting with Cauchy's first law of motion, which we write as

$$B: \quad div\, S + b = \rho\, \ddot{u}, \quad \forall\, t \geq 0 \tag{1}$$

where time is denoted by *t*, and the incremental motion starts at $t = 0$. The density of mass is denoted by ρ, being taken per unit volume of the natural state in the Lagrange description, and per unit volume of the intermediate state in the up-dated Lagrange version. The rate of time change with respect to an inertial frame is indicated by a superimposed dot, and *div* stands for the divergence operator with respect to the place in the natural state or in the intermediate state, depending on whether the Lagrange or the up-dated Lagrange description is used. The incremental stress S represents an incremental 1. Piola-Kirchhoff stress, namely the difference between the stress in the actual position of *B* and in the intermediate state. In the Lagrange description, this incremental stress difference must be referred to the natural state. In contrast, the incremental 1. Piola-Kirchhoff stress S is to be referred to the intermediate state in the up-dated Lagrange version. Subsequently, such differences in the mechanical meaning of symbols used in the two descriptions will be mentioned only when it appears to be necessary. Of course, the intermediate state may represent a natural state itself, such that the Lagrange and the updated-Lagrange descriptions do coincide. In this case one deals with the linear theory of elasticity, the stress then representing the symmetric Cauchy stress, see e.g., Gurtin (1972). The reader who is interested in problems of linear elasticity only, thus may follow our derivations directly by leaving aside any remark concerning the reference state.

Since we deal with a small deformation from the intermediate state, we take the incremental stress S to be linearly related to the gradient of the incremental displacement. In accordance with the literature, we take into account the presence of incremental actuating stresses by writing this linear mapping in the form of the following generalization of Hooke's law, see e.g., Knops and Wilkes (1973) for the case of thermal eigenstrains

$$B: \quad S = C\left[\nabla u\right] + S_A, \quad \forall\, t \geq 0 \tag{2}$$

Here, ∇ denotes the gradient with respect to the place in the natural state, or with respect to the place in the intermediate state, respectively, depending on whether the Lagrange or the up-dated Lagrange formulation is used. The fourth order tensor of elastic parameters is denoted by C, and $C\left[\nabla u\right]$ stands for the second order tensor that represents the linear mapping of ∇u by means of C. For the different physical meanings of C in the Lagrange and in the up-dated Lagrange description, we refer to Knops and Wilkes (1973). Following the derivations presented in the latter reference, we take the following major symmetry property of C to be valid in both, the Lagrange and the up-dated Lagrange description

$$C = C^T \quad \Rightarrow \quad E \cdot C\left[F\right] = F \cdot C\left[E\right] \tag{3}$$

for any two second-order tensors E and F. The transpose of the fourth-order tensor C is denoted by the superscript T. The dot product in Eq. (3) indicates the contraction of two second order tensors to a scalar quantity. The dot product will be subsequently used also for the scalar product of two vectors. For further details of the single dot and square bracket notations, see Gurtin (1972). Note that we do not use bold face symbols for indicating vectors or tensors. Note also that in the case of a pre-strained intermediate state, C depends on the place in the respective reference configuration, even when the body is homogeneous in the natural state. If the intermediate state itself represents a natural state, such that one deals with the linear theory of elasticity, the fourth-order tensor of elasticities C admits additional symmetry conditions, see Gurtin (1972).

The set of field equations governing our problem is formed by Eqs. (1) and (2). The latter relations are to be accompanied by means of boundary conditions. We consider the following generalized formulation for describing conditions at the boundary ∂B of the body B, see Gurtin (1972) for the linear theory

$$\partial B: \; Pu = \hat{u} \quad \textit{and} \quad (1 - P)S\, n = \hat{s}, \quad \forall\, t \geq 0 \tag{4}$$

The entities $\hat{u}$ and $\hat{s}$ are vector fields on ∂B, which are designed such that they reflect the above mentioned three types of boundary conditions on the portions ∂B_1, ∂B_2 and ∂B_3, respectively. In each point of ∂B the perpendicular projection tensor P accordingly must be formed by one of the three following second-order tensors. If the incremental displacement u is prescribed, i.e. on ∂B_1, P is to be taken as the unit tensor

$$\partial B_1 : \quad P = 1 \tag{5}$$

such that one has no prescribed traction, $\hat{s} = 0$ on ∂B_1, see Eq. (4). When the incremental surface traction is prescribed, the tensor *P* must be chosen as the zero tensor

$$\partial B_2 : \quad P = 0 \tag{6}$$

such that there is no prescribed displacement, $\hat{u} = 0$ on ∂B_2, see Eq. (4). Finally, on ∂B_3, where the normal component of the displacement and the tangential component of the surface traction are prescribed, one sets

$$\partial B_3 : \quad P = n \otimes n \tag{7}$$

The unit outer normal vector at ∂B is denoted by *n*. In the Lagrange description, *n* must be taken perpendicular to the boundary of the body in the natural state, while in the up-dated Lagrange formulation, *n* is to be understood as perpendicular to the boundary in the intermediate state. The sign $\otimes$ indicates the dyadic product of two vectors. No multiplication sign is utilized for the linear mapping of a vector by means of a second-order tensor. This convention is also used for the second order tensor that follows as the product of two second order tensors, compare Gurtin (1972) for notation. Recall that the incremental surface traction is represented by the vector $S\,n$ according to the Cauchy fundamental theorem on stresses, see again Gurtin (1972), which in Eq. (4) however is to be understood as being formulated in the Lagrange or in the up-dated Lagrange formulation, respectively. Recall that the second order tensor *S* represents an incremental 1. Piola-Kirchhoff stress tensor. For further details, see Knops and Wilkes (1973). Note also that the linear mapping of a vector by means of the tensor $n \otimes n$ results in the vectorial component of that vector in the direction of n, from which the notion of a perpendicular projection stems. It is thus seen from Eqs. (4) and (7) that $\hat{u}$ must be taken perpendicular to the boundary on ∂B_3, while $\hat{s}$ must be a tangential vector there. The portions ∂B_1, ∂B_2 and ∂B_3 must represent complementary subsets of ∂B.

As mentioned above, the initial boundary value problem has to be completed by prescribing initial displacements u_0 and initial velocities v_0 from the intermediate configuration

$$B: \quad u = u_0 , \quad \dot{u} = v_0 , \quad t = 0 \tag{8}$$

The initial data u_0 and v_0 must be compatible with the incremental displacement boundary data $\hat{u}$ in Eq. (4). Since we here deal with a statically pre-deformed intermediate configuration, one requires $u_0 = 0$ and also $v_0 = 0$.

3. Tracking of displacements by means of actuation stresses

The present contribution is devoted to the solution of the following problem. In the initial boundary value problem stated in Section 2 above, assume that the quantities *b*, u_0, v_0, as well as $\hat{u}$, $\hat{s}$, or proper components of the latter two, are prescribed as functions of time and the place in the respective reference state of the Lagrange or the up-dated Lagrange description. Seek a distribution of incremental actuation stresses S_A, such that the incremental displacements u do coincide with some given vector field z everywhere and during the whole observation period

$$B: \quad u = z, \quad \forall\, t \geq 0 \tag{9}$$

The magnitude of ∇z in the respective description is assumed to be sufficiently small, such that the formulations given in Section 2 above for small deformations superimposed upon the intermediate state do apply for z also.

We call the problem introduced by the requirement stated in Eq. (9) as displacement tracking problem. The special case of tracking a zero displacement, $z = 0$, i.e. of designing an incremental actuation stress field S_A such that the displacements do vanish in Eq. (9), $u = 0$, is denoted as shape control in the literature.

In order that the non-zero tracking problem stated in Eq. (9) is solvable from a kinematical point of view, we must require, first, that the fields u and z are initially coinciding

$$B: \quad z = u_0, \quad \dot{z} = v_0, \quad t = 0 \tag{10}$$

see Eq. (8) and the subsequent remark. Secondly, it must be assured that there is no conflict with the kinematical boundary conditions prescribed on ∂B_1 and ∂B_3 in Eq. (4)

$$\partial B_1, \partial B_3: \quad P\, z = \hat{u}, \quad \forall\, t \geq 0 \tag{11}$$

We furthermore have to require that the shape of the body B is sufficiently regular, and the fields under consideration are sufficiently smooth, such that the subsequently utilized mathematical formulations make sense. Under many circumstances, these requirements may appear to be rather mild. However, they must be carefully considered, and they should be checked from case to case, particularly, when the occurrence of a singular surface is to be expected, on which some of the fields under consideration may take on different values when approaching from different sides.

Last but not least, in order to make the goal of tracking the incremental displacement field z from a pre-deformed intermediate equilibrium position mechanically meaningful, we require that this intermediate state is stable in a proper sense. A substantiation of the latter requirement will be given subsequently. In order to keep the corresponding arguments short, we once and for ever assume that the kinematical boundary conditions in Eqs. (4) and (11) are such that the case of a rigid body motion superimposed upon the intermediate state is excluded.

As already mentioned, as physical effects for producing the incremental actuation stresses S_A in Eq. (2), we consider the class of eigenstrains. From a technological point of view, the latter class includes various serious candidates for achieving the goal of displacement tracking, such as thermal actuation. Due to its rapid action, a technologically important physical effect for producing eigenstrains, or actuation stresses S_A, respectively, is piezoelectricity, see e.g., Rao and Sunar (1994), (1999) for some reviews. The case of small deformations superimposed upon an intermediate state of piezoelectric bodies has been thoroughly treated in the memoir by Eringen and Maugin (1989), see also Tiersten (1978). It becomes clear from these studies that the influence of the electromagnetic fields in the reference states under consideration should be negligible in order that the initial boundary value problem for the superimposed motion can be written in the form presented in Section 2 above, i.e. without so-called ponderomotive forces. For a further discussion of the latter assumption, which is however often made in the literature, we refer to the review article by Kamlah (2001), and to the literature cited there.

The present contribution is devoted to develop general methods for determining distributions of actuation stresses S_A that are able to achieve the goal of displacement tracking. We do not go further into the details of the specific physical mechanisms for producing those actuation stresses. We only mention that, once a distribution of actuation stresses S_A has been found that solves the above displacement tracking problem, in principle a further inverse problem should be tackled, namely to find a distribution of temperature, or of electric fields, or of similar physical or mechanical mechanisms, respectively, that is able to produce this distribution. Since the solution of the displacement tracking problem guarantees the displacement of the body to be known, the solution of the mentioned further inverse problem should be feasible, particularly when temperature or electric field can be treated as being uncoupled from the incremental deformation. When the fields b, $\hat{u}$, $\hat{s}$ and z are separable in space and time, the corresponding solution of course is particularly straightforward. We present a numerical example at the end of the paper.

4. Solution of the displacement tracking problem

In order to solve the displacement tracking problem stated in Eq. (9), we extend a method originally developed by Neumann for proving the uniqueness of solutions of linear elasto-dynamic problems. For a contemporary presentation of Neumann's proof in the context of a homogeneous and isotropic body, see Chandrasekharaiah and Debnath (1994). In a previous contribution by Irschik and Pichler (2004), a Neumann-type strategy has been used for solving the elastodynamic shape control problem, i.e. the problem of tracking zero displacements, from a natural state. Subsequently, we extend the latter derivations with respect to the following two general aspects. First, we consider the case of a small deformation superimposed upon a statically pre-deformed intermediate state of a hyperelastic body, this intermediate state possibly representing a large deformation from a natural state, see the formulations given in Section 2 above. Secondly, and this is our main concern, we treat the case of a non-zero field to be tracked, $z \neq 0$.

We start our derivations by introducing an extended Neumann function $N(t)$ in the form of a sum of the following spatial integrals

$$N = \int_B \left(\nabla y \cdot C\left[\nabla y\right]\right) dv + \int_B \rho\, \dot{y} \cdot \dot{y}\, dv \tag{12}$$

where y denotes the difference between the incremental displacement u and the field z to be tracked

$$y = u - z \tag{13}$$

The integration in Eq. (12) is to be performed over the body B in the natural state in the Lagrange description, and over the intermediate state in the up-dated Lagrange description. In order to assure sufficient stability of the intermediate state, we require the tensor of elastic constants C and the mass density ρ to be distributed such that the Neumann function is non-negative for all non-vanishing fields y and $\dot{y}$

$$y \neq 0, \quad \dot{y} \neq 0 \Rightarrow \quad N > 0 \tag{14}$$

Note that the second integral on the right hand side of Eq. (12), referring to $\dot{y}$ is positive definite. In order that Eq. (14) does hold in general, this then should hold true for the first integral also, which involves *C*. If this latter requirement is satisfied, the intermediate state is said to be infinitesimally superstable, see Knops and Wilkes (1973) and Wang and Truesdell (1973) for further discussions concerning the definition of superstability. Recall also that we have excluded the case of rigid body motions superimposed upon the intermediate state. Since the intermediate configuration may represent a large deformation from the natural state, infinitesimal superstability may not be guaranteed. To require this form of stability of course is not necessary when the intermediate configuration itself represents a natural one. That is why this requirement did not play a role in the work on shape control by Irschik and Pichler (2004).

We now perform the rate of time-change of *N*. Taking into account the symmetry condition of Eq. (3), and noting Eq. (13), we obtain from Eq. (12) that

$$\frac{1}{2}\dot{N} = \int_B \Big(\nabla \dot{y} \cdot C\big[\nabla(u-z)\big] + \rho(\ddot{u}-\ddot{z}) \cdot \dot{y} \Big) dv \tag{15}$$

From Eq. (2), we have

$$\nabla \dot{y} \cdot C[\nabla u] = \nabla \dot{y} \cdot (S - S_A) \tag{16}$$

A well-known relation of tensor analysis proves that

$$\nabla \dot{y} \cdot (S - S_A) = -\dot{y} \cdot div(S - S_A) + div\Big((S - S_A)^T \dot{y}\Big) \tag{17}$$

where *T* denotes the transpose of a tensor. Substituting Eqs. (1) and (17) into Eq. (16) gives

$$\nabla \dot{y} \cdot C[\nabla u] = \dot{y} \cdot (div\, S_A + b - \rho \ddot{u}) + div\Big((S - S_A)^T \dot{y}\Big) \tag{18}$$

Analogous to Eq. (17), there is

$$\nabla \dot{y} \cdot C[\nabla z] = -\dot{y} \cdot div\big(C[\nabla z]\big) + div\Big(\big(C[\nabla z]\big)^T \dot{y}\Big) \tag{19}$$

Substituting Eqs. (18) and (19) into Eq. (15) and applying the divergence theorem, leads to

$$\frac{1}{2}\dot{N} = \int_B \dot{y} \cdot \Big(div\, S_A + b + div\big(C[\nabla z]\big) - \rho \ddot{z}\Big) dv - \int_{\partial B} \Big(\big(S_A + C[\nabla z] - S\big)^T \dot{y}\Big) \cdot n\, ds \tag{20}$$

From tensor algebra, we know that

$$\left(\left(S_A + C[\nabla u] - S\right)^T \dot{y}\right) \cdot n = \left(S_A + C[\nabla u] - S\right) n \cdot \dot{y} \tag{21}$$

An identical expansion and substitution of Eq. (4) shows that

$$(S\,n) \cdot \dot{y} = \left((1 - P + P) S\,n\right) \cdot \dot{y} = \left(\hat{s} + P\,S\,n\right) \cdot \dot{y} \tag{22}$$

Analogously, we obtain

$$\dot{y} = \dot{u} - \dot{z} = (1 - P)(\dot{u} - \dot{z}) + \dot{\hat{u}} - P\,\dot{z} \tag{23}$$

Substituting Eqs. (21)-(23) into Eq. (20) leads to a form that is appropriate for obtaining a solution of the tracking problem

$$\frac{1}{2}\dot{N} = \int_B \dot{y} \cdot \left(div\,S_A + b + div\left(C[\nabla z]\right) - \rho\,\ddot{z}\right) dv$$

$$- \int_{\partial B} \left(S_A n + C[\nabla_z] n - \hat{s} - PSn\right) . \left((1 - P)(\dot{u} - \dot{z}) + \dot{u} - P\dot{z}\right) ds \tag{24}$$

Namely, if we assume in Eq. (24) that

$$B: \quad div\,S_A + b^* = 0, \quad \forall\, t \geq 0 \tag{25}$$

$$\partial B_2, \partial B_3: \quad S_A\,n = \hat{s}^*, \quad \forall\, t \geq 0 \tag{26}$$

with

$$b^* = b + div\left(C[\nabla z]\right) - \rho\,\ddot{z} \tag{27}$$

$$\hat{s}^* = \hat{s} - C[\nabla z] n \tag{28}$$

then it follows that

$$\dot{N} = 0 \tag{29}$$

in Eq. (24). This is explained subsequently in more detail. That the volume integral in Eq. (24) vanishes, is a consequence of Eqs. (25) and (27). In order to see that the boundary integral in Eq. (24) vanishes, we consider first the portion ∂B_1 of the boundary. There we have $P = 1$, see Eq. (5), which, together with Eq. (11) guarantees vanishing of the boundary integral. At ∂B_2, vanishing of the boundary integral follows from Eqs. (26) and (28), together with the fact that there is $P = 0$, see Eq. (6). Finally, at ∂B_3, both, Eq. (11) and (26), do apply. Moreover, $P\,S\,n$ and $(1 - P)(\dot{u} - \dot{z})$ are perpendicular vectors, such that the boundary integral in Eq. (24) vanishes also on ∂B_3.

We conclude from Eq. (29) that the Neumann function becomes constant in time, when Eqs. (25) - (28) are satisfied, and thus is determined by the initial data

$$N = N_0 = \int_B \Big(\nabla\left(u_0 - z_0\right) \cdot C\left[\nabla\left(u_0 - z_0\right)\right] + \rho\left(v_0 - \dot{z}_0\right) \cdot \left(v_0 - \dot{z}_0\right)\Big) dv \tag{30}$$

It is then seen from Eq. (30) that, due to the initial conditions for z stated in Eq. (10), the Neumann function itself does vanish. From the requirement of infinitesimal superstability of the intermediate state, see Eq. (14), it is thus concluded that y in Eq. (13) must vanish. Hence, we have arrived at the following solution of the displacement tracking problem: If the actuation stress S_A is chosen according to Eqs. (25)-(28), and if Eqs. (10) and (11) do hold for *z*, together with the condition of superstability of the intermediate state, then the displacement becomes the field to be tracked:

$$N = 0 \quad \Rightarrow \quad B: \quad y = 0, \quad u \equiv z, \quad t \geq 0 \tag{31}$$

We end the present Section with some explanatory remarks. In the above solution represented by Eqs. (25)-(28), the actuation stress S_A, although it is time-dependent in general, can be interpreted as a statically admissible stress with respect to the auxiliary body force b^* and the auxiliary stress-boundary data $\hat{s}^*$, meaning that S_A is in temporal equilibrium with the latter auxiliary forces to be computed from Eqs. (27) and (28), see Eqs. (25) and (26). Note that the surface tractions $\hat{s}^*$ are only prescribed at the portions ∂B_2 and ∂B_3 of the boundary ∂B. The required distribution of actuation stresses S_A for displacement tracking is therefore not unique, with the rare exception of statically determinate problems. Any divergence-less stress with a vanishing boundary stress vector at ∂B_2 and ∂B_3 may be added to S_A in Eqs. (25) and (26), without changing the validity of the solution. However, uniqueness can be achieved by assigning an auxiliary constitutive relation, and auxiliary displacement boundary data on ∂B_1 and ∂B_3 to the statically admissible stress problem. As a main result, it is only necessary to treat an auxiliary quasi-static problem in order to solve the above dynamic displacement-tracking problem. The auxiliary quasi-static problem needs not to correspond to the more complex original hyperelastic initial-boundary-value problem, in the framework of which the displacement tracking problem has been stated. E.g., the auxiliary body needs not to be hyperelastic, and the kinematic boundary conditions need not to coincide with the original problem. No initial conditions of course need to be introduced in the auxiliary quasi-static problem. We come back to the auxiliary problem below, when presenting a numerical example.

As a special case of our above solution, consider now zero-tracking, i.e., $z = 0$. This requires

$$B: \quad u = 0, \quad \forall\, t \geq 0 \tag{32}$$

in Eq. (9). In order that the shape control problem is geometrically compatible with the original initial boundary value problem, the kinematical boundary conditions and the initial conditions must vanish

$$\partial B_1, \partial B_3: \quad Pu = 0, \quad \forall\, t \geq 0 \tag{33}$$

$$B: \quad u_0 = 0, \quad v_0 = 0, \quad t = 0 \tag{34}$$

see Eqs. (10) and (11). Introducing $z = 0$ into Eqs. (25)-(28) gives

$$B: \quad div\, S_A + b = 0, \quad \forall\, t \geq 0 \tag{35}$$

$$\partial B_2, \partial B_3: \quad S_A\, n = \hat{s}, \quad \forall\, t \geq 0 \tag{36}$$

such that S_A must be in temporal equilibrium with the original forces b and $\hat{s}$ only, in order that the goal of zero-tracking is reached. Formally, the same result has been obtained earlier by our group for shape control under the assumption of a small deformation from a natural state, see the references cited above.

5. Numerical example

In order to demonstrate the validity of the analytic solution for (non-zero) displacement tracking presented in Section 4 above, the following example problem is considered. We study an irregularly shaped plane four-corner domain being situated in the (e_1, e_2)- plane, see Fig. 1. Unit vectors are denoted by e, and x_1 is the coordinate in the direction of e_1. The four corners are given by $P_1: (0, h)$, $P_2: (0, 0)$, $P_3: (h, 0.7h)$, $P_4: (1.2h, 0.2h)$. This irregular domain has been used in Irschik and Pichler (2004) for the sake of shape control from a natural state, in order to demonstrate that the presented concept also does hold, when there are no geometric symmetries present. A state of plane strain is considered with respect to the coordinate perpendicular to the (e_1, e_2)- plane. The body is fixed at the left boundary $\overline{P_1P_2}$, i.e. at $x_1 = 0$, see Eqs. (4) and (5) with $\hat{u} = 0$. The edges $\overline{P_2P_4}$ and $\overline{P_3P_4}$ are free of stress, see Eq. (6) with $\hat{s} = 0$. At the edge $\overline{P_1P_3}$, the domain is loaded by a distributed surface traction in the following form

$$\hat{s} = -pH(t)n_{13} \tag{37}$$

Here, $\hat{p}$ is a space- and time-wise constant pressure, $H(t)$ is the Heaviside function, and n_{13} denotes the outer unit normal vector at this edge.

In the present example, the following displacement field shall be tracked by a proper smart actuation of the eigenstrain type

$$z = u_0 \frac{x_1}{h} q(t) e_2 \tag{38}$$

Hence, we wish that the desired displacement of the domain takes place in the e_2 direction only, and that it is separable in space and time, the space-wise dependence being taken as linear in the coordinate x_1. The factor u_0 carries the dimension of length. For the time-wise dependency in Eq. (38), we require a smooth ramp-type behavior

$$q(t)=\left[\frac{1}{2}\left(1-cos\frac{\pi t}{T}\right)\left(H(t)-H(t-T)\right)+H(t-T)\right] \tag{39}$$

where T denotes a characteristic time. The gradient of z becomes

$$\nabla z=\frac{u_0}{h}\,q(t)\,e_2\otimes e_1 \tag{40}$$

In the following, we restrict ourselves to the case of an incremental deformation from a natural state, which we take as isotropic and homogeneous. In this case, the tensor of linear elastic constants C yields, see e.g. Gurtin (1972)

$$C\left[\nabla z\right]=C\left[sym\nabla z\right]=G\frac{u_0}{h}\,q(t)\left(e_2\otimes e_1+e_1\otimes e_2\right) \tag{41}$$

where G is the shear modulus. Moreover, since in the homogeneous case G does not depend on the place in the four-corner domain, there is

$$div C\left[\nabla z\right]=0 \tag{42}$$

Also, we have

$$\ddot{z}=u_0\frac{x_1}{h}\frac{\pi^2}{T^2}cos\frac{\pi t}{T}\left(H(t)-H(t-T)\right)e_2 \tag{43}$$

Now, substituting Eqs. (37)-(43) into Eqs. (25)-(28), which constitute our proposed solution of the tracking problem, and setting $b=0$, we obtain a quasi-static problem with an auxiliary body force of the form

$$b^*=-\rho\ddot{z}=-\rho u_0\frac{x_1}{h}\frac{\pi^2}{T^2}\cos\frac{\pi t}{T}\left(H(t)-H(t-T)\right)e_2 \tag{44}$$

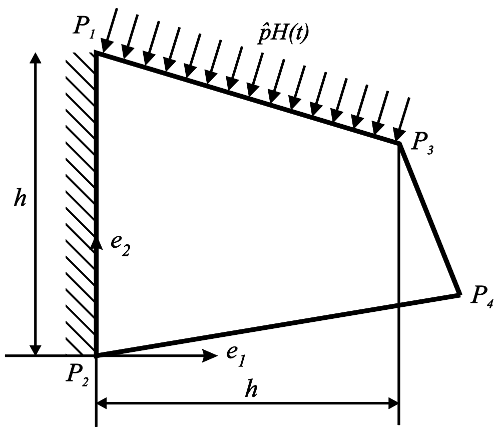

Fig. 1 Example problem: Four-corner domain with imposed loading

Additionally, non-vanishing auxiliary surface tractions $\hat{s}^*$ are to be applied at the edges $\overline{P_2P_4}$, $\overline{P_3P_4}$ and $\overline{P_1P_3}$, in accordance with Eqs. (28), (37) and (41). It is to be noted that the time-evolution of the surface tractions $\hat{s}^*$ and of the body forces b^* are different. In order to close the corresponding quasi-static auxiliary problem stated in Eqs. (25) and (26), we assume that the edge $\overline{P_1P_2}$ is fixed, like in the original problem, and that the auxiliary problem is linear elastic with the tensor of elastic constants C of the original problem. The four-corner domain under consideration then is modeled using the commercial Finite Element code ABAQUS, Version 6.12, where CPE4R type Finite Elements are used, and a numerically converged solution for the stress tensor in the auxiliary problem eventually is derived. In a self-written code formulated in C++, the resulting auxiliary stress distribution afterwards is assigned to the ABAQUS Finite Elements as a distribution of anisotropic thermal actuation stresses, where the same procedure as described in Irschik and Pichler (2004) for shape control is utilized. The fictitious thermal actuation stress is used as a smart actuation. A dynamic Finite Element computation is performed, where the transient thermal actuation stresses were superposed to the original force loading applied at the edge $\overline{P_1P_3}$ according to Eq. (37).

As a characteristic result, the time-wise evolution of the horizontal displacement u_1 and the vertical displacement u_2 of point P_4 in Fig.1 are presented in Figs. 2 and 3, respectively. The displacements are scaled by means of the static displacements due to the pressure $\hat{p}$ applied at the edge $\overline{P_1P_3}$, and time is scaled by the fundamental vibration period T_1. Both the displacements for the uncontrolled case and in the presence of the additional eigenstrain actuation are shown. Excellent agreement of the results and the desired displacements to be tracked, Eqs. (38) and (39), is obtained; the vertical displacement shows the desired ramp-type behavior, while the horizontal displacement does vanish, as it should be. Note that in this example shape control is required for the horizontal displacement only, while a non-vanishing, ramp-type displacement is tracked in the vertical direction.

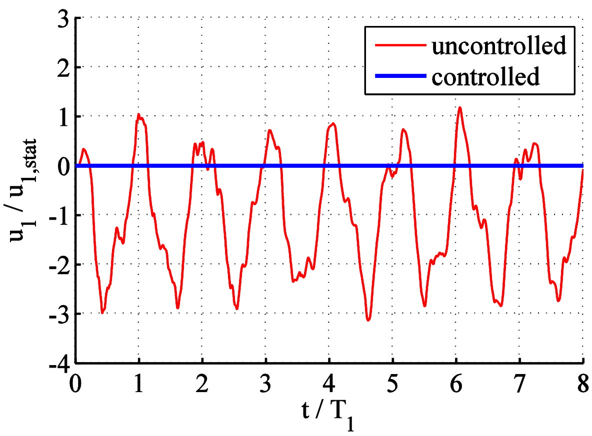

Fig. 2 Dimensionless horizontal tip deflection u_1 as a function of time

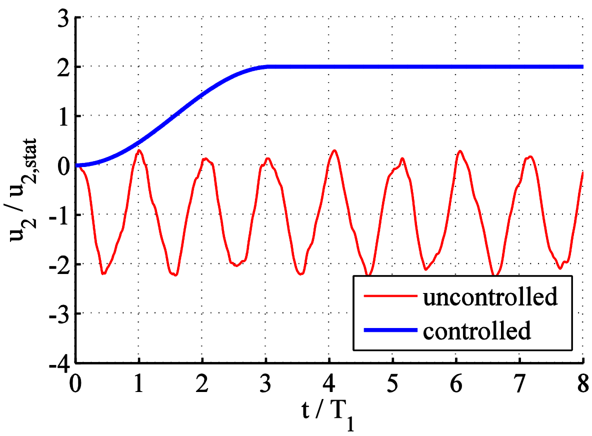

Fig. 3 Dimensionless horizontal tip deflection u_2 as a function of time

6. Conclusions

A general solution for displacement tracking of small displacements from a statically pre-deformed configuration of a hyperelastic body has been derived, see Eqs. (25)-(28). Smart actuation by eigenstrain-type actuation stresses has been considered. It turns out that the actuation stresses only need to satisfy a quasi-static equilibrium problem. This result contains earlier results of our group concerning zero-displacement tracking, or shape-control, as special cases, see Eqs. (35) and (36). The presented exact result for stress tracking has been derived assuming that the actuation stresses can be tailored freely and applied everywhere within the body. This exact solution can be used as a proper starting point for the actuator design when restrictions with respect to spatial distribution and intensity of the actuation have to be taken into account. The presented results have been successfully confirmed by a numerical study.

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