Smart Structures and Systems, *Vol. 17, No. 6 (2016) 903-915*
DOI: http://dx.doi.org/10.12989/sss.2016.17.6.903

Improved Kalman filter with unknown inputs based on data fusion of partial acceleration and displacement measurements

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(Received January 5, 2016, Revised April 12, 2016, Accepted April 16, 2016)

Abstract. The classical Kalman filter (KF) provides a practical and efficient state estimation approach for structural identification and vibration control. However, the classical KF approach is applicable only when external inputs are assumed known. Over the years, some approaches based on Kalman filter with unknown inputs (KF-UI) have been presented. However, these approaches based solely on acceleration measurements are inherently unstable which leads poor tracking and so-called drifts in the estimated unknown inputs and structural displacement in the presence of measurement noises. Either on-line regularization schemes or post signal processing is required to treat the drifts in the identification results, which prohibits the real-time identification of joint structural state and unknown inputs. In this paper, it is aimed to extend the classical KF approach to circumvent the above limitation for real time joint estimation of structural states and the unknown inputs. Based on the scheme of the classical KF, analytical recursive solutions of an improved Kalman filter with unknown excitations (KF-UI) are derived and presented. Moreover, data fusion of partially measured displacement and acceleration responses is used to prevent in real time the so-called drifts in the estimated structural state vector and unknown external inputs. The effectiveness and performance of the proposed approach are demonstrated by some numerical examples.

Keywords: Kalman filter; unknown inputs; input estimation; response prediction; data fusion

1. Introduction

The state estimation of a dynamic system in a stochastic frame is important for structural health monitoring and vibration control (Azam *et al.* 2015). In practical cases, it is impossible to measure all structural responses; hence a state estimation of partially observed dynamic system is essential. In this regard, the Kalman filter (KF), which was proposed by R. E. Kalman in the early sixties (Kalman 1960), provides a particularly practical and efficient state estimation algorithm with partial measurements of structural responses. Moreover, KF has the ability to inherently take the uncertainty in the model into account, which is not possible in the deterministic approaches (Naets *et al.* 2015). However, in the classical KF approach, the external input forces are assumed either known or broadband, so that they can be modeled as a zero mean stationary white process. In many cases, no measurements of the input forces are available or the broadband assumption is violated.

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http://www.techno-press.org/?journal=sss&subpage=8 ISSN: 1738-1584 (Print), 1738-1991 (Online)

Over the years, some researchers have proposed various improved KF with unknown inputs to circumvent the above limitation of the classical KF approach, e.g., Gillijns and Moor (2007) derived a recursive filter with the structure of the Kalman filter for joint input and state identification using linear minimum-variance unbiased estimation for optimal control applications; Pan *et al.* (2010) also derived a Kalman filter with unknown inputs approach by the weighted least-squares estimation method. The least-squares estimators for states and unknown inputs are proven inherently optimal in the minimum-variance and unbiased sense. Wu *et al.* (2009) employed the Kalman filter to establish a regression model between the residual innovation and the input excitation. Based on the regression model, a recursive least-squares estimator is proposed to identify the input excitation forces. Lin *et al.* (Lin *et al.* 2010, Ma *et al.* 2003) also studied input force estimation of linear and nonlinear structural systems based on the Kalman filter (KF) with a recursive estimator, in which the KF generates the residual innovation sequences and the estimator uses a least-squares algorithm to evaluate the time histories of the exciting forces; Lourens *et al.* (2012) developed an augmented Kalman filter (AFK) for force identification in structural dynamics, in which the unknown forces are included in the state vector and estimated in conjunction with the states. Ding *et al.* (2013) presented a discrete force identification method based on average acceleration discrete algorithm. The method is formulated in state space and the external excitation acting on a structure is estimated with regularization method; Liu *et al.* (2014) transferred the implicit Newmark- algorithm to an explicit form in the inverse analysis of dynamic force identification with better performance; Wang *et al.* (2015) developed a novel method for force identification based on the Galerkin weak formulation in which the conventional implicit Newmark method for the forward dynamic analysis was transformed into an equivalent explicit form for more accurate force identification. The authors (Lei *et al.* 2012, 2014, 2015) also investigated the identification of structures as well as the unknown external excitations. However, many of previous approaches are not based on the direct extension of the classical KF and the derivations of the analytical solutions are quite complex (Gillijns and Moor 2007; Pan *et al.* 2010). Moreover, it has been demonstrated that most previous KF-UI approaches based solely on acceleration measurements are inherently unstable which leads poor tracking and so-called drifts in the estimated unknown external inputs and structural displacements. These drifts are caused by acceleration's insensitivity to any quasi-static component in the inputs (Azam *et al.* 2015). Although regularization approaches (Mao *et al.* 2010, Liu *et al.* 2015, Wang and Xie 2015, Sun *et al.* 2015) or post-signal processing schemes (Lei *et al.* 2012, 2014, 2015) can be used to treat the drift in the identified results, these treatments prohibits the on-line and real-time identification of coupled structural state and unknown inputs. Naets *et al.* (2015) introduced the addition of dummy-measurements on a position level to circumvent the unreliable identification results. Azam *et al.* (2015) proposed a dual Kalman filter approach, in which a fictitious process equation serving for calibration of the input force is introduced. However, the selections of proper dummy-measurements and the fine-tuning the regularization parameters in fictitious process are quite subjective.

In this paper, it is aimed to extend the classical KF approach to circumvent the limitation of the classical KF and the drawbacks of previous KF-UI based approaches for real time estimation of structural states and unknown inputs. Based on the scheme of the classical KF, an improved Kalman filter with unknown excitations (KF-UI) is derived. Since acceleration and displacement measurements contains high and low frequencies vibration characteristics respectively, data fusion of measured acceleration and displacement can make full use of these two types of measured signals (Jiang *et al.* 2011). Therefore, data fusion of partially measured displacement and acceleration responses is used to avoid in real time the so-called drifts in the identified state and unknown external inputs in the presence of measurement noises. Some numerical examples are used to demonstrate the effectiveness and versatilities of the proposed approach.

2. Brief review of the classical KF

The derivation of the improved Kalman filter with unknown excitations (KF-UI) is based on the scheme of the classical KF. Therefore, the classical KF is briefly reviewed in this section.

The equation of motion for a linear structural system can be expressed in the discrete form in state space as

$$\boldsymbol{X}_{k+1} = \boldsymbol{A}_k \boldsymbol{X}_k + \boldsymbol{B}_k \boldsymbol{f}_k + \boldsymbol{w}_k \tag{1}$$

where $\boldsymbol{X}_k$ is the state vector at time $t = k\Delta t$ with Δt being the sampling time step. $\boldsymbol{A}_k$ is the state transformation matrix, $\boldsymbol{f}_k$ is the external excitation vector with influence matrix $\boldsymbol{B}_k$, and $\boldsymbol{w}_k$ is the model noise (uncertainty) with zero mean and a covariance matrix $\boldsymbol{Q}_k$.

In practice, only partial structural responses can be measured. The discrete form of the observation equation can be expressed as

$$\boldsymbol{Y}_k = \boldsymbol{C}_k \boldsymbol{X}_k + \boldsymbol{D}_k \boldsymbol{f}_k + \boldsymbol{v}_k \tag{2}$$

where $\boldsymbol{Y}_k$ is the measured response vector, $\boldsymbol{C}_k$ and $\boldsymbol{D}_k$ are two known measurement matrices associated with structural state and external force vectors, respectively, and $\boldsymbol{v}_k$ is the measurement noise vector, which is assumed a Gaussian white noise vector with zero mean and a covariance matrix $\boldsymbol{R}_k$.

The classical KF consists of the two procedures. The first one is the time update (prediction) procedure, in which

$$\tilde{\boldsymbol{X}}_{k+1|k} = \boldsymbol{A}_k \hat{\boldsymbol{X}}_{k|k} + \boldsymbol{B}_k \boldsymbol{f}_k \tag{3}$$

where $\tilde{\boldsymbol{X}}_{k+1|k}$ and $\hat{\boldsymbol{X}}_{k|k}$ denote the predicted $\boldsymbol{X}_{k+1}$ and estimated $\boldsymbol{X}_k$ at time at time $t = k\Delta t$, respectively. Then, the prediction error of $\tilde{\boldsymbol{X}}_{k+1|k}$ is $\tilde{\boldsymbol{e}}_{k+1|k} = \boldsymbol{X}_{k+1} - \tilde{\boldsymbol{X}}_{k+1|k}$ with the prediction error covariance matrix $\tilde{\boldsymbol{P}}_{k+1|k} = \boldsymbol{E}\left[\tilde{\boldsymbol{e}}_{k+1|k}\ \tilde{\boldsymbol{e}}_{k+1|k}^T\right]$. From Eqs. (1) and (3), it is known that

$$\tilde{\boldsymbol{P}}_{k+1|k} = \boldsymbol{A}_k \hat{\boldsymbol{P}}_{k|k} \boldsymbol{A}_k^T + \boldsymbol{Q}_k \tag{4}$$

where $\hat{\boldsymbol{P}}_{k|k} = \boldsymbol{E}\left[\hat{\boldsymbol{e}}_{k|k}\ \hat{\boldsymbol{e}}_{k|k}^T\right]$ and $\hat{\boldsymbol{e}}_{k|k} = \boldsymbol{X}_k - \hat{\boldsymbol{X}}_{k|k}$.

The second process of KF is the measurement update (correction) procedure, in which

$$\hat{\boldsymbol{X}}_{k+1|k+1} = \tilde{\boldsymbol{X}}_{k+1|k} + \boldsymbol{K}_{k+1}(\boldsymbol{Y}_{k+1} - \boldsymbol{C}_{k+1}\tilde{\boldsymbol{X}}_{k+1|k} - \boldsymbol{D}_{k+1}\boldsymbol{f}_{k+1}) \tag{5}$$

where $\hat{\boldsymbol{X}}_{k+1|k+1}$ is the estimated $\boldsymbol{X}_{k+1}$ given the observations ($\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots, \boldsymbol{Y}_{k+1}$), $\boldsymbol{K}_{k+1}$ is the Kalman gain matrix which can be derived as

$$\boldsymbol{K}_{k+1} = \tilde{\boldsymbol{P}}_{k+1|k}\boldsymbol{C}_{k+1}^{T}\left(\boldsymbol{C}_{k+1}\tilde{\boldsymbol{P}}_{k+1|k}\boldsymbol{C}_{k+1}^{T} + \boldsymbol{R}_{k+1}\right)^{-1} \tag{6}$$

and the covariance matrix of the error $\hat{\boldsymbol{e}}_{k+1|k+1} = \boldsymbol{X}_{k+1} - \hat{\boldsymbol{X}}_{k+1|k+1}$ is obtained by

$$\hat{\boldsymbol{P}}_{k+1|k+1} = (\boldsymbol{I} - \boldsymbol{K}_{k+1}\boldsymbol{H}_{k+1})\tilde{\boldsymbol{P}}_{k+1|k} \tag{7}$$

in which $\boldsymbol{I}$ denotes a unit matrix

In the above scheme of the classical KF, the external input vector $\boldsymbol{f}$ is assumed to be known. This is the limitation of the classical KF.

3. Data fusion based KF-UI

When the external inputs to a linear structural system are unknown, the state equation of the system in the discrete form can be expressed as

$$\boldsymbol{X}_{k+1} = \boldsymbol{A}_k\boldsymbol{X}_k + \boldsymbol{B}_k\boldsymbol{f}_k^u + \boldsymbol{w}_k \tag{8}$$

where $\boldsymbol{f}^u$ denotes the unmeasured external input vector.

3.1 Derivation of the improved KF-UI from the direct extension of the classical KF

Analogous to the scheme of the classical KF described in the above section, $\tilde{\boldsymbol{X}}_{k+1|k}$ is first predicted as

$$\tilde{\boldsymbol{X}}_{k+1|k} = \boldsymbol{A}_k\hat{\boldsymbol{X}}_{k|k} + \boldsymbol{B}_k\hat{\boldsymbol{f}}_{k|k}^u \tag{9}$$

where $\hat{\boldsymbol{f}}_{k|k}^u$ denotes the estimated $\boldsymbol{f}^u$ at time at time $t = k\Delta t$.

Based on the observation equation at time $t=(k+1)\Delta t$

$$\boldsymbol{Y}_{k+1} = \boldsymbol{C}_{k+1}\boldsymbol{X}_{k+1} + \boldsymbol{D}_{k+1}\boldsymbol{f}_{k+1}^u + \boldsymbol{v}_{k+1} \tag{10}$$

and the estimated $\boldsymbol{X}_{k+1}$ in the measurement update (correction) procedure is derived as

$$\hat{\boldsymbol{X}}_{k+1|k+1} = \tilde{\boldsymbol{X}}_{k+1|k} + \boldsymbol{K}_{k+1}(\boldsymbol{Y}_{k+1} - \boldsymbol{C}_{k+1}\tilde{\boldsymbol{X}}_{k+1|k} - \boldsymbol{D}_{k+1}\hat{\boldsymbol{f}}_{k+1|k+1}^u) \tag{11}$$

where $\hat{\boldsymbol{X}}_{k+1|k+1}$ is the estimated $\boldsymbol{X}_{k+1}$ given the observations ($\boldsymbol{Y}_1, \boldsymbol{Y}_2, \ldots, \boldsymbol{Y}_{k+1}$) and $\boldsymbol{K}_{k+1}$ is the Kalman gain matrix.

Under the condition that the number of response measurements (sensors) is larger than the number of unknown external excitations in Eq. (10), $\hat{\boldsymbol{f}}_{k+1|k+1}^u$ can be estimated by minimizing the following error vector as

$$\begin{aligned}\boldsymbol{\Delta}_{k+1} &= \boldsymbol{y}_{k+1} - \boldsymbol{C}_{k+1}\hat{\boldsymbol{X}}_{k+1|k+1} - \boldsymbol{D}_{k+1}\hat{\boldsymbol{f}}^{u}_{k+1|k+1} \\ &= \left(\mathbf{I} - \boldsymbol{C}_{k+1}\boldsymbol{K}_{k+1}\right)\left(\boldsymbol{Y}_{k+1} - \boldsymbol{C}_{k+1}\tilde{\boldsymbol{X}}_{k+1|k}\right) - \left(\mathbf{I} - \boldsymbol{C}_{k+1}\boldsymbol{K}_{k+1}\right)\boldsymbol{D}_{k+1}\hat{\boldsymbol{f}}^{u}_{k+1|k+1}\end{aligned} \tag{12}$$

Then, $\hat{\boldsymbol{f}}^{u}_{k+1|k+1}$ can be estimated from Eq.(12) based on least-squares estimation as

$$\hat{\boldsymbol{f}}^{u}_{k+1|k+1} = \boldsymbol{S}_{k+1}\boldsymbol{D}^{\mathrm{T}}_{k+1}\boldsymbol{R}^{-1}_{k+1}\left(\boldsymbol{I} - \boldsymbol{C}_{k+1}\boldsymbol{K}_{k+1}\right)\left(\boldsymbol{Y}_{k+1} - \boldsymbol{C}_{k+1}\tilde{\boldsymbol{X}}_{k+1|k}\right) \tag{13}$$

in which $\boldsymbol{S}_{k+1} = \left[\boldsymbol{D}^{\mathrm{T}}_{k+1}\boldsymbol{R}^{-1}_{k+1}\left(\boldsymbol{I} - \boldsymbol{C}_{k+1}\boldsymbol{K}_{k+1}\right)\boldsymbol{D}_{k+1}\right]^{-1}$

By inserting $\boldsymbol{Y}_{k+1}$ in Eq. (10) into Eq. (13), the error of the estimated $\hat{\boldsymbol{f}}^{u}_{k+1|k+1}$ can be derived as

$$\hat{\boldsymbol{e}}^{\boldsymbol{f}}_{k+1|k+1} = -\boldsymbol{S}_{k+1}\boldsymbol{D}^{\mathrm{T}}_{k+1}\boldsymbol{R}^{-1}_{k+1}\left(\boldsymbol{I} - \boldsymbol{C}_{k+1}\boldsymbol{K}_{k+1}\right)\left(\boldsymbol{C}_{k+1}\tilde{\boldsymbol{e}}^{\boldsymbol{X}}_{k+1|k} + \boldsymbol{v}_{k+1}\right) \tag{14}$$

where $\hat{\boldsymbol{e}}^{\boldsymbol{f}}_{k+1|k+1} = \boldsymbol{f}^{u}_{k+1} - \hat{\boldsymbol{f}}^{u}_{k+1|k+1}$ and $\tilde{\boldsymbol{e}}^{\boldsymbol{X}}_{k+1|k} = \boldsymbol{X}_{k+1} - \tilde{\boldsymbol{X}}_{k+1|k}$.

From Eqs. (10) and (11), the error $\hat{\boldsymbol{e}}^{\boldsymbol{X}}_{k+1|k+1}$ can be estimated as

$$\begin{aligned}\hat{\boldsymbol{e}}^{\boldsymbol{X}}_{k+1|k+1} &= \left(\boldsymbol{I} + \boldsymbol{K}_{k+1}\boldsymbol{D}_{k+1}\boldsymbol{S}_{k+1}\boldsymbol{D}^{\mathrm{T}}_{k+1}\boldsymbol{R}^{-1}_{k+1}\boldsymbol{C}_{k+1}\right)\left(\boldsymbol{I} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\right)\tilde{\boldsymbol{e}}^{\boldsymbol{X}}_{k+1|k} \\ &\quad - \boldsymbol{K}_{k+1}\left[\boldsymbol{I} - \boldsymbol{D}_{k+1}\boldsymbol{S}_{k+1}\boldsymbol{D}^{\mathrm{T}}_{k+1}\boldsymbol{R}^{-1}_{k+1}\left(\boldsymbol{I} - \boldsymbol{C}_{k+1}\boldsymbol{K}_{k+1}\right)\right]\boldsymbol{v}_{k+1}\end{aligned} \tag{15}$$

Then

$$\begin{aligned}\hat{\boldsymbol{P}}^{\boldsymbol{X}}_{k+1|k+1} &= \left(\boldsymbol{I} + \boldsymbol{K}_{k+1}\boldsymbol{D}_{k+1}\boldsymbol{S}_{k+1}\boldsymbol{D}^{\mathrm{T}}_{k+1}\boldsymbol{R}^{-1}_{k+1}\boldsymbol{C}_{k+1}\right)\left(\boldsymbol{I} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\right)\tilde{\boldsymbol{P}}^{\boldsymbol{X}}_{k+1|k}\left(\boldsymbol{I} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\right)^{T}\left(\boldsymbol{I} + \boldsymbol{K}_{k+1}\boldsymbol{D}_{k+1}\boldsymbol{S}_{k+1}\boldsymbol{D}^{\mathrm{T}}_{k+1}\boldsymbol{R}^{-1}_{k+1}\boldsymbol{C}_{k+1}\right)^{T} \\ &\quad + \boldsymbol{K}_{k+1}\left[\boldsymbol{I} - \boldsymbol{D}_{k+1}\boldsymbol{S}_{k+1}\boldsymbol{D}^{\mathrm{T}}_{k+1}\boldsymbol{R}^{-1}_{k+1}\left(\boldsymbol{I} - \boldsymbol{C}_{k+1}\boldsymbol{K}_{k+1}\right)\right]\boldsymbol{R}_{k+1}\left[\boldsymbol{I} - \boldsymbol{D}_{k+1}\boldsymbol{S}_{k+1}\boldsymbol{D}^{\mathrm{T}}_{k+1}\boldsymbol{R}^{-1}_{k+1}\left(\boldsymbol{I} - \boldsymbol{C}_{k+1}\boldsymbol{K}_{k+1}\right)\right]^{\mathrm{T}}\end{aligned} \tag{16}$$

To minimize the error covariance matrix $\hat{\boldsymbol{P}}^{\boldsymbol{X}}_{k+1|k+1}$, $\boldsymbol{K}_{k+1}$ should be selected as

$$\boldsymbol{K}_{k+1} = \tilde{\mathbf{P}}^{\boldsymbol{X}}_{k+1|k}\boldsymbol{C}^{\mathrm{T}}_{k+1}(\boldsymbol{C}_{k+1}\tilde{\mathbf{P}}^{\boldsymbol{X}}_{k+1|k}\boldsymbol{C}^{\mathrm{T}}_{k+1} + \boldsymbol{R}_{k+1})^{-1} \tag{17}$$

Therefore, the estimated $\hat{\boldsymbol{P}}^{\boldsymbol{X}}_{k+1|k+1}$ in Eq. (19) can be simplified as

$$\hat{\boldsymbol{P}}^{\boldsymbol{X}}_{k+1|k+1} = \left(\boldsymbol{I} + \boldsymbol{K}_{k+1}\boldsymbol{D}_{k+1}\boldsymbol{S}_{k+1}\boldsymbol{D}^{\mathrm{T}}_{k+1}\boldsymbol{R}^{-1}_{k+1}\boldsymbol{C}_{k+1}\right)\left(\boldsymbol{I} - \boldsymbol{K}_{k+1}\boldsymbol{C}_{k+1}\right)\tilde{\boldsymbol{P}}^{\boldsymbol{X}}_{k+1|k} \tag{18}$$

The covariance matrix for error $\hat{\boldsymbol{e}}^{\boldsymbol{f}}_{k+1|k+1}$ can be derived as

$$\begin{aligned}\hat{\boldsymbol{P}}^{\boldsymbol{f}}_{k+1|k+1} &= \boldsymbol{S}_{k+1}\boldsymbol{D}^{\mathrm{T}}_{k+1}\boldsymbol{R}^{-1}_{k+1}\left(\boldsymbol{I} - \boldsymbol{C}_{k+1}\boldsymbol{K}_{k+1}\right)\left(\boldsymbol{C}_{k+1}\tilde{\boldsymbol{P}}^{\boldsymbol{X}}_{k+1|k}\boldsymbol{C}^{T}_{k+1} + \boldsymbol{R}_{k+1}\right)\left(\boldsymbol{I} - \boldsymbol{C}_{k+1}\boldsymbol{K}_{k+1}\right)^{T}\boldsymbol{R}^{-T}_{k+1}\boldsymbol{D}_{k+1}\boldsymbol{S}^{\mathrm{T}}_{k+1} \\ &= \boldsymbol{S}_{k+1}\boldsymbol{D}^{\mathrm{T}}_{k+1}\left(\boldsymbol{I} - \boldsymbol{C}_{k+1}\boldsymbol{K}_{k+1}\right)^{T}\boldsymbol{R}^{-T}_{k+1}\boldsymbol{D}_{k+1}\boldsymbol{S}^{\mathrm{T}}_{k+1} = \boldsymbol{S}_{k+1}\end{aligned} \tag{19}$$

Also, the error covariance matrix $\hat{\boldsymbol{P}}^{Xf}_{k+1|k+1}$ can be estimated as

$$\hat{\boldsymbol{P}}^{Xf}_{k+1|k+1} = \left(\hat{\boldsymbol{P}}^{Xf}_{k+1|k+1}\right)^T = E\left[\hat{\boldsymbol{e}}^{X}_{k+1|k+1}\left(\hat{\boldsymbol{e}}^{f}_{k+1|k+1}\right)^T\right] = -\boldsymbol{K}_{k+1}\boldsymbol{D}_{k+1}\boldsymbol{S}_{k+1} \tag{20}$$

From Eqs. (8) and (9), the error $\tilde{\boldsymbol{e}}^{X}_{k+1|k}$ is derived as

$$\tilde{\boldsymbol{e}}^{X}_{k+1|k} = \boldsymbol{A}_k\hat{\boldsymbol{e}}^{X}_{k|k} + \boldsymbol{B}_k\hat{\boldsymbol{e}}^{f}_{k|k} + \boldsymbol{w}_k \tag{21}$$

and the error covariance matrix $\tilde{\boldsymbol{P}}^{X}_{k+1|k}$ can be obtained by

$$\tilde{\boldsymbol{P}}^{X}_{k+1|k} = \left[\boldsymbol{A}_k \;\; \boldsymbol{B}_k\right]\begin{bmatrix}\hat{\boldsymbol{P}}^{X}_{k|k} & \hat{\boldsymbol{P}}^{Xf}_{k|k}\\ \hat{\boldsymbol{P}}^{fX}_{k|k} & \hat{\boldsymbol{P}}^{f}_{k|k}\end{bmatrix}\begin{bmatrix}\boldsymbol{A}_k^T\\ \boldsymbol{B}_k^T\end{bmatrix} + \boldsymbol{Q}_k \tag{22}$$

In summary, the analytical derivation of the above KF-UI is completely based on the classical KF and the recursive procedures of the proposed KF-UI are analogous to those of the classical KF described in the above section. The resulting filter has the structure of the classical Kalman filter, except that the true value of the input is replaced by an optimal estimate. When the inputs are available, the proposed KF-UI reduces to the classical Kalman filter Therefore, the proposed KF-UI is a direct extension of the classical KF, which greatly simplifies the complex derivations in previous KF-UI approaches (Gillijns and Moor 2007, Wu *et al.* 2009, Pan *et al.* 2010).

3.2 Data fusion of acceleration and displacement measurement in the improved KF-UI

In practice, accelerometers are most often used in structural dynamics applications. However, previous KF-UI approaches by using sparse noisy acceleration measurements are inherently unstable which leads poor tracking and the so-called spurious low-frequency drifts in the estimated of unknown inputs and structural displacement. These drifts are caused by acceleration's insensitivity to any quasi-static component in the inputs.

The detectability analysis shows that the typical case where forces are reconstructed from acceleration measurements through Kalman filtering on a structure will inevitably lead to divergence issues (Naets *et al.* 2015). This conclusion can also be considered from a physical point of view. At any given time it is not clear whether acceleration is the effect of external forces or from the elastic restoring force due to a certain position. Therefore, the joint input and state estimation based solely on to sparse noisy acceleration measurements is an ill-posed identification problem.

Although either regularization scheme or post-signal processing can be used to treat the spurious low-frequency drift problem, these approaches prohibit the on-line identification of structural state and unknown inputs. In this paper, it is proposed to add partial measured displacements to the acceleration measurements since acceleration and displacement measurements contains high and low frequencies vibration characteristics, respectively. Data fusion of measured displacement and acceleration responses, are used in the observation equation, i.e., Eq. (10) is expressed as:

$$\boldsymbol{Y}_k = \begin{Bmatrix} \boldsymbol{Y}_k^a \\ \boldsymbol{Y}_k^d \end{Bmatrix} = \begin{bmatrix} \boldsymbol{E}_k^a & \boldsymbol{F}_k^a \\ \boldsymbol{L}_s & 0 \end{bmatrix} \begin{Bmatrix} \boldsymbol{x}_k \\ \dot{\boldsymbol{x}}_k \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{G}_k \\ 0 \end{Bmatrix} \boldsymbol{f}_k^u + \boldsymbol{v}_k = \boldsymbol{C}_k \boldsymbol{X}_k + \boldsymbol{D}_k \boldsymbol{f}_k^u + \boldsymbol{v}_k \tag{23}$$

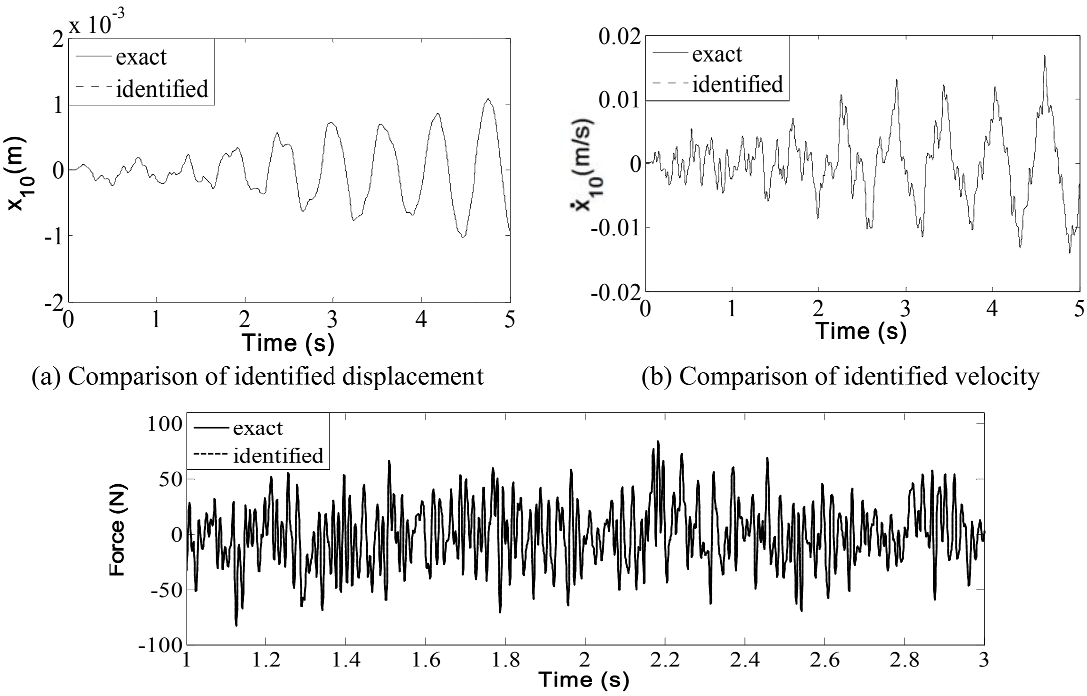
in which, $\boldsymbol{E}_k^a$ and $\boldsymbol{F}_k^a$ are two measurement matrices associated with the measured accelerations, $\boldsymbol{G}_k$ is the influence matrix of external inputs on the acceleration responses, and $\boldsymbol{L}_s$ denotes the locations of measured displacements.

4. Numerical validations of the proposed KF-UI

To validate the proposed KF-UI, two numerical examples are used for real time estimation of structural states and unknown inputs using data fusion of partially measured accelerations and displacements. The theoretically computed displacement and acceleration responses are superimposed with corresponding white noises to consider the influence of measurement noises. These polluted responses are treated as “measured responses” for the identification problem.

4.1 Identification of multi-story shear building and unknown input

A twenty-story shear building is used as an example. Parameters of the building are assumed as: floor mass m_i=60 kg, floor stiffness k_i=1.2x10^6 N/m, floor damping c_i=1000 Ns/m (i=1,2,..,20), respectively. An input of wide-banded white noise is applied to the top floor of the building.

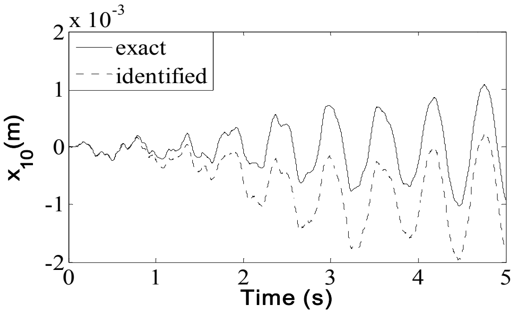


(a) Comparison of identified displacement

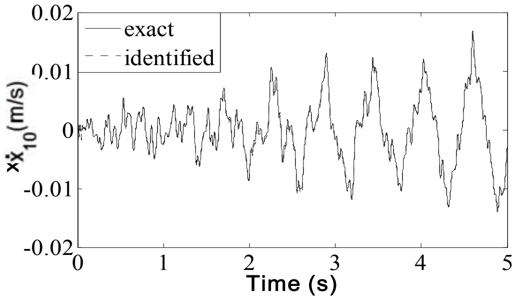
(b) Comparison of identified velocity

(c) Comparison of identified unknown input

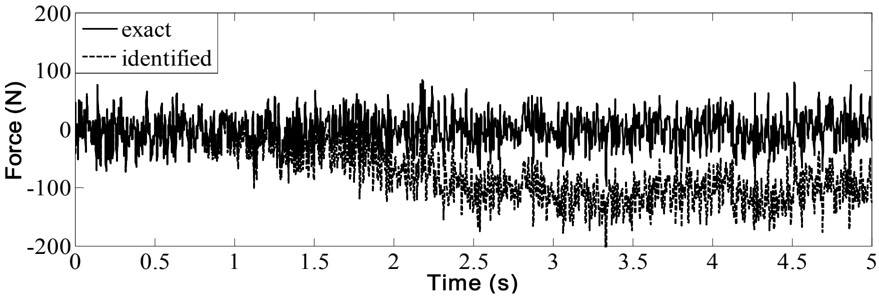
Fig. 1 Comparisions of identified structural state and input with unpolluted acceleration measuremnts



(a) Drift in the identified displacement

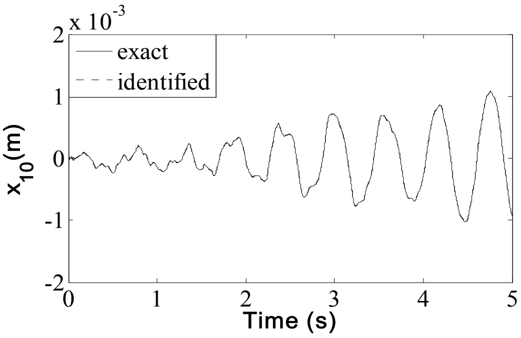


(b) Comparison of identified velocity

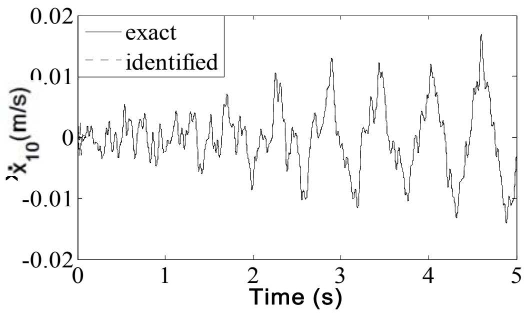


(c) Drift in the identified unknown input

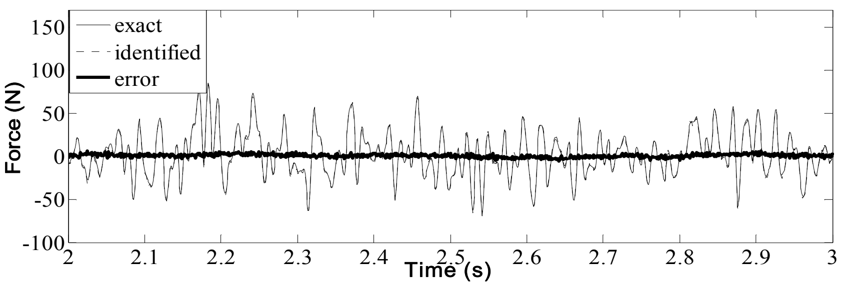
Fig. 2 Comparisions of identified structural state and input with noisy acceleration measuremnts



(a) Comparison of identified displacement



(b) Comparison of identified velocity



(c) Comparison of identified unknown input

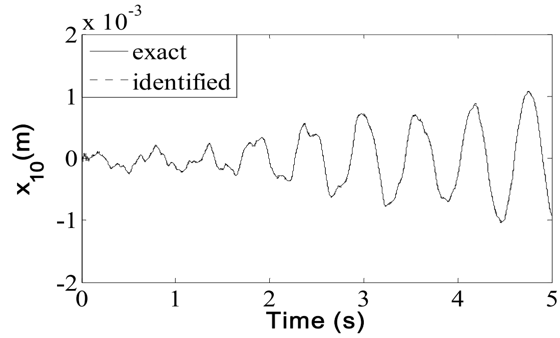
Fig. 3 Comparisions of identified results with data fusion of 5% noisy measuremnts

First, only three acceleration measurements at the 6th, 14th and 20th story floors are used in the proposed KF-UI. If measurement noises are not considered, the joint identification results of structural displacement and velocity responses as well as the unknown input are very accurate with comparisons to their exact values as shown in Fig. 1.

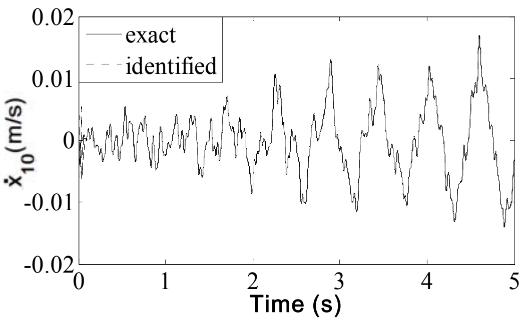
However, in the presence of measurement noise in the measured accelerations, significant drifts occur in the identified displacement and unknown input, as shown in Fig. 2 where measured accelerations contains noises with a 5% noise- to- signal ratio in RMS.

To circumvent the spurious low-frequency drift problem, partial measured displacements are added in the improved KF-UI. For this relatively simple structural model, the displacement at the 2nd floor of the shear building is added in the measured signals. Fig. 3 shows the comparisons of identified structural state and input with their exact values. It is clearly demonstrated that the so-called drift can be avoided by the improved KF-UI with data fusion of 5% noisy acceleration and displacement measurements.

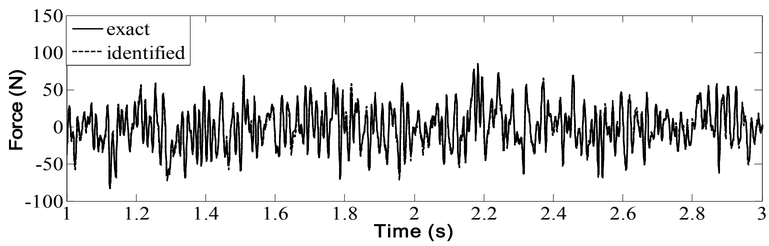
To further validate the performances of the proposed KF-UI, measurement noise level is increased to 10% in RMS. From the comparisons shown in Fig. 4, it is noted that the identified structural state and the unknown input are still in good agreements with the exact values.



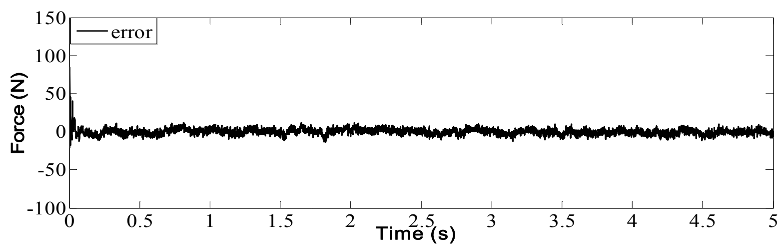
(a) Comparison of identified displacement



(b) Comparison of identified velocity



(c) Comparison of identified unknown input



(d) Error of identified unknown input

Fig. 4 Comparisions of identified results with data fusion of 10% noisy measuremnts

4.2 Identification of a truss-structure and unknown input

To validate the proposed KF-UI for the identification of other type structures with unknown input, the identification of a plane truss and unknown input is studied. As shown in Fig. 5, the truss consists of 11 uniform members. The length of each horizontal and inclined bar are $2m, \sqrt{2}$ m, respectively. Other parameters of the truss are: cross section area $A = 7.854 \times 10^{-5}\,\mathrm{m}^2$, Young's module $E = 2 \times 10^{11}$ pa, mass density of truss member $\rho = 7.8 \times 10^3 kg/m^3$ and the mass is concentrated on each node. The truss is subjected to an unknown input in the vertical direction at node 4. In this example, Rayleigh damping $\boldsymbol{C}=\alpha\boldsymbol{M}+\beta\boldsymbol{K}$ is employed with α=0.6993 and β=0.0011.

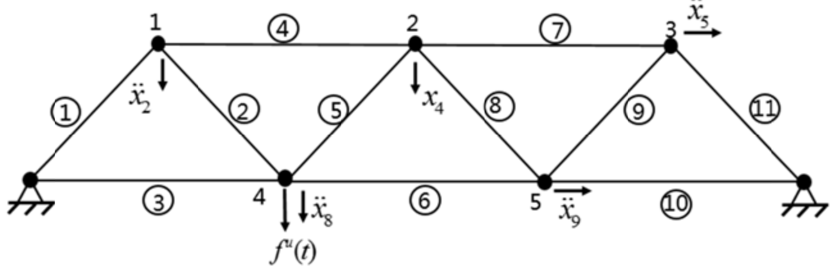
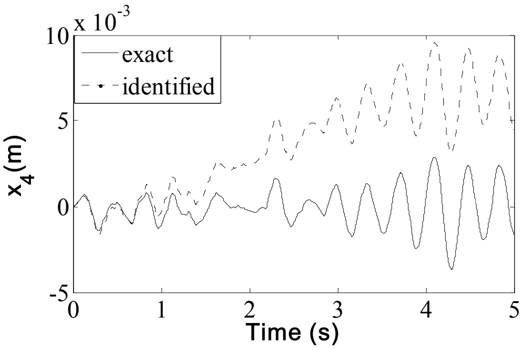
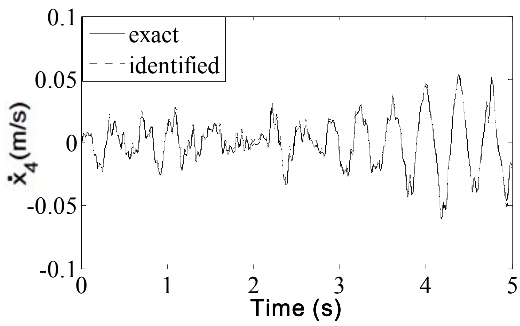


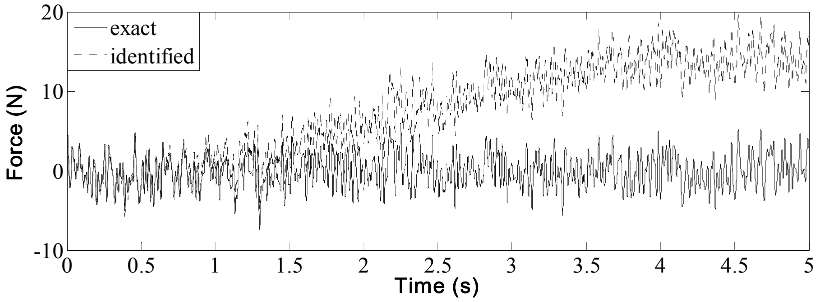
Fig. 5 A plane truss under unknown input



(a) Drift in the identified displacement

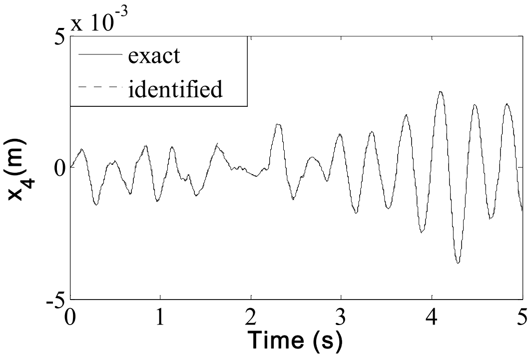


(b) Comparison of identified velocity

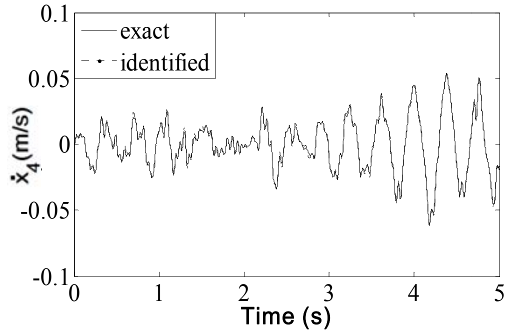


(c) Drift in the identified unknown input

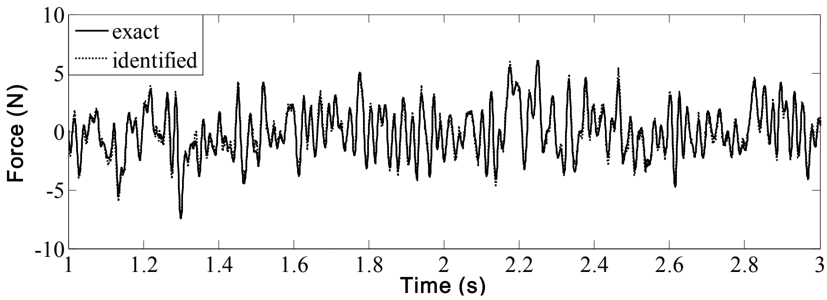
Fig. 6 Comparisions of identified structural state and input with noisy acceleration measuremnts



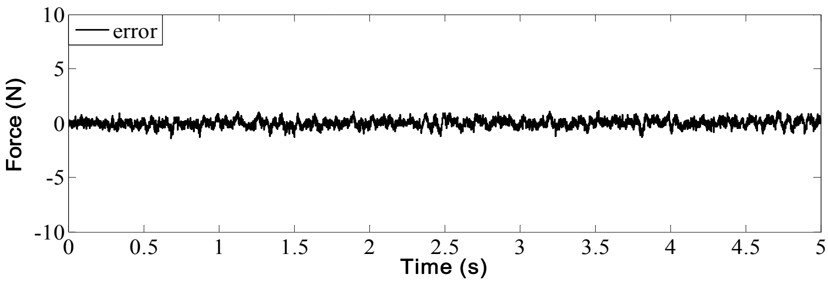
(a) Comparison of identified displacement



(b) Comparison of identified velocity



(c) Comparison of identified unknown input



(d) Error of identified unknown input

Fig. 7 Comparisions of identified results with data fusion of 10% noisy measuremnts

As indicted in Fig. 5, acceleration responses in the vertical directions of nodes 1 and 4 and the lateral directions of node 3 and 5 are measured. If only these four measured noisy acceleration responses are used in the identification by the KF-UI, significant drifts in the identified displacement and unknown input are shown in Fig. 6 where measured accelerations contains noises with a 5% noise- to- signal ratio in RMS.

To prevent the above drift problem, partial measured displacements are added in combination with acceleration measurements for the joint identification problem by the proposed KF-UI. For this relatively small size structural model, the displacement in the vertical direction at node 2 in Fig. 5 is added as the measured displacement signals. Data fusion of this measured displacement and the above four accelerations are used in the observation equation, i.e.,Eq.(27). As shown by the comparisons of identified structural state and input with their exact values in Fig. 7, it is clearly demonstrated that the so-called drifts in estimated structural state and input are avoided by the improved KF-UI with data fusion of 10% noisy acceleration and displacement measurements.

5. Conclusions

In this paper, an improved KF-UI algorithm using data fusion of partial acceleration and displacement measurements is proposed. The analytical derivation of the proposed KF-UI is a direct extension of the classical KF and the resulting filter has the structure of the Kalman filter, except that the true value of the input is replaced by an optimal estimate, so it greatly simplifies the complex derivations in previous KF-UI approaches. Moreover, the data fusion based KF-UI prevents the so-called drifts in the estimated structural state vector and unknown external inputs by previous approaches. Therefore, the proposed KF-UI provides an efficient algorithm of real time joint estimation of structural states and the unknown inputs, which is also important for optimal structural vibration control under known external inputs. Such analytical recursive solution for date fusion based KF-UI is not available in the previous literature. Some numerical examples have demonstrated the effectiveness and performance of the proposed approach. Even in the presence of high level of measurement noises, the joint estimations of structural states and the unknown inputs are still quite accurate.

In the proposed KF-UI, it is requested that (i) the number of response measurements (sensors) should be larger than the total number of unknown external inputs, and (ii) the acceleration responses at unknown excitation locations should be measured, The first requirement is reasonable to avoid the ill-posed identification of unknown inputs while the removal of the later requirement needs further investigations. Also, displacement measurement may be absent in practical engineering, but strain data can be easily measured. Displacement measurement can be replaced by strain data in the proposed KF-UI based on data fusion. These researches are undertaken by the authors.

Acknowledgments

This research is supported by the National Natural Science Foundation of China (NSFC) through Grant Nos. 51308482 and 51378445.

References

Azam, S.E., Chatzi, E. and Costas Papadimitriou, C. (2015), "A dual Kalman filter approach for state estimation via output-only acceleration measurements", *Mech. Syst. Signal Pr.*, **60-61**, 866-886.

Ding, Y., Law, S.S., Wu, B., Xu, G.S., Lin, Q., Jiang, H.B. and Miao, Q.S. (2013), "Average acceleration discrete algorithm for force identification in the state space", *Eng. Struct.*, **56**, 1880-1892.

Gillijns, S. and Moor, B.D. (2007), "Unbiased minimum-variance input and state estimation for linear discrete-time systems with direct feed through", *Automatica*, **43**, 934-937.

Jiang, S.F., Fu, C. and Zhang, C.M. (2011), "A hybrid data-fusion system using modal data and probabilistic neural network for damage detection", *Adv. Eng. Softw.*, **42**(6), 368-374.

Kalman, R.E. (1960), "A new approach to linear filtering and prediction problems", *J. Basic Eng.*, **82**(1), 35-45.

Lei, Y., Chen, F. and Zhou H. (2015), "A two-stage and two-step algorithm for the identification of structural damage and unknown excitations: Numerical and experimental studies", *Smart Struct. Syst*., **15**(1), 57-80.

Lei, Y., Jiang, Y.Q. and Xu, Z.Q. (2012), "Structural damage detection with limited input and output measurement signals", *Mech. Syst. Signal Pr.*, **28**, 229-243.

Lei, Y., Liu, C. and Liu, L.J. (2014), "Identification of multistory shear buildings under unknown earthquake excitation using partial output measurements: numerical and experimental studies", *Struct. Control Health Monit.*, **21**(5), 774-783.

Lin, D.C. (2010), "Input estimation for nonlinear systems", *Inverse Probl. Sci. En.*, **18(**5), 673-689.

Liu, J., Sun, X.S., Han, X., Jiang, C. and Yu, D.J. (2014), "A novel computational inverse technique for load identification using the shape function method of moving least square fitting", *Comput. Struct*., **144**, 127-137.

Liu, K., Law, S.S., Zhu, X.Q. and Xia, Y. (2014), "Explicit form of an implicit method for inverse force identification", *J. Sound Vib*., **333**, 730-744.

Lourens, E., Reynders, E., Roeck, G.De, Degrande, G. and Lombaert, G. (2012), "An augmented Kalman filter for force identification in structural dynamics", *Mech. Syst. Signal Pr.*, **27**, 446-460.

Ma, C.K., Chang, J.M. and Lin, D.C. (2003), "Input forces estimation of beam structures by an inverse method", *J. Sound Vib.*, **259**(2), 387-407.

Mao, Y.M., Guo, X.L. and Zhao, Y. (2010), "A state space force identification method based on Markov parameters precise computation and regularization technique", *J. Sound Vid.*, **329**(15), 3008-3019.

Naets, F., Cuadrado, J. and Desmet, W. (2015), "Stable force identification in structural dynamics using Kalman filtering and dummy-measurements", *Mech. Syst. Signal Pr*., **50-51**, 235-248.

Pan, S.W., Su, H.Y., Wang, H. and Chu, J. (2010), "The study of input and state estimation with Kalman filtering", *Inst. Measurement Control*, **33**(8), 901-918.

Sun, H., Feng, D.M., Liu, Y. and Feng, M.Q. (2015), "Statistical regulari for identification of structural parameters and external loadings using state space model", *Comput. - Aided Civil Infrastruct. Eng*., **30**(11), 843-858.

Wang, L.J. and Xie, Y.X. (2015), "A novel regularization method and application to load identification of composite laminated cylindrical shell", *J. Appl. Anal. Comput*., **5**(4), 570-580.

Wang, T., Wan, Z.M., Wang, X.L. and Hu Y.J. (2015), "A novel state space method for force identification based on the Galerkin weak formulation", *Comput. Struct.*, **157**, 132-141.

Wu, A.L., Loh, C.H., Yang, J.N., Weng, J.H., Chen, C.H. and Ueng, T.S. (2009), "Input force identification: application to soil-pile interaction", *Struct. Control Health Monit.,* **16**, 223-240.