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# Mode shape expansion with consideration of analytical modelling errors and modal measurement uncertainty

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**Abstract.** Mode shape expansion is useful in structural dynamic studies such as vibration based structural health monitoring; however most existing expansion methods can not consider the modelling errors in the finite element model and the measurement uncertainty in the modal properties identified from vibration data. This paper presents a reliable approach for expanding mode shapes with consideration of both the errors in analytical model and noise in measured modal data. The proposed approach takes the perturbed force as an unknown vector that contains the discrepancies in structural parameters between the analytical model and tested structure. A regularisation algorithm based on the Tikhonov solution incorporating the L-curve criterion is adopted to reduce the influence of measurement uncertainties and to produce smooth and optimised expansion estimates in the least squares sense. The Canton Tower benchmark problem established by the Hong Kong Polytechnic University is then utilised to demonstrate the applicability of the proposed expansion approach to the actual structure. The results from the benchmark problem studies show that the proposed approach can provide reliable predictions of mode shape expansion using only limited information on the operational modal data identified from the recorded ambient vibration measurements.

**Keywords:** mode shape expansion; modelling errors; perturbed force; measurement uncertainty; regularisation algorithm; Canton Tower benchmark problem

## 1. Introduction

Mode shape expansion has many applications in structural dynamic studies, such as model updating (Mottershead and Friswell 1993), test-analysis correlation study (Avitabile 1999, Kammer 1991), structural system identification (Tee *et al*. 2009) and structural damage detection (Chen 2008, Chen and Bicanic 2010). In general, the analytical mode shapes consisting of a full set of degrees of freedom (DOFs) can be obtained from finite element analysis for the constructed analytical model. The measured data set of a dynamic test for the actual structure however is usually incomplete and only exists at the DOFs associated with the positions where sensors are installed, because the number of sensors for measurement is often limited. To overcome this problem, mode shape expansion should be considered to eliminate the requirement of complete measurements of the actual tested structure.

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Most existing mode shape expansion methods involve a model reduction transformation matrix as an expansion mechanism to obtain the unmeasured mode components. A review of published literature shows that the first major step toward a method of reducing or condensing the dimension of the eigenproblem of a structural dynamic system appeared in the paper published by Guyan (1965). Guyan static expansion method assumes that the inertia forces at the unmeasured DOFs are negligible. Since the dynamic effects were ignored in this method, the error can be large for dynamic problems with large mass inertia. Hence, many methods have subsequently been proposed to improve the accuracy of expanded mode shapes. The inertia terms were considered partially by Kidder (1975), and also included statically in the Improved Reduced System (IRS) method (O'Callahan 1989). The more accurate System Equivalent Reduction Expansion Process (SEREP) technique can preserve the correct dynamic structural response (O'Callahan *et al.* 1989). However, it may produce poor expansion estimates if the experimental mode shapes are not well correlated with the corresponding analytical mode shapes. Also, the expansion of individual modes depends on the entire set of measured mode shapes, and the accuracy of the expansion estimates may be affected by the uncertainty of unrelated measured mode shapes.

Another category of expansion technique can be expressed in terms of a constrained minimization problem. For example, Kammer (1987) used an unconstrained least-squares minimization approach to minimize the error between the expanded mode shape and the paired analytical mode shape. To relax the hard constraints and to incorporate the uncertainties in the measurements and in the analytical model, the penalty method and least-squares minimization with quadratic inequality constraints have been proposed (Levine-West *et al.* 1996). In general, these methods require information about the modal properties or structural parameters such as mass and stiffness of the analytical model in the mode shape expansion processes. They do not consider the modelling errors due to the discrepancies in structural parameters between the analytical model and the tested structure. In addition, they are unable to implement effective measures to reduce the influence of inevitable measurement uncertainties on the estimates of mode shape expansion. Recently, a new method for expanding experimental mode shapes has been proposed by Chen (2010) using the perturbed force approach which considers the modelling errors of the analytical model in the expansion processes. The results show that the proposed approach has better performance than the commonly used existing expansion methods, particularly in the cases with limited modal data measurements, large modelling errors and severe measurement noise.

This paper presents a reliable approach for expanding incomplete experimental mode shapes by considering the modelling errors in the analytical model and reducing the influence of uncertainties in measured modal data. The proposed approach takes the perturbed force as an unknown vector that contains the difference in structural parameters between the analytical model and the tested structure. The unknown perturbed force is then obtained from an inverse prediction of the governing equations that are constructed from the analytical and measured modal data, without requiring information on the structural parameters of the tested structure. A regularisation algorithm based on the Tikhonov solution incorporating the L-curve criterion is employed to reduce the effect of measurement noise and to produce smooth and reliable expansion estimates. The Canton Tower benchmark problem established by the Hong Kong Polytechnic University (HKPU) is employed to demonstrate the effectiveness and applicability of the proposed approach for expanding experimental mode shapes of the actual civil engineering structure (Chen *et al.* 2011, Ni *et al.* 2012). Modal properties of the structure such as frequencies and incomplete experimental mode shapes are identified from the recorded ambient acceleration measurements by using the stochastic subspace identification technique. The proposed expansion technique is then performed to expand the incomplete experimental data set onto the associated analytical coordinate set. The results for the benchmark problem show that the proposed approach with consideration of both modelling errors and measurement uncertainty produces reliable estimates of the expanded mode shapes for the actual complex civil engineering structure.

## 2. SHM benchmark problem

The Canton Tower is located in Guangzhou, China, with a total height of 610 m, i.e., a 454 m high main tower and a 156 m high antenna mast. The structure is one of the tallest completed towers in the world, and became operational in September 2010. The tower comprises a reinforced concrete interior tube and a steel external tube which consists of 24 concrete-filled-tube columns. A sophisticated long-term Structural Health Monitoring (SHM) system, consisting of more than 700 sensors, has been designed and implemented for the real-time monitoring of the structure at both in-construction and in-service stages. A total number of 20 uni-axial accelerometers were installed at eight different levels and mounted firmly to the shear wall of the inner structure. Four uni-axial accelerometers were placed in the 4th and 8th floors and two uni-axial accelerometers were equipped in each of the remaining six floors, as shown in Fig. 1(a). The tower has been established as an international benchmark problem for SHM studies by the HKPU. Details of the SHM system and the tower can be found in the studies by Ni *et al*. (2009, 2011).

In order to undertake SHM and associated studies, a reduced-order 3D beam model was developed on the basis of the complex 3D full finite element model, as shown in Fig. 1(b) (Ni *et al*. 2012). In the reduced analytical model, the tower is modelled as a cantilever beam with 38 nodes (nodes with installed sensors are marked in the figure) and 37 beam elements, i.e., 27 elements for the main tower and 10 elements for the upper mast. The vertical displacement of the structure is ignored in the reduced analytical model, giving a total of 5 DOFs for each node, i.e., two horizontal translational DOFs and three rational DOFs. Therefore, each beam element has 10 DOFs and the reduced analytical model has a total of 185 DOFs with a fixed end at the base. The reduced model was fine tuned so that its dynamic characteristics agree with those of the full model as closely as possible. The frequencies and mode shapes of the first 15 modes of the reduced and full models were compared and they are very close to each other.

The ambient vibration measurements, recorded through the SHM system for 24 hours from 18:00 pm on 19 January to 18:00 pm on 20 January 2010 with a sampling frequency of 50 Hz, were released by the HKPU for the Phase I of the Benchmark Problem for SHM of High-Rise Slender Structures. The acceleration measurements recorded by accelerometers 1 and 2 during the first 60 seconds after 19:00 pm on 19 January are plotted in Fig. 1(c). In this study, the ambient vibration measurements recorded from 19:00 pm to 20:00 pm on 19 January 2010 are adopted for identifying operational modal properties of the tower. The stochastic subspace identification technique is then utilised to extract operational modal properties of the tested structure, such as frequencies, damping ratios and mode shapes (DOF's readings). The technique can identify the state space matrices on the basis of the acceleration measurements by using robust numerical techniques, such as singular value decomposition (Peeters and De Roeck 1999), and produce reliable operational modal properties from ambient vibration measurements.

From the operational modal analysis, the natural frequencies of the tower can be extracted from

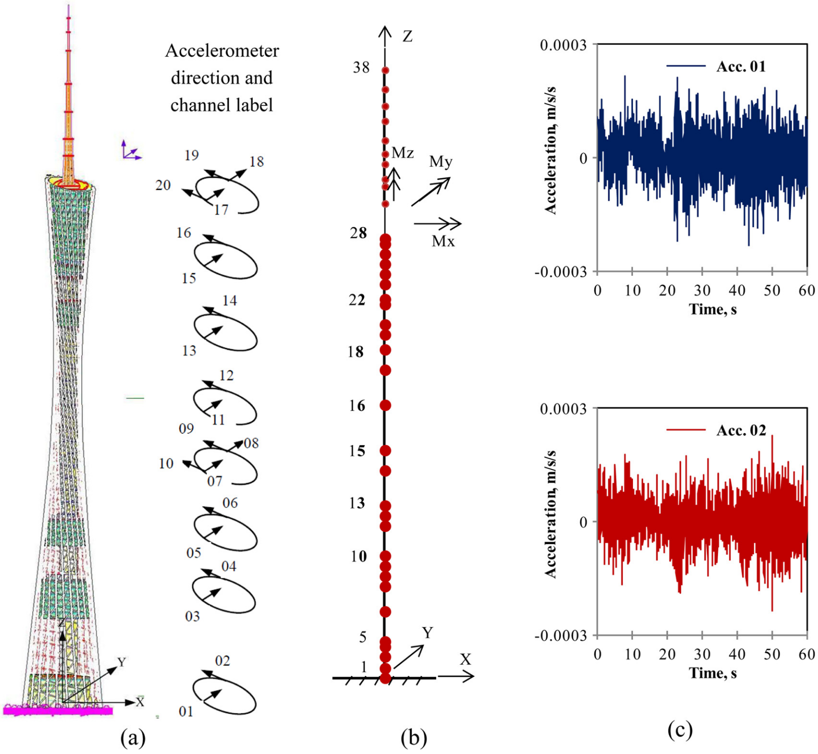



Fig. 1 The Canton Tower benchmark problem: (a) positions of installed accelerometers (after Chen *et al*. 2011), (b) reduced-order finite element model and (c) acceleration measurements of accelerometers 01 and 02

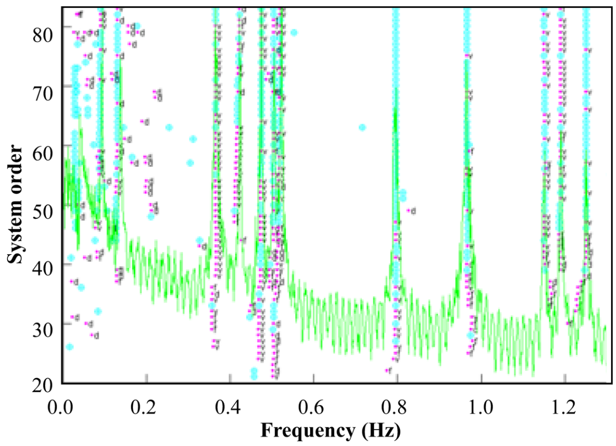



Fig. 2 The stabilisation diagram of measured acceleration data used for the stochastic subspace identification technique

the recorded output only vibration measurements, as indicated in the stabilisation diagram shown in Fig. 2. The identified operational modal properties including frequencies, damping ratios and mode shapes, together with the modal data from the finite element analysis, are summarised in Table 1.

Table 1 Operational modal properties identified from the recorded ambient vibration measurements and modal data calculated from the reduced finite element model

| Mode | Analytical frequency (Hz) | Measured frequency (Hz) | Difference (%) | Damping (%) | MAC value | Mode description |
|---|---|---|---|---|---|---|
| 1 | 0.111 | 0.090 | 23.81 | 2.97 | 0.904 | Short-axis bending |
| 2 | 0.159 | 0.131 | 21.19 | 6.18 | 0.938 | Long-axis bending |
| 3 | 0.347 | 0.366 | -5.17 | 0.24 | 0.888 | Short-axis bending |
| 4 | 0.369 | 0.422 | -12.50 | -1.50 | 0.888 | Long-axis bending |
| 5 | 0.400 | 0.474 | -15.59 | 0.07 | 0.869 | Short-axis bending |
| 6 | 0.462 | 0.504 | -8.46 | 0.38 | 0.104 | Torsion |
| 7 | 0.487 | 0.520 | -6.31 | 0.07 | 0.783 | Long and short-axis bending |
| 8 | 0.738 | 0.796 | -7.21 | 0.20 | 0.797 | Short-axis bending |
| 9 | 0.904 | 0.966 | -6.44 | 0.33 | 0.771 | Long-axis bending |
| 10 | 0.997 | 1.151 | -13.34 | 0.10 | 0.701 | Short-axis bending |
| 11 | 1.037 | 1.191 | -12.86 | 0.03 | 0.753 | Long-axis bending |
| 12 | 1.121 | 1.251 | -10.38 | 0.16 | 0.161 | Torsion |

The identified damping ratios may have relatively higher uncertainty by the covariance-driven stochastic subspace identification technique, with possible negative damping occasionally due to an over-estimation of the model order (De Roeck and Peeters 2011). The results show that the difference between the frequencies identified from the ambient vibration measurements and those calculated from the analytical model is relatively large, with the highest relative difference of 23.81% for the fundamental frequency. The MAC diagonal values, calculated from the identified incomplete mode shapes and the analytical eigenvectors restricting to the same DOFs, indicate good correlation between the identified and analytical modes, except two torsion modes, i.e., the $6^{th}$ and $12^{th}$ modes.

The significant difference between the measured and analytical frequencies indicates that there could be large modelling errors in the reduced analytical model. In addition, only a limited number of DOF's readings measured by 20 installed accelerometers are available for SHM and related studies, comparing with a total number of 185 DOFs of the analytical model. These identified operational modal data have inevitable measurement noise due to uncertainties in the recorded vibration data and the system identification process. From the modal analysis for the benchmark problem, two important issues, i.e., analytical modelling errors and modal measurement uncertainty, need to be addressed in the mode shape expansion processes. Therefore, a reliable approach is required to be developed to provide accurate results of mode shape expansion by considering the modelling errors and measurement uncertainty in the expansion processes.

## 3. Expansion considering modelling errors

In mode shape expansion processes, a reference analytical model, i.e., a finite element model, is usually required to represent the dynamic characteristics of the actual tested structure. Assume that the global stiffness and mass matrices for the analytical model, $\boldsymbol{K}$ and $\boldsymbol{M}$, are obtained from the details of the design and construction of the structure. Thus, the modal parameters for the analytical model such as $i$th eigenvalue and the corresponding unit mass normalized eigenvector, $\lambda_i$ and $\phi_i$, can be calculated by solving its characteristic equations. However, the obtained analytical modal properties may not agree well with those identified from the field measurements due to uncertainty in modelling the associated tested structure. These modeling errors could be represented by the unknown perturbations of stiffness ($\Delta\boldsymbol{K}$) and mass ($\Delta\boldsymbol{M}$) between the analytical model and tested structure. The global stiffness matrix ($\hat{\boldsymbol{K}}$) and mass matrix ($\hat{\boldsymbol{M}}$) of the tested dynamic system then can be expressed as

$$\hat{\boldsymbol{K}} = \boldsymbol{K} + \Delta\boldsymbol{K} \tag{1}$$

$$\hat{\boldsymbol{M}} = \boldsymbol{M} + \Delta\boldsymbol{M} \tag{2}$$

The characteristic equation for the tested structure is now given as follows

$$(\hat{\boldsymbol{K}} - \hat{\omega}_i^2 \hat{\boldsymbol{M}})\hat{\boldsymbol{\phi}}_i = 0 \tag{3}$$

where modal parameters $\hat{\omega}_i$ and $\hat{\boldsymbol{\phi}}_i$ are the $i$th frequency and the corresponding mode shape for the tested structure, respectively, and $i$ ranges from 1 to $n$ where $n$ represents the total number of DOFs for the structure. It should be noted that the analytical eigenvectors are linearly independent of each other, because the analytical global stiffness and mass matrices are symmetric and positive definite. The $i$th mode shape for the tested structure then can be expressed as a linear combination of the analytical eigenvectors

$$\hat{\boldsymbol{\phi}}_i = \sum_{k=1}^{n} C_{ik}\boldsymbol{\phi}_k \tag{4}$$

where $C_{ik}$ are mode participation factors. Premultiplying Eq. (4) by $\phi_k{}^T\boldsymbol{M}$, and using the mass normalisation of the analytical eigenvectors, yields

$$C_{ik} = \boldsymbol{\phi}_k^T \boldsymbol{M}\hat{\boldsymbol{\phi}}_i \tag{5}$$

In order to obtain a unique solution for the expanded mode shape of the tested structure, the $i$th mode shape for the tested structure is assumed to be mass normalised in the form $\boldsymbol{\phi}_k^T\boldsymbol{M}\hat{\boldsymbol{\phi}}_i = 1$, i.e., the mode participation factor in Eq. (5) $C_{ii} = 1$ (Chen and Bicanic 2010). Therefore, Eq. (4) is rewritten here as

$$\hat{\boldsymbol{\phi}}_i = \boldsymbol{\phi}_i + \sum_{k=1,\, k\neq 1}^{n} C_{ik}\boldsymbol{\phi}_k \tag{6}$$

Pre-multiplying Eq. (3) by $\boldsymbol{\phi}_k^T$ and using the stiffness and mass of the tested structure in Eq. (1) and the mode shape of the tested structure in Eq. (4), leads to

$$\boldsymbol{\phi}_k^T(\Delta\boldsymbol{K} - \hat{\omega}_i^2\Delta\boldsymbol{M})\hat{\boldsymbol{\phi}}_i + \sum_{l=1}^{n} C_{il}\boldsymbol{\phi}_k^T(\boldsymbol{K} - \hat{\omega}_i^2\boldsymbol{M})\boldsymbol{\phi}_l = 0 \tag{7}$$

Define a perturbed force for the $i$th mode, representing a residual force due to the modelling errors in the analytical model, as

$$\boldsymbol{f}_i = (\Delta\boldsymbol{K} - \hat{\omega}_i^2 \Delta\boldsymbol{M})\hat{\boldsymbol{\phi}}_i \tag{8}$$

By using the mass normalisation of the analytical eigenvectors, the mode participation factors are obtained from Eq. (7) as

$$C_{ik} = \frac{1}{(\hat{\omega}_i^2 - \omega_k^2)} \boldsymbol{\phi}_k^T \boldsymbol{f}_i \tag{9}$$

From Eq. (9), the mode participation factors, adopted here for calculating the fully expanded mode shapes, depend on the unknown perturbed force $\boldsymbol{f}_i$, if the frequency of the tested structure is available. Once the perturbed force is determined by the measured modal data, the fully expanded mode shapes will be estimated from Eq. (4), without requiring the information on the structural parameters (i.e., stiffness and mass) of the tested structure. By utilising Eqs. (9) and (6) can be rewritten as

$$\sum_{k=1, k\neq i}^{n} \frac{1}{(\hat{\omega}_i^2 - \omega_k^2)} \boldsymbol{\phi}_k^T \boldsymbol{f}_i \boldsymbol{\phi}_k = \hat{\boldsymbol{\phi}}_i - \boldsymbol{\phi}_i \tag{10}$$

In structural dynamic testing, modal data about natural frequency $\hat{\omega}_i$ and limited number of measured DOF's readings $\hat{\boldsymbol{\psi}}_i$ of dimension $m$ could be identified from the recorded vibration measurements by system identification techniques. The measured incomplete mode shape then could be paired to the analytical eigenvector (restricted to the same dimensions as $\hat{\boldsymbol{\psi}}_i$), $\phi_k^a$, by using Modal Assurance Criterion (MAC) factors, defined as

$$MAC(k,i) = \frac{\left|\boldsymbol{\phi}_k^{aT} \hat{\boldsymbol{\psi}}_i\right|^2}{\left|\boldsymbol{\phi}_k^{aT} \boldsymbol{\phi}_k^a\right| \left|\hat{\boldsymbol{\psi}}_i^T \hat{\boldsymbol{\psi}}_i\right|} \tag{11}$$

Large MAC factors indicate a high degree of similarity between two mode shapes and small MAC factors represent little or even no correlation between two vectors. In order to make the incomplete set of measured DOF's readings, $\hat{\boldsymbol{\psi}}_i$, close to the corresponding part of the analytical eigenvector, $\boldsymbol{\phi}_i^a$, the measured mode shapes need to be scaled with a factor as

$$\hat{\boldsymbol{\phi}}_i^a = \beta_i \hat{\boldsymbol{\psi}}_i \tag{12}$$

in which mode scale factor $\beta_i$ is defined as

$$\beta_i = \frac{\boldsymbol{\phi}_i^{aT} \boldsymbol{M} \hat{\boldsymbol{\psi}}_i}{\hat{\boldsymbol{\psi}}_i^T \boldsymbol{M} \hat{\boldsymbol{\psi}}_i} \tag{13}$$

By considering only measured DOF's readings on the right hand side in Eq. (10), the governing equation for solving for the unknown perturbed force is now expressed in a matrix format as

$$\boldsymbol{A}_i \boldsymbol{f}_i = \beta_i \hat{\boldsymbol{\psi}}_i - \boldsymbol{\phi}_i^a \tag{14}$$

in which the sensitivity coefficient matrix of dimension $m \times n$ for the $i$th experimental mode $\boldsymbol{A}_i$ is defined as

$$\boldsymbol{A}_i = \sum_{k=1,k\neq i}^{n} \frac{\boldsymbol{\phi}_k^a \boldsymbol{\phi}_k^T}{(\hat{\omega}_i^2 - \omega_k^2)} \tag{15}$$

The sensitivity coefficient matrix is only associated with the analytical modal properties and the corresponding measured frequency. From Eq. (14) and by using the Moore-Penrose pseudoinverse $\boldsymbol{A}_i^+ = \boldsymbol{A}_i^T [\boldsymbol{A}_i \boldsymbol{A}_i^T]^{-1}$ (Tikhonov and Arsenin 1977), the perturbed force for the $i$th experimental mode is directly calculated from

$$\boldsymbol{f}_i = \boldsymbol{A}_i^+ (\beta_i \hat{\boldsymbol{\psi}}_i - \boldsymbol{\phi}_i^a) \tag{16}$$

Once the mode participation factors $C_{ik}$ in Eq. (9) are determined by using the obtained perturbed force in Eq. (16), the unmeasured part of the $i$th mode shape of the tested structure can be calculated from Eq. (6), namely

$$\tilde{\boldsymbol{\phi}}_i^u = \boldsymbol{\phi}_i^u + \sum_{k=1,k\neq i}^{n} \frac{\boldsymbol{\phi}_k^u \boldsymbol{\phi}_k \boldsymbol{A}_i^+}{(\hat{\omega}_i^2 - \omega_k^2)} (\beta_i \hat{\boldsymbol{\psi}}_i - \boldsymbol{\phi}_i^a) \tag{17}$$

Finally, the $i$th full experimental mode shape, consisting of the measured part $\tilde{\boldsymbol{\phi}}_i^a$ and the estimated unmeasured part $\tilde{\boldsymbol{\phi}}_i^u$, is obtained from

$$\tilde{\boldsymbol{\phi}}_i = \begin{Bmatrix} \boldsymbol{I} \\ \boldsymbol{T}_f^i \end{Bmatrix} \tilde{\boldsymbol{\phi}}_i^a + \begin{Bmatrix} \boldsymbol{0} \\ \boldsymbol{\phi}_i^u - \boldsymbol{T}_f^i \boldsymbol{\phi}_i^a \end{Bmatrix} \tag{18}$$

where $\boldsymbol{I}$ is the identity matrix and the transformation matrix $\boldsymbol{T}_f^i$ of the proposed approach is defined as

$$\boldsymbol{T}_f^i = \sum_{k=1,k\neq i}^{n} \frac{\boldsymbol{\phi}_k^u \boldsymbol{\phi}_k^T \boldsymbol{A}_i^+}{(\hat{\omega}_i^2 - \omega_k^2)} \tag{19}$$

In this expansion process, the measured DOF's readings are reproduced in the expanded mode shapes of the tested structure. It is found that the proposed transformation matrix depends only on the analytical modal properties and the corresponding experimental frequency. The proposed approach thus expands individual mode shapes independently, without requiring the modal data measurements of other modes.

## 4. Regularized mode shape expansion

Due to inevitable noise in the measured DOF's readings identified from the vibration data of the tested structure, the solution of the perturbed force obtained from the Moore-Penrose pseudoinverse in Eq. (16) may not be stable. In order to reduce the influence of the uncertainty in the measured modal data on the performance of mode shape expansion, a regularisation method is now employed to obtain a reliable solution for the unknown perturbed force from the linear system in Eq. (14). It should be noted that the sensitivity coefficient matrix $\boldsymbol{A}_i$ is only related to the analytical modal properties and the corresponding measured frequency and it can be estimated at a relatively high level of accuracy. The total number of measured noisy DOF's readings in the incomplete mode shape of the tested structure is not greater than the total number of DOFs, i.e., $m \le n$. The singular value decomposition (SVD) of the sensitivity coefficient matrix $\boldsymbol{A}_i$, defined in Eq. (15), then can be expressed in the form

$$\boldsymbol{A}_{i(m\times n)} = \boldsymbol{U}_{(m\times n)}\boldsymbol{\Sigma}_{(m\times n)}\boldsymbol{V}^T_{(ne\times n)} = \sum_{j=1}^{m}\sigma_j \boldsymbol{u}_j \boldsymbol{v}_j^T \tag{20}$$

where $\Sigma$ is the diagonal matrix of singular values $\sigma_j$ that are non-negative and non-increasing numbers, i.e., $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_m \ge 0$. $\boldsymbol{U}$ and $\boldsymbol{V}$ are the matrices of orthonormal left and right vectors $\boldsymbol{u}_j$ and $\boldsymbol{v}_j$, respectively. Here, one of the most commonly used regularisation methods, i.e., the Tikhonov regularisation algorithm with a continuous regularisation parameter, is adopted to obtain a regularised solution. The Tikhonov regularisation replaces the original operation with a better-conditioned but related one and produces a regularised solution to the original problem (Tikhonov and Arsenin 1977). The Tikhonov regularised solution is given in terms of the SVD by

$$\boldsymbol{f}_i(\alpha) = \sum_{j=1}^{m} t_j(\alpha)\frac{\boldsymbol{u}_j^T(\beta_i\hat{\psi}_i-\boldsymbol{\phi}_i^a)}{\sigma_j}\boldsymbol{v}_j \tag{21}$$

where $t_j(\alpha)$ are the Tikhonov filter factors, which depend on singular values $\sigma_j$ and regularisation parameter $a$ through the expression

$$t_j(\alpha) = \frac{\sigma_j^2}{\sigma_j^2+\alpha^2} \approx \begin{cases} 1, & if \quad \sigma_j \gg \alpha \\ \dfrac{\sigma_j^2}{\alpha^2}, & if \quad \sigma_j \ll \alpha \end{cases} \tag{22}$$

A smooth solution then can be obtained since the Tikhonov regularised solution coefficients $t_j(\alpha)\left|\boldsymbol{u}_j^T(\beta_i\hat{\psi}_i-\boldsymbol{\phi}_i^a)\right|/\sigma_j$ gradually damp out as singular values decrease. The filter factors $t_j(\alpha)$ gradually filter out the contributions to $\boldsymbol{f}_i(\alpha)$ corresponding to the small singular values, while the contributions corresponding to the large singular values are almost unaffected.

The Tikhonov regularisation parameter $\alpha$ needs to be properly chosen in order to filter out enough noise without losing too much measurement information in the regularised solution. The L-curve criterion has been proven to be a robust and useful method for choosing a regularisation parameter, without requiring the *priori* knowledge of noise in the measured data (Hansen and O'Leary 1993), such as applications to FE model updating (Hua *et al.* 2009). The L-curve is a plot in log-log scale of corresponding values of the residual norm $\rho(\alpha)$ and solution norm $\eta(\alpha)$ as a function of the regularisation parameter $a$, defined in terms of the SVD as

$$\rho(\alpha) = \left\|\boldsymbol{A}_i\boldsymbol{f}_i(\alpha)-(\beta_i\hat{\psi}_i-\boldsymbol{\phi}_i^a)\right\|_2^2 = \sum_{j=1}^{m}[(1-t_j(\alpha))\boldsymbol{u}_j^T(\beta_i\hat{\psi}_i-\boldsymbol{\phi}_i^a)]^2 \tag{23a}$$

$$\eta(\alpha) = \left\|\boldsymbol{f}_i(\alpha)\right\|_2^2 = \sum_{j=1}^{m}\left[t_j(\alpha)\frac{\boldsymbol{u}_j^T(\beta_i\hat{\psi}_i-\boldsymbol{\phi}_i^a)}{\sigma_j}\right]^2 \tag{23b}$$

The $L$-curves basically consist of two parts, i.e., the flat part and the steep part. The flat part corresponds to the regularised solutions where the regularisation parameter is too large and the solution is dominated by regularisation errors. The steep part corresponds to the solutions where the regularisation parameter is too small and the solution is dominated by the noise in the measured data. The balance between the two errors must occur near the $L$-curve's corner, where the curvature of the $L$-curve approximately has a maximum value. From Eq. (23), the curvature of the $L$-curve $\kappa(\alpha)$, as a function of $\alpha$, is expressed by

$$\kappa(\alpha) = \frac{2\eta\rho}{\eta'}\frac{(\alpha^2\eta'\rho + 2\alpha\eta\rho + \alpha^4\eta\eta')}{(\alpha^4\eta^2 + \rho^2)^{3/2}} \tag{24}$$

where $\eta'$ denotes the first derivative of $\eta$ with respect to $\alpha$. A one-dimensional optimization procedure is utilised to determine the regularisation parameter $\alpha$ corresponding to the maximum curvature. Once the regularisation parameter $\alpha$ is determined, the regularised perturbed force $\boldsymbol{f}_i(\alpha)$ is calculated from Eq. (21), and then the regularised estimates of fully expanded mode shapes are obtained from

$$\hat{\phi}_i(\alpha) = \phi_i + \sum_{k=1,k\neq i}^{n} \frac{\boldsymbol{\phi}_k\boldsymbol{\phi}_k^T}{(\hat{\omega}_i^2 - \omega_k^2)}\boldsymbol{f}_i(\alpha) \tag{25}$$

Note that the measured DOF's readings of the tested structure will not be reproduced in the regularised expansion process, which is different from the expansion obtained by most existing techniques. The proposed regularised expansion approach provides the smooth predictions of mode shape expansion in the least squares sense by minimising the difference between the expanded and measured modal properties.

## 5. Benchmark problem results

The Canton Tower benchmark problem is now used to demonstrate the applicability of the proposed approach for mode shape expansion to actual complex structure. The operational modal data such as frequencies and incomplete mode shapes are identified by the stochastic subspace identification technique from the recorded ambient acceleration measurements, as summarised in Table 1. The reduced 3D beam model shown in Fig. 1(b) is used for the analytical model for mode shape expansion. First 10 bending modes are considered for the regularised model shape expansion in this study, since the two measured torsion modes, i.e., the $6^{th}$ and $12^{th}$ modes, do not correlate well with the corresponding analytical modes.

The results in Fig. 3 show the applications of the Tikhonov regularisation incorporating the $L$-curve criterion to the mode shape expansion for measured incomplete modes 1 to 5. The optimum regularisation parameter $\alpha$ is evaluated from the $L$-curve, where the curvature of the curve has a maximum value. The corner of each $L$-curve with a maximum curvature is clearly identified and marked by a square in the figure. The values of determined optimum regularisation parameters for the first five bending modes range from $\alpha_4 = 5.14\times10^{-9}$ for mode 4 to $\alpha_2 = 2.32\times10^{-7}$ for mode 2. These optimum regularisation parameters are then utilised to obtain the regularised predictions for expanding the corresponding measured incomplete mode shapes.

The regularised expansion results for the first 10 bending modes of the benchmark problem are

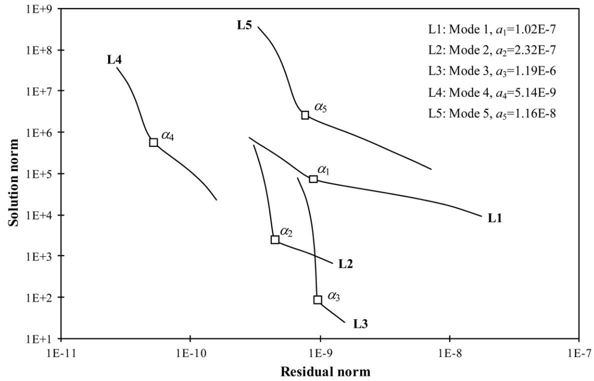
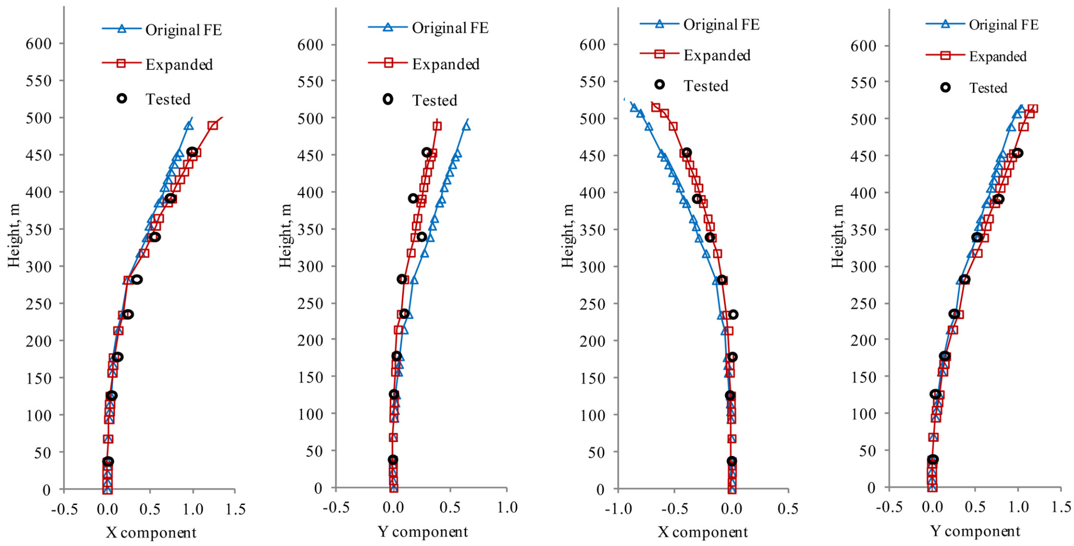


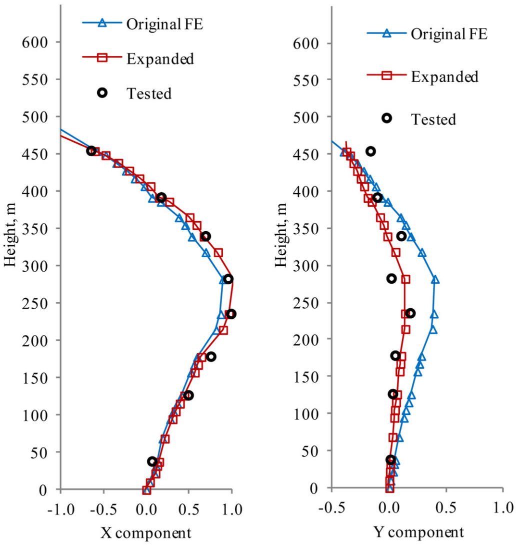

Fig. 3 *L*-curves for Tikhonov regularisation for expanding modes 1 to 5, with optimum regularisation parameters determined at points marked by squares

illustrated in Figs. 4(a)-(j). The proposed approach provides optimised and smooth estimates of mode shape expansion, without just simply reproducing the measured DOF's readings for the known part of the complete mode shape concerned. The results of expanded mode shapes obtained by the proposed approach are then compared with the results calculated from the analytical model and the modal data identified from the ambient vibration measurements. In order to give more obvious comparisons with the tested modal data results, the expanded mode shapes are plotted within the height range similar to the installed accelerometers on the tower. The MAC diagonal values between the expanded mode shapes and the associated original analytical eigenvectors as well as incomplete experimental mode shapes are listed Table 2. The MAC values are calculated for

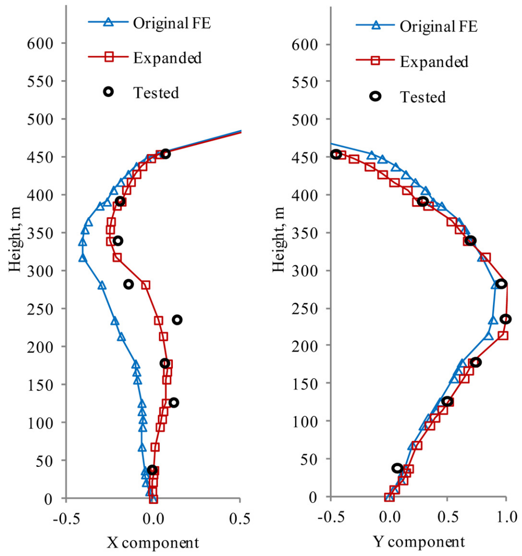



**(a)** Mode 1 (Short-axis bending). **(b)** Mode 2 (Long-axis bending).

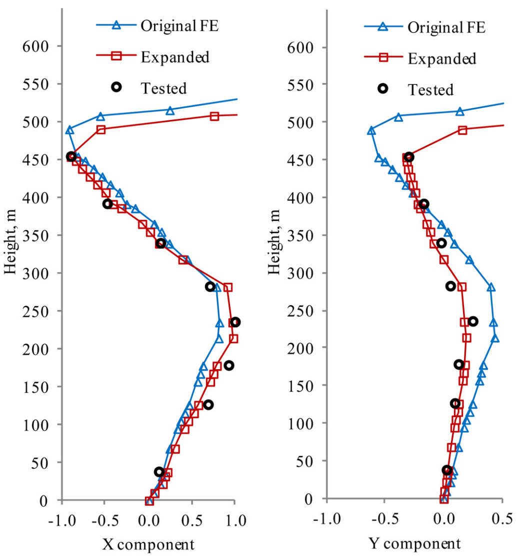
Fig. 4 Expanded first 10 bending modes, compared with the modal readings identified from vibration data and the eigenvectors of the reduced analytical model



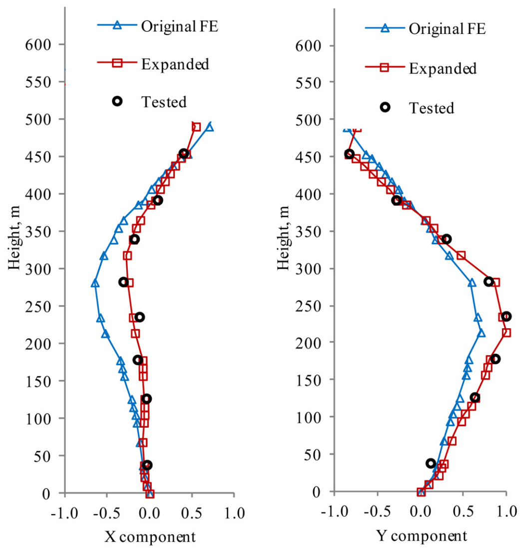

**(c)** Mode 3 (Short-axis bending).




**(d)** Mode 4 (Long-axis bending).



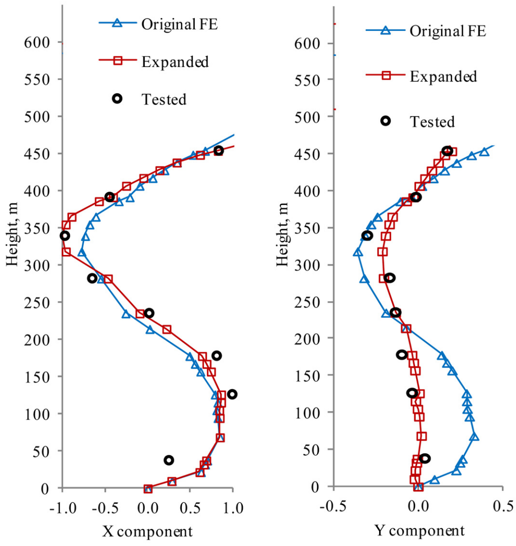

**(e)** Mode 5 (Short-axis bending).



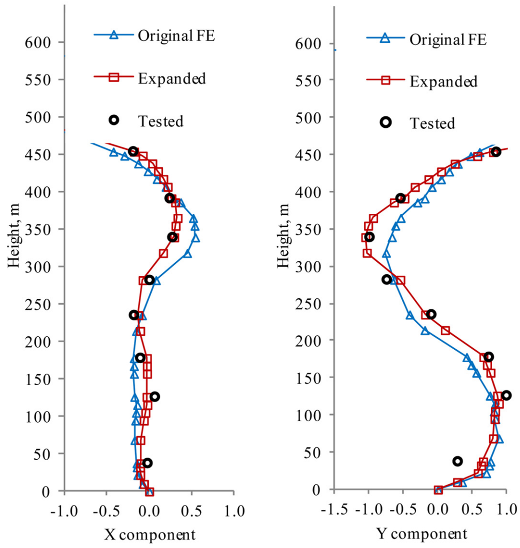

**(f)** Mode 7 (Long and short-axis bending).

Fig. 4 (Continued)

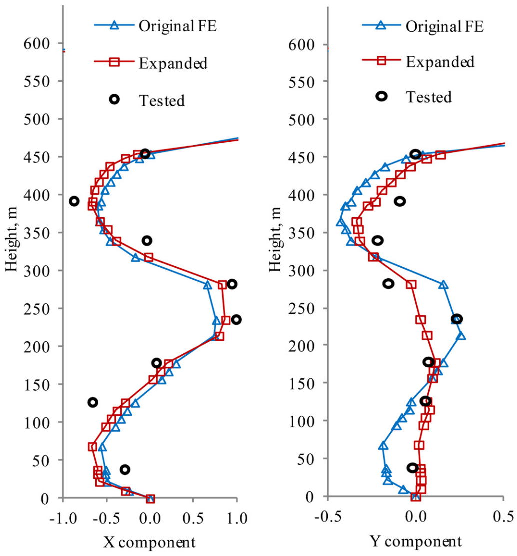
the expanded and original analytical vectors restricting the same dimensions as the measured incomplete mode shapes. The results indicate that the expanded mode shapes correlate very well with the corresponding original analytical eigenvectors, with an average of MAC diagonal values of 0.906. In the meantime, the average of MAC diagonal values with respect to the measured modes has a significant improvement after expansion, increasing from a value of 0.829 for the original analytical and measured modes to a value of 0.934 for the expanded and measured modes. All expanded mode shapes have better correlation with and are closer to the measured incomplete mode shapes.



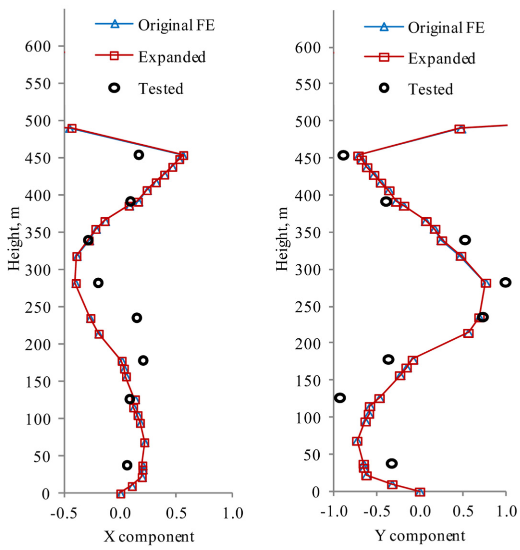

**(g)** Mode 8 (Short-axis bending).




**(h)** Mode 9 (Long-axis bending).




**(i)** Mode 10 (Short-axis bending).




**(j)** Mode 11 (Long-axis bending).

Fig. 4 (Continued)

## 6. Conclusions

A reliable approach is proposed for effectively expanding mode shapes of the actual complex tower structure with limited available modal data measurements, modelling errors and measurement uncertainties present. The proposed expansion approach considers the influence of discrepancies in structural parameters between the finite element model and the actual tested dynamic structural system. The regularization algorithm based on the Tikhonov solution incorporating the *L*-curve

Table 2 MAC diagonal values between the expanded modes, the original analytical eigenvectors and the measured incomplete modes

| Mode | Between original and measured | Between expanded and original | Between expanded and measured |
|---|---|---|---|
| 1 | 0.904 | 0.927 | 0.985 |
| 2 | 0.938 | 0.949 | 0.993 |
| 3 | 0.888 | 0.922 | 0.969 |
| 4 | 0.888 | 0.905 | 0.988 |
| 5 | 0.869 | 0.906 | 0.972 |
| 7 | 0.783 | 0.803 | 0.986 |
| 8 | 0.797 | 0.886 | 0.926 |
| 9 | 0.771 | 0.854 | 0.944 |
| 10 | 0.701 | 0.905 | 0.825 |
| 11 | 0.753 | 0.999 | 0.755 |
| Average | 0.829 | 0.906 | 0.934 |

criterion is employed to filter out the effect of measurement uncertainties on the solutions for the chosen unknown residual force. The optimised and smooth estimates of the fully expanded mode shapes of the tested structure are then given by the obtained regularised solutions.

Based on the verification studies on the benchmark problem, the following conclusions are noted: (1) The proposed approach is capable of successfully expanding mode shapes for the actual complex structure and produces reliable expansion estimates, where large modelling errors in the analytical model may exist. (2) The knowledge of the structural parameters of the tested structure is not required but included in the expansion processes, which avoids the shortcoming in most existing expansion techniques. (3) Information about only limited number of modal data measurements of the tested structure (measured less than 9% of full DOFs in the benchmark problem) is required to expand the measured DOF data set onto full analytical DOF set. (4) The proposed approach gives optimised and smooth expansion estimates in the least squares sense and reduces the influence of measurement uncertainties in the expansion processes. (5) Each mode shape can be expanded independently, and the expanded results depend on the corresponding measured frequency and DOF's readings of the tested structure. (6) The expanded mode shapes become closer to the measured incomplete mode shapes with improved correlations, and also match well the corresponding original analytical eigenvectors.

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