Structural Engineering and Mechanics, Vol. 9, No. 5 (2000) 449-467
DOI: http://dx.doi.org/10.12989/sem.2000.9.5.449

Free vibration analysis of a non-uniform beam with multiple point masses

Jong-Shyong Wu† and Mang Hsieh‡

Institute of Naval Architecture and Marine Engineering, National Cheng-Kung University, Tainan, Taiwan 701, R.O.C.

Abstract. The natural frequencies and the corresponding mode shapes of a non-uniform beam carrying multiple point masses are determined by using the analytical-and-numerical-combined method. To confirm the reliability of the last approach, all the presented results are compared with those obtained from the existing literature or the conventional finite element method and close agreement is achieved. For a "uniform" beam, the natural frequencies and mode shapes of the "clamped-hinged" beam are exactly equal to those of the "hinged-clamped" beam so that one eigenvalue equation is available for two boundary conditions, but this is not true for a "non-uniform" beam. To improve this drawback, a simple transformation function $\varphi(\xi)=(e+\xi\alpha)^2$ is presented. Where $\xi=x/L$ is the ratio of the axial coordinate x to the beam length L, α is a taper constant for the non-uniform beam, e=1.0 for "positive" taper and $e=1.0+|\alpha|$ for "negative" taper (where $|\alpha|$ is the absolute value of α). Based on the last function, the eigenvalue equation for a non-uniform beam with "positive" taper (with increasingly varying stiffness) is also available for that with "negative" taper (with decreasingly varying stiffness) so that half of the effort may be saved. For the purpose of comparison, the eigenvalue equations for a positively-tapered beam with five types of boundary conditions are derived. Besides, a general expression for the "normal" mode shapes of the non-uniform beam is also presented.

Key words: non-uniform beam; natural frequencies; normal mode shapes; transformation function.

1. Introduction

For the "uniform" beams carrying various concentrated elements, the free vibration problem has been studied by a lot of researchers (Laura *et al.* 1975, 1977, Gurgoze 1984, Laura, Fillipich and Cortinez 1987, Wu and Lin 1990, Hamdan and Jubran 1991, Rossi *et al.* 1993, Gurgoze 1998). But for the "non-uniform" beams, even without any attachments, the researches on their dynamic behaviors are relatively fewer (Housner and Keightley 1962, Heidebrecht 1967, Gupta 1985, Abrate 1995, Naguleswaran 1996). As to the free vibration analysis of the "non-uniform" beams carrying multiple concentrated elements, the information concerned is rare and this is one of the reasons why the problem in this aspect is studied.

The analytical-and-numerical-combined method (ANCM) has been found to be an effective approach for the free vibration analysis of the "uniform" beams carrying any number of concentrated elements (Wu and Lin 1990), hence this paper tries to apply the ANCM to determine the natural frequencies and the corresponding mode shapes of a "non-uniform" beam with multiple

† Professor
‡ Research Student

point masses. Since the ANCM is available for the cases that the closed-form solution for the natural frequencies and the associated normal mode shapes of the "unconstrained" beams (without any attachment) are obtainable, any non-uniform beams having closed-form solutions for the natural frequencies and the normal mode shapes will be suitable for the application of the ANCM. Hence, the non-uniform beam with constant depth and biquadratic variation in breadth reported by Abrate (1995) is studied in this paper. The most predominant feature of this non-uniform beam is that the equation of motion for the "non-uniform" beam can be transformed into that for the "uniform" beam so that the closed-form solutions for the natural frequencies and the mode shapes of the "non-uniform" beam with various boundary conditions may easily be obtained.

Fertis (1973, 1995) has presented a method to replace a "non-uniform" beam of variable stiffness by an equivalent "uniform" beam of constant stiffness. It seems that the Fertis' method should be one of the best approaches to incorporate with the ANCM. However, this is not true because the Fertis' theory is obtained based on the assumption that the static deflection of the "non-uniform" beam is equal to the one of the equivalent "uniform" beam. Therefore, the natural frequencies of the equivalent "uniform" beam obtained from the Fertis' method diverge the exact values of the "non-uniform" beam to some degree and the error becomes larger for the case of the non-uniform beam carrying multiple point masses.

One of the heaviest tasks for the ANCM is the derivation of the "normal" mode shapes of the unconstrained beams. A general expression of the "normal" mode shapes is presented for the non-uniform beam of Abrate (1995) with five types of boundary conditions. It is evident that the last general expression for the "non-uniform" beam will be also available for the "uniform" beam if one sets the taper constant to be zero, i.e., α=0.

In the works of Lindberg (1963) and To (1979), the property matrices of the linearly tapered beam elements were derived and then the natural frequencies of the non-uniform beams were solved. Since the foregoing property matrices for the "non-uniform" beam elements are much complicated than the ones for the conventional "uniform" beam elements, the latter are used to determine the natural frequencies and the associated mode shapes of the non-uniform beams in this paper. It is found that satisfactory results may also be achieved if the smaller conventional "uniform" beam elements are adopted.

2. Closed-form solution for the natural frequencies and normal mode shapes of a non-uniform beam

Since the ANCM requires the closed-form solution for the natural frequencies and the corresponding normal mode shapes of the non-uniform beam, the latter is determined first in this section.

The transverse motion of a non-uniform beam is governed by

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right]+\rho A(x)\frac{\partial^2 w(x,t)}{\partial t^2}=0 \tag{1}$$

where E is the Young's modulus, ρ is the mass density of the beam, $A(x)$ is the cross-sectional area at the position x, $I(x)$ is the moment of inertia of $A(x)$ and t is time.

According to Abrate (1995), if $A(x)$ and $I(x)$ take the following forms

$$A(x)=A_o\varphi^2(x)=A_o\left[1+\alpha\left(\frac{x}{L}\right)\right]^4 \tag{2}$$

$$I(x)=I_o\varphi^2(x)=I_o\left[1+\alpha\left(\frac{x}{L}\right)\right]^4 \tag{3}$$

where

$$\varphi(x)=\left[1+\alpha\left(\frac{x}{L}\right)\right]^2 \tag{4}$$

then Eq. (1) may be transformed to

$$EI_o\frac{\partial^4(\varphi w)}{\partial x^4}+\rho A_o\frac{\partial^2(\varphi w)}{\partial t^2}=0 \tag{5}$$

In the last expressions, A_o and I_o are the values of $A(x)$ and $I(x)$ at position $x=0$, L is the beam length, while α is a "positive" constant to represent the taper of the beam.

The non-uniform beam defined by Eqs. (2)-(5) denotes a tapered beam with increasingly varying stiffness. It is evident that the natural frequencies and mode shapes of such a beam with "clamped-hinged" boundary conditions are different from those with "hinged-clamped" ones and the divergence is dependent upon the magnitude of the taper α. To improve the drawback in this aspect in the existing approaches for the non-uniform beams (Housner *et al.* 1962, Heidebrecht 1967, Gupta 1985, Abrate 1995 and Naguleswaren 1996), the transformation function given by Eq. (4) is replaced by

$$\varphi(x)=\left[e+\left(\frac{x}{L}\right)\alpha\right]^2 \tag{6a}$$

or

$$\varphi(\xi)=(e+\xi\alpha)^2 \tag{6b}$$

where

$$e=1.0 \text{ if } \alpha\geq 0;\ e=1.0+|\alpha| \text{ if } \alpha<0 \tag{7}$$

$$\xi=x/L \tag{8}$$

From Eq. (7) one sees that the taper constant (α) may be "positive" or "negative". Positive taper ($\alpha>0$) means that the cross-sectional area ($A(x)$) and the moment of inertia ($I(x)$) of the non-uniform beam increase with increasing the coordinate x (or $\xi=x/L$), and negative taper ($\alpha<0$) means that the values of $A(x)$ and $I(x)$ decrease with increasing x (or $\xi=x/L$).

It is noted that for the xy-coordinate systems and the configurations of the non-uniform beams shown in Figs. 1 and 2 together with the transformation function defined by Eqs. (6)~(8), the symbols A_o and I_o appearing in Eq. (5) now denote the minimum cross-sectional area $A(x)$ and the minimum moment of inertia $I(x)$ at the "left" end for a positively-tapered beam (see Fig. 1), and denote those at the "right" end for a negatively-tapered beam (see Fig. 2). Therefore, for a specified values of A_o and I_o, one may obtain a positively-tapered beam by setting the taper constant α to be a "positive" value and obtain a negatively-tapered beam by setting α to be a "negative" value. By using this property of the transformation function $\varphi(\xi)$ defined by Eq. (6b), if the boundary

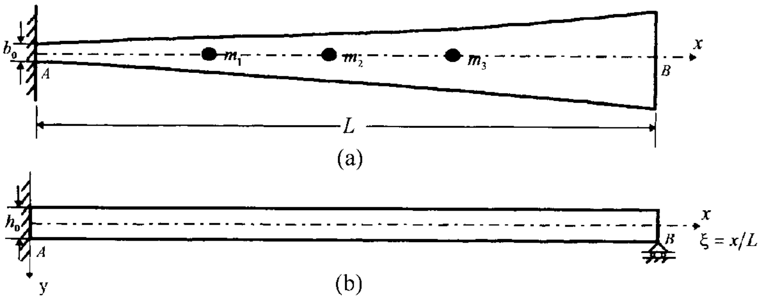

Fig. 1 A constrained non-uniform beam with "positive" taper constant a=0.5: (a) Top view and (b) Front view

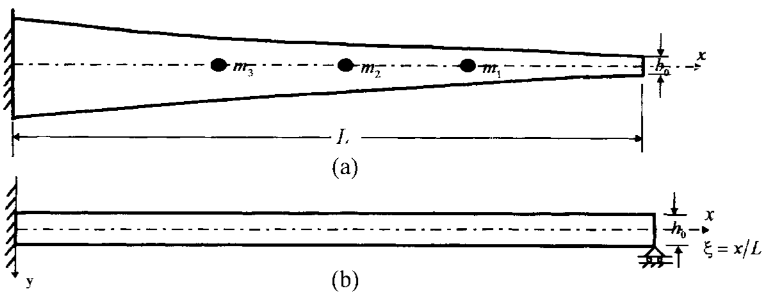

Fig. 2 A constrained non-uniform beam with "negative" taper constant α=-0.5: (a) Top view and (b) Front view

conditions of the positively-tapered beam as shown in Fig. 1 are changed to the "hinged-clamped" ones, then the natural frequencies of the new beam may also be obtained from the original positively-tapered beam with the original "clamped-hinged" boundary conditions by setting the taper constant α to be a negative value (see Figs. 2).

Eq. (5) is the equation of motion for a uniform beam (with area A_o and moment of inertia I_o) with the transverse displacement defined by

$$v(x,t)=\varphi(x)w(x,\ t) \tag{9}$$

For free vibration, one has

$$v(x,t)=V(x)e^{i\omega t} \tag{10}$$

where ω is the natural frequency of the beam (uniform or non-uniform) and $V(x)$ is the corresponding mode shape of the equivalent beam. The substitution of Eqs. (9) and (10) into Eq. (5) yields

$$V''''-\beta^4 V=0 \tag{11}$$

where

$$\beta^4=\frac{\rho A_o}{EI_o}\omega^2 \tag{12}$$

and $V''''=\partial^4 V/\partial x^4$. The solution of Eq. (11) takes the form (Meirovitch 1967)

$$V(x)=A(\cos\beta x+\cosh\beta x)+B(\cos\beta x-\cosh\beta x) \tag{13}$$

$$+C(\sin\beta x+\sinh\beta x)+D(\sin\beta x-\sinh\beta x)$$

Where the constants A, B, C and D are determined from the specified boundary conditions.

2.1. For the clamped-hinged boundary conditions

For a clamped-hinged beam, the boundary conditions are:

$$w(x,t)=0, \quad \frac{\partial w(x,t)}{\partial x}=0 \text{ at } x=0 \tag{14a}$$

$$w(x,t)=0, \quad \frac{\partial^2 w(x,t)}{\partial x^2}=0 \text{ at } x=L \tag{14b}$$

From Eqs. (6), (9) and (14), one obtains

$$v(x,t)=0, \quad \frac{\partial v(x,t)}{\partial x}=0 \text{ at } x=0 \tag{15a}$$

$$v(x,t)=0, \quad \frac{\partial^2 v(x,t)}{\partial x^2}=\left(\frac{4C^*}{L}\right)\frac{\partial v(x,t)}{\partial x} \text{ at } x=L \tag{15b}$$

where

$$C^*=\frac{\alpha}{e+\alpha} \tag{16}$$

It is noted that

$$\varphi(0)=e^2, \quad \frac{\partial\varphi(0)}{\partial x}=2e\left(\frac{\alpha}{L}\right), \quad \frac{\partial^2\varphi(0)}{\partial x^2}=2\left(\frac{\alpha}{L}\right)^2 \tag{17a}$$

$$\varphi(L)=(e+\alpha)^2, \quad \frac{\partial\varphi(L)}{\partial x}=2(e+\alpha)\left(\frac{\alpha}{L}\right), \quad \frac{\partial^2\varphi(L)}{\partial x^2}=2\left(\frac{\alpha}{L}\right)^2 \tag{17b}$$

By using Eq. (10), Eqs. (15a) and (15b) are transformed to

$$V(x)=0, \quad V'(x)=0 \text{ at } x=0 \tag{18a}$$

$$V(x)=0, \quad V''(x)=\frac{4C^*}{L}V'(x) \text{ at } x=L \tag{18b}$$

From Eqs. (13), (18a) and (18b) one obtains

$$A=C=0 \tag{19a}$$

$$B(\cos\beta L-\cosh\beta L)+D(\sin\beta L-\sinh\beta L)=0 \tag{19b}$$

$$B[-\beta L(\cos\beta L+\cosh\beta L)+4C^*(\sin\beta L+\sinh\beta L)] \tag{19c}$$
$$-D[(\beta L(\sin\beta L+\sinh\beta L))+4C^*(\cos\beta L-\cosh\beta L)]=0$$

Non-trivial solution of Eqs. (19b) and (19c) leads to

$$\frac{a_r(\sin a_r+\sinh a_r)+4C^*(\cos a_r-\cosh a_r)}{\sin a_r-\sinh a_r} \tag{20}$$

$$+\frac{-a_r(\cos a_r + \cosh a_r) + 4C^*(\sin a_r + \sinh a_r)}{\cos a_r - \cosh a_r} = 0$$

where

$$a_r = \beta_r L \tag{21}$$

Eq. (20) is the frequency equation, which is in a different form from the one given by Eq. (41) of Abrate (1995). It was found that one can not obtain the correct values of the non-dimensional parameters $a_r = \beta_r L$ (r=1, 2, ...) from Eq. (41) of Abrate (1995), unless some transformation was made on that frequency equation. However, it is easy to find the values of $a_r = \beta_r L$ (r=1, 2, ...) from Eq. (20) and the associated natural frequencies of the non-uniform beam are given by (c.f. Eq. 12)

$$\omega_r = (a_r)^2 \sqrt{\frac{EI_o}{\rho A_o L^4}} \quad \text{(rad/sec)}, \qquad (r=1,2,...) \tag{22}$$

Now, from Eqs. (13) and (19), one obtains the mode shapes

$$V_r(\xi) = B_r[(\cos a_r\xi - \cosh a_r\xi) - Q_{1r}(\sin a_r\xi - \sinh a_r\xi)] \tag{23}$$

where

$$Q_{1r} = \frac{\cos a_r - \cosh a_r}{\sin a_r - \sinh a_r}$$

For convenience of calculating the "normal" mode shapes of the beam in various boundary conditions (see the Appendix), Eq. (23) is rewritten in the "general" form below

$$V_r(\xi) = B_r(E_1 \sin a_r\xi + F_1 \cos a_r\xi + G_1 \sinh a_r\xi + H_1 \cosh a_r\xi) \tag{24a}$$

$$= B_r \bar{V}_{1r}(\xi) \tag{24b}$$

where

$$E_1 = -Q_{1r}, \quad F_1 = 1, \quad G_1 = Q_{1r}, \quad H_1 = -1 \tag{25}$$

$$\bar{V}_{1r}(\xi) = E_1 \sin a_r\xi + F_1 \cos a_r\xi + G_1 \sinh a_r\xi + H_1 \cosh a_r\xi \tag{26}$$

The value of B_r appearing in Eq. (23) was determined from

$$\int_0^1 \rho A_o L V_r^2(\xi) d\xi = B_r^2 \int_0^1 \rho A_o L \bar{V}_{1r}^2(\xi) d\xi = 1.0 \tag{27}$$

Hence the "normal" mode shapes of the "clamped-hinged" non-uniform beam are given by

$$V^*_{1r}(\xi) = B_{1r}(E_1 \sin a_r\xi + F_1 \cos a_r\xi + G_1 \sinh a_r\xi + H_1 \cosh a_r\xi), \quad r = 1, 2, ... \tag{28}$$

where

$$B_{1r} = \frac{1}{\sqrt{\rho A_o L R_{1r}}} \tag{29}$$

$$R_{1r} = \int_0^1 \bar{V}_{1r}^2(\xi) d\xi \tag{30a}$$

$$= \frac{1}{2}(E_1^2 + F_1^2 - G_1^2 + H_1^2) + \frac{1}{4a_r}\sin 2a_r(-E_1^2 + F_1^2) + \frac{1}{4a_r}\sinh 2a_r(G_1^2 + H_1^2)$$

$$+\frac{\sin a_r \cosh a_r}{a_r}(E_1G_1+F_1H_1)+\frac{\cos a_r \cosh a_r}{a_r}(-E_1G_1+F_1H_1)$$

$$+\frac{\sin a_r \cosh a_r}{a_r}(E_1H_1+F_1G_1)+\frac{\cos a_r \cosh a_r}{a_r}(-E_1H_1+F_1G_1)$$

$$+\frac{1}{a_r}(E_1H_1-F_1G_1)+\frac{E_1F_1}{a_r}\sin^2 a_r+\frac{G_1H_1}{a_r}\sinh^2 a_r \tag{30b}$$

In this paper, five types of boundary conditions were studied, hence the first subscripts i of the symbols V_{ir}*, B_{ir} and R_{ir} appearing in Eqs. (28)-(30) denote the i-th type of boundary condition.

The free vibration response of the non-uniform beam takes the form

$$w(x,t)=W(x)e^{i\omega t} \tag{31}$$

The substitution of Eqs. (10) and (31) into Eq. (9) gives the "normal" mode shapes of the non-uniform beam to be

$$W_{ir}(\xi)=V_{ir}^*(\xi)/\varphi(\xi)=V_{ir}^*(\xi)/(e+\xi\alpha)^2 \tag{32}$$

2.2. For the hinged-clamped boundary conditions

Although, by using the transformation function given by Eq. (6), one may determine the natural frequencies and the corresponding mode shapes of a non-uniform "hinged-clamped" beam with the formulation for the non-uniform "clamped-hinged" beam, in order to check the reliability of the presented theory, the closed-form solutions for the natural frequencies and the corresponding normal mode shapes of a non-uniform beam with the "hinged-clamped" boundary conditions were also derived in the following.

For a hinged-clamped beam, one has

$$w(x,t)=0,\quad \frac{\partial^2 w(x,t)}{\partial x^2}=0 \quad \text{at} \quad x=0 \tag{33a}$$

$$w(x,t)=0,\quad \frac{\partial w(x,t)}{\partial x}=0 \quad \text{at} \quad x=L \tag{33b}$$

From Eqs. (6), (9) and (33), one obtains

$$v(0,t)=0,\quad \frac{\partial^2 v(0,t)}{\partial x^2}=\left(\frac{4\bar{\alpha}}{L}\right)\frac{\partial v(0,t)}{\partial x} \tag{34a),(34b}$$

$$v(L,t)=0,\quad \frac{\partial v(L,t)}{\partial x}=0 \tag{35a),(35b}$$

where

$$\bar{\alpha}=\alpha/e \tag{36}$$

To insert Eq. (10) into the last expressions yields

$$V(0)=0,\quad V''(0)=\frac{4\bar{\alpha}}{L}V'(0) \tag{37a),(37b}$$

$$V(L)=0,\quad V'(L)=0 \tag{38a),(38b}$$

By using the similar derivation steps as shown in Eqs. (19)~(30), one obtains the frequency equation of the "hinged-clamped" non-uniform beam

$$2(1-\cos\alpha_r\cosh\alpha_r)+\frac{a_r}{2\bar{\alpha}}(\sin a_r\cosh a_r-\cos a_r\sinh a_r)=0 \tag{39}$$

and the "normal" mode shapes

$$V_{2r}{}^*=\frac{1}{\sqrt{\rho A_o L R_{2r}}}(E_2\sin a_r\xi+F_2\cos a_r\xi+G_2\sinh a_r\xi+H_2\cosh a_r\xi) \tag{40}$$

where

$$E_2=-\frac{a_r}{4\bar{\alpha}}-Q_{2r},\quad F_2=1,\quad G_2=-\frac{a_r}{4\bar{\alpha}}+Q_{2r},\quad H_2=-1 \tag{41}$$

$$Q_{2r}=\frac{(\cos a_r-\cosh a_r)-\frac{a_r}{4\bar{\alpha}}(\sin a_r+\sinh a_r)}{\sin a_r-\sinh a_r} \tag{42}$$

The values of R_{2r} appearing in Eq. (40) may be obtained from Eq. (30b) by replacing the values of E_1, F_1, G_1, H_1 by those of E_2, F_2, G_2, H_2 defined by Eqs. (41) and (42), respectively. Where the values of $\alpha_r=\beta_r L$ (r=1, 2, ...), are the roots of the frequency Eq. (39).

3. Solution for a non-uniform beam carrying point masses

According to the analytical-and-numerical-combined method (ANCM), the eigenvalue equation for a beam carrying p point masses with magnitudes m_j (j=1~p) located at x_j (j=1~p) is to take the form (see Eq. 11 of Wu and Lin 1990)

$$(\omega_r^2-\bar{\omega}^2)\bar{\eta}_r-\sum_{j=1}^{p}\sum_{s=1}^{n}m_j W_r(x_j)W_s(x_j)\bar{\omega}^2\bar{\eta}_s=0,\quad r=1,2,\ldots,n \tag{43}$$

where ω_r is the r-th natural frequency of the "unconstrained" beam (without any attachment) with "normal" mode shape $W_r(x)$, $\bar{\omega}$ is the natural frequency of the "constrained" beam (carrying any attachments) with mode shape $\overline{W}(x)$, n is the total number of modes considered, $\bar{\eta}$ is the amplitude of the generalized coordinate $\eta(t)$, i.e.,

$$\eta(t)=\bar{\eta}e^{i\bar{\omega}t} \tag{44}$$

and

$$W(x_j)=W(x)\cdot\delta(x-x_j) \tag{45}$$

In the last two equations, $\delta(\cdot)$ is the Dirac delta function, x is the axial coordinate, t is time and $i=\sqrt{-1}$.

Eq. (43) is derived from the equation of motion for the "unconstrained" beam by considering the inertia forces of the p point masses, $m_j W(x_j)\bar{\omega}^2$ (j=1~p), as the external exciting forces and applying the property of orthogonality of the "normal" mode shapes $W_r\,(x)$ $(r=1, 2, ...)$.

For convenience, Eq. (43) is rewritten in matrix form

$$[A]\{\bar{\eta}\}=\bar{\omega}^2[B]\{\bar{\eta}\} \tag{46}$$

where

$$\begin{aligned}
&\{\bar{\eta}\}=\{\bar{\eta}_1\bar{\eta}_2\ldots\bar{\eta}_n\}\\
&[A]=\lceil\omega_1\,\omega_2\ldots\omega_n\rfloor_{n\times n}\\
&[B]=[I]_{n\times n}+[B']_{n\times n}\\
&[I]=\lceil 1\;\,1\ldots1\rfloor_{n\times n}\\
&[B']=\sum_{j=1}^{p}m_j[W(x_j)]_{n\times n}\\
&[W(x_j)]=\{W(x_j)\}_{n\times1}\cdot\{W(x_j)\}^T_{n\times1}\\
&\{W(x_j)\}_{n\times1}=\{W_1(x_j)W_2(x_j)\ldots W_n(x_j)\}_{n\times1}
\end{aligned} \tag{47}$$

In the last equations, the symbols [ ], { } and $\lceil\;\rfloor$ denote the square matrix, column vector and diagonal matrix, respectively.

Nontrivial solution of Eq. (46) requires that

$$\left|[A]-\bar{\omega}^2[B]\right|=0 \tag{48}$$

Eq. (46) is a standard eigenvalue equation, here the half-interval method (Carnahan, Luther and Wilker 1969) is used to determine the natural frequencies of the "constrained" beam, $\bar{\omega}_r$ $(r=1\sim n)$, from Eq. (48) and the substitution of the values of $\bar{\omega}_r$ $(r=1\sim n)$ into Eq. (46) will determine the corresponding generalized coordinates $\{\bar{\eta}\}^{(r)}$ $(r=1\sim n)$. Finally, the corresponding mode shapes of the constrained beam are given by

$$\bar{W}_r(x)=\{W(x)\}^T_{n\times1}\{\bar{\eta}\}^{(r)},\quad r=1, 2, \ldots, n \tag{49}$$

From the foregoing formulation for the ANCM, one sees that the natural frequencies $\bar{\omega}_r$ $(r=1\sim n)$ and the corresponding modes shapes $\bar{W}_r\,(x)$ $(r=1\sim n)$ of a "constrained" non-uniform beam (with any attachments) are easily obtained if the natural frequencies ω_r $(r=1\sim n)$ and the corresponding modes shapes $W_r(x)$ $(r=1\sim n)$ of the "unconstrained" non-uniform beam (without any attachments) are obtainable. Since the accuracy of the first $n-1$ natural frequencies, $\bar{\omega}_r$ $(r=1\sim n)$, will be satisfactory if the total number of modes used by the ANCM is n, the order of the two square matrices in Eq. (46) or Eq. (48), $[A]_{n\times n}$ and $[B]_{n\times n}$, is much lower than the order of the property matrices for the conventional finite element method (FEM). Hence, the numerical calculations with the ANCM will be faster than those with the FEM.

4. Numerical results and discussions

The dimensions and physical properties for the non-uniform beam studied in the following are: minimum height h_o=1.5 in, minimum width b_o=1.0 in, minimum cross-sectional area $A_o=b_oh_o$=1.5 in^2, minimum moment of inertia $I_o=b_o h_o^3/12=0.28125$ in^4, total beam length L=30.0 in, Young's modulus $E=30\times10^6$ psi and mass density of the beam material $\rho=0.73386\times10^{-3}$lb-s^2/in/in^3. For convenience, the five compound adjectives for the boundary conditions of the non-uniform beam studied in this paper, clamped-hinged, hinged-clamped, clamped-free, free-clamped and hinged-hinged, will be represented by the five two-letter acronyms CH, HC, CF, FC and HH, respectively, hereafter.

4.1. Check with the existing and FEM results

For the CH non-uniform beam shown in Fig. 1 without carrying any point mass, the lowest six frequency coefficients $a_r^2=(\beta_r L)^2$ (r =1~6) are shown in Table 1(a) for the taper constant α=0, Table 1(b) for α=±1.0 and Table 1(c) for α=±2.0. In addition to the ANCM results, those obtained from the existing literature (Abrate 1995) and the conventional FEM are also listed in Table 1. The FEM model is shown in Fig. 3, where the entire non-uniform beam is replaced by a stepped beam composed of 24 uniform beam segments. The cross-sectional area A_i and the moment of inertia I_i of the i-th "uniform beam segment" are equal to the average values of the corresponding ones for the i-th "non-uniform beam segment", respectively, and the mass per unit length of the i-th uniform beam segment is evaluated by ρA_i. The length of each uniform beam segment is $l=L/24=1.25$ in. From Tables 1(a), 1(b) and 1(c) one finds that the results of ANCM and those of FEM are all very

Table 1 The lowest six non-dimensional frequency coefficients $a_r^2=(\beta_r L)^2$ (r=1~6) for the unconstrained "clamped-hinged" non-uniform beam (p=0) shown in Fig. 1 with taper constants (a) α=0.0; (b) α=±1.0; (c) α=±2.0

(a)

Methods	α	Non-dimensional frequency coefficients					
		a_1^2	a_2^2	a_3^2	a_4^2	a_5^2	a_6^2
Abrate (1995)		15.4182	49.9649	104.248	178.270	272.032	385.533
FEM	0.0	15.4183	49.9654	104.251	178.282	272.074	383.651
ANCM		15.4186	49.9654	104.247	178.269	272.031	385.531

(b)

Methods	α	Non-dimensional frequency coefficients					
		a_1^2	a_2^2	a_3^2	a_4^2	a_5^2	a_6^2
Abrate (1995)	1.0	12.3635	47.6265	102.025	176.105	269.904	383.423
FEM	1.0	12.3755	47.6581	102.089	176.220	270.096	383.756
ANCM	1.0	12.3633	47.6259	102.025	176.105	269.901	383.421
	* −1.0	12.3633	47.6259	102.025	176.105	269.901	383.421

(c)

Methods	α	Non-dimensional frequency coefficients					
		a_1^2	a_2^2	a_3^2	a_4^2	a_5^2	a_6^2
Abrate (1995)	2.0	10.5984	46.6678	101.174	175.304	269.136	382.669
FEM	2.0	10.6256	46.7203	101.270	175.463	269.382	383.074
ANCM	2.0	10.5986	46.6673	101.174	175.304	269.129	382.669
	* −2.0	10.5986	46.6673	101.174	175.304	269.129	382.669

*For the "hinged -clamped" boundary conditions.

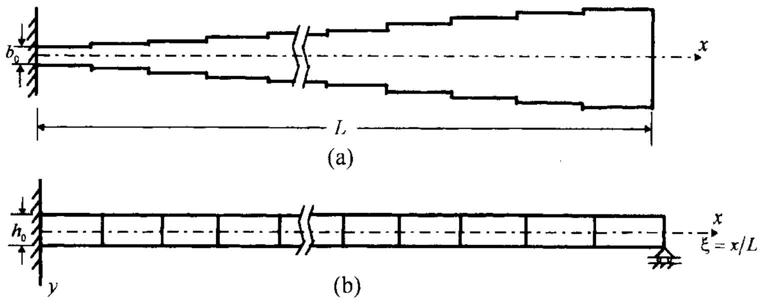

Fig. 3 The finite element model for the non-uniform beam with: (a) Top view and (b) Front view

close to those of Abrate (1995), but the accuracy of ANCM is better than that of FEM, particularly for the higher modes of the higher taper beam (e.g., a_6^2 for α=2.0). It is noted that the values of $a_r^2=(\beta_r L)^2$ (r=1~6) obtained from the CH beam with "positive" taper (α=+1, +2) are exactly equal to those obtained from the HC beam with "negative" taper (α=−1, −2) as one may see from the final two rows in each of Tables 1(a), 1(b) and 1(c). In the next subsections, all results are obtained based on the "positive" taper except those for the cases indicated by stars (*).

4.2. Free vibration analysis of the "unconstrained" non-uniform beam

For convenience of comparison, the lowest five natural frequencies ω_r (r=1~5) and some of the corresponding mode shapes $W_r(\xi)$ (r=1~5) of the "unconstrained" non-uniform beam (without any

Table 2 The lowest five natural frequencies for the "unconstrained" non-uniform beam (p=0) with five boundary conditions and taper constant α=0.5

Cases	Methods	Boundary conditions	Natural frequencies $\bar{\omega}_i$ (rad/sec)				
			ω_1	ω_2	ω_3	ω_4	ω_5
1	FEM	CH	1328.20687	4718.77248	10005.97164	17212.65567	26341.18345
	ANCM	CH / *HC	1327.59225	4716.81231	10001.92912	17204.94875	26327.19985
2	FEM	HC	1656.80298	5025.75128	10311.38537	17513.05824	26634.34892
	ANCM	HC / *CH	1657.76545	5028.65459	10317.05808	17522.05884	26645.33977
3	FEM	CF	204.07943	1837.20129	5732.17181	11500.97222	19191.37908
	ANCM	CF / *FC	203.83525	1835.61577	5727.57575	11491.78064	19175.07545
4	FEM	FC	547.06614	2493.55986	6356.31530	12117.73476	19795.45096
	ANCM	FC / *CF	547.62025	2496.31653	6363.39762	12131.22403	19816.24566
5	FEM	HH	935.71446	3862.22951	8675.33854	15402.33232	24047.99504
	ANCM	HH	935.89198	3862.96437	8676.91792	15404.44705	24049.59868

*α=−0.5

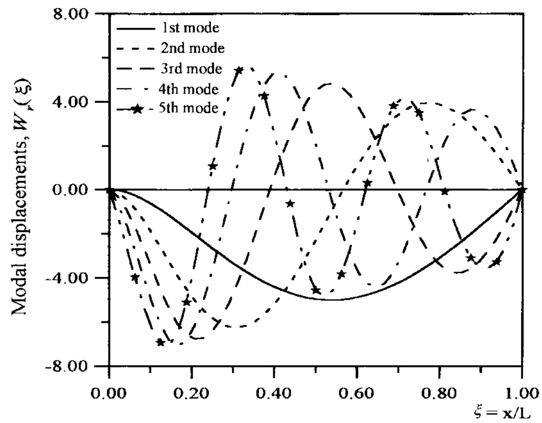

Fig. 4 The lowest five mode shapes for the "unconstrained" non-uniform CH beam (p=0)

attachment) are shown in Table 2 and Fig. 4, respectively.

In Table 2 and the subsequent tables, the lowest five natural frequencies for the same non-uniform beam with taper α=0.5 and five boundary conditions (i.e., CH, HC, CF, FC and HH) are listed. Since the natural frequencies obtained based on the "positive" taper are equal to those based on the "negative" taper, the last results are placed in the same row for each type of boundary conditions in each table. From Table 2 one sees that the results of ANCM and those of FEM are very close to each other. From Fig. 4 one sees that the node number N_r for the r-th mode shape of the CH beam

Table 3 The lowest five natural frequencies for the constrained non-uniform beam carrying one point mass (p=1) m_1=m_b=0.0522874 lb-s^2/in located at ξ_1=x_1/L=0.5

Cases	Methods	Boundary conditions	Natural frequencies $\bar{\omega}_i$ (rad/sec)				
			$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
1	FEM	CH	872.64439	4435.74015	8355.01791	16664.28935	23160.40196
	ANCM	CH / *HC	872.27284	4436.91706	8380.31183	16699.68756	23483.69626
2	FEM	HC	1058.66779	4864.20255	8534.50548	17105.39814	23302.21228
	ANCM	HC / *CH	1059.39315	4867.81992	8572.96803	17141.68163	23661.13839
3	FEM	CF	184.94691	1259.41166	5705.76392	9273.90864	19182.62619
	ANCM	CF / *FC	184.72330	1258.68294	5701.60635	9312.03428	19167.31272
4	FEM	FC	453.91068	1830.52352	6341.08473	9884.70803	19777.73302
	ANCM	FC / *CF	454.29062	1833.11086	6348.21408	9945.29591	19799.81812
5	FEM	HH	617.84920	3838.80069	6878.38984	15388.78028	20464.68372
	ANCM	HH	617.93874	3839.67257	6907.15213	15391.92198	20827.75045

*α=−0.5

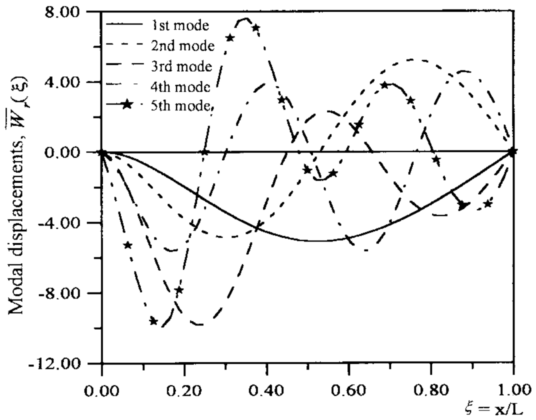

Fig. 5 The lowest five mode shapes for the constrained CH non-uniform beam carrying "one" point mass (p=1) m_1=m_b=0.0522874 lb-s^2/in located at ξ_1=x_1/L=0.5

is given by N_r=r−1, besides, the modal displacements near the left end of the non-uniform CH beam are larger than those near the right end of the beam in spite of the fact that the left end is clamped and the right end is simply supported. This is a reasonable result, because the stiffness of the left end is much smaller than that of the right end of the non-uniform beam as shown in Fig. 1.

Table 4 The lowest five natural frequencies for the constrained non-uniform beam carrying "three" point masses (p=3) m_j=1/3m_b=0.0174291 lb-s^2/in (j=1~3) located at ξ_1=1/3, ξ_2=1/2, ξ_3=2/3, respectively

Cases	Methods	Boundary conditions	Natural frequencies $\bar{\omega}_i$ (rad/sec)				
			$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
1	FEM	CH	930.67927	3472.06962	8107.75115	14681.39763	19612.75112
	ANCM	CH / *HC	930.12399	3473.28332	8141.60333	14796.32219	20101.84511
2	FEM	HC	1113.78387	3769.21121	8588.64471	14531.50330	20203.97146
	ANCM	HC / *CH	1114.30805	3774.37784	8625.28129	14697.43825	20650.43981
3	FEM	CF	181.67449	1356.67554	4228.70132	9284.83949	17256.18292
	ANCM	CF / *FC	181.45834	1355.48769	4230.73834	9333.00813	17313.13449
4	FEM	FC	433.25328	1980.49308	4812.84403	9722.97249	17986.60657
	ANCM	FC / *CF	433.58070	1982.70024	4823.70589	9791.57865	18085.90083
5	FEM	HH	648.86403	2954.50278	7359.25356	13085.86686	17042.32834
	ANCM	HH	648.90805	2956.39971	7386.61726	13141.65576	17665.21596

*α=−0.5

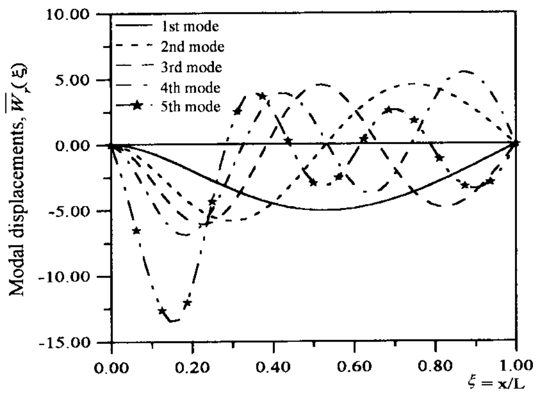

Fig. 6 The lowest five mode shapes for the constrained CH non-uniform beam carrying "three" point masses (p=3) m_j=m_b/3=0.0174291 lb-s^2/in (j=1~3) located at ξ_1=1/3, ξ_2=1/2, ξ_3=2/3, respectively

4.3. A non-uniform beam carrying "one" point mass

All the situations of the present example are the same as the last one, the only difference is that a single point mass with magnitude m_1=m_b=0.0522874 lb-s^2/in is attached to the center of the beam (i.e., ξ_1=x_1/L=0.5). The lowest five natural frequencies of the constrained non-uniform beam, $\overline{\omega}_r$ (r= 1~5), are shown in Table 3 for the five types of boundary conditions. The corresponding mode shapes of the CH beam, $\overline{W}_r$ (r=1~5), are shown in Fig. 5. From Tables 2 and 3 one sees that the single point mass m_1 reduces the lowest five natural frequencies of the constrained beam significantly and so does the corresponding mode shapes as may be seen from Figs. 4 and 5.

4.4. A non-uniform beam carrying "three" point masses

If the single point mass in the last example is equally divided into three point masses (i.e., m_1=m_2=m_3=m_b/3=0.0174291 lb-s^2/in) located at ξ_1=x_1/L=1/3, ξ_2=x_2/L=1/2, and ξ_3=x_3/L=2/3, respectively, then the lowest five natural frequencies of the constrained beam, $\overline{\omega}_r$ (r=1~5), are

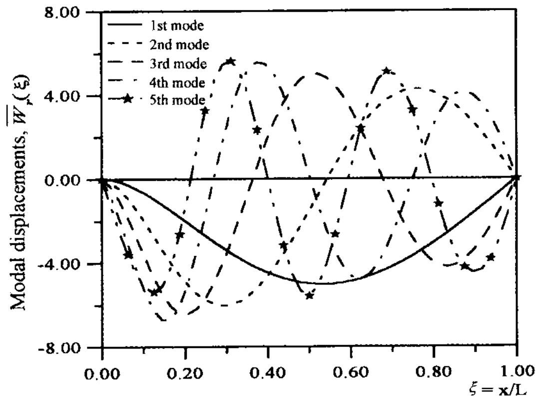

Fig. 7 The lowest five mode shapes for the constrained CH non-uniform beam carrying "five" point masses (p=5) m_j=1/5m_b=0.01045748 lb-s^2/in (j=1~5) located at ξ_j= x_j/L =j/6, respectively

Table 5 The lowest five natural frequencies for the constrained non-uniform beam carrying "five" point masses (p=5) m_j= 1/5m_b=0.01045748 lb-s^2/in (j = 1~5) located at ξ_j=x_j/L=j/6, respectively

Cases	Methods	Boundary conditions	Natural frequencies $\bar{\omega}_i$ (rad/sec)				
			$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\omega}_3$	$\bar{\omega}_4$	$\bar{\omega}_5$
1	FEM	CH	1014.36228	3498.23044	7330.88251	12302.84740	17979.73215
	ANCM	CH / *HC	1013.75599	3497.38616	7343.03177	12401.36931	18398.50774
2	FEM	HC	1204.86422	3674.37284	7532.83742	12710.82456	19278.90953
	ANCM	HC / *CH	1205.40915	3677.67665	7557.39337	12873.70416	19627.95120
3	FEM	CF	175.85005	1455.15056	4374.34074	8587.67473	13762.07416
	ANCM	CF / *FC	175.64694	1453.82883	4372.75513	8609.76994	13937.26143
4	FEM	FC	399.51491	1978.40731	5119.81284	9648.30479	14905.52460
	ANCM	FC / *CF	399.77693	1980.21645	5127.99381	9686.75699	15116.15563
5	FEM	HH	701.03674	2834.50828	6353.27981	11228.60299	17438.03585
	ANCM	HH	701.09247	2835.42321	6365.44802	11327.69400	17985.17861

*α=−0.5

shown in Table 4 and the lowest five mode shapes of the CH beam are shown in Fig. 6. From Tables 3 and 4 one sees that the lowest five natural frequencies of the non-uniform beam carrying "three" point masses are not much different from those carrying one single point mass. This may be due to the summation of the "three" point masses is equal to the "one" single point mass. But the associated mode shapes are different to some degree as may be seen from Figs. 5 and 6. This may have something to do with the distribution of the point masses.

4.5. A non-uniform beam carrying "five" point masses

Similarly, if the single point mass in the previous example was replaced by five identical point masses, each with magnitude m_j=m_b/5=0.01045748 lb-s^2/in (j=1~5) and located at ξ_j=x_j/L=j/6 (j=1~5), respectively, then the lowest five natural frequencies of the constrained beam are listed in Table 5, while the lowest five mode shapes of the CH beam are plotted in Fig. 7. From Tables 5 and 2 one sees that the lowest five natural frequencies of the "constrained" beam, $\bar{\omega}_r$ (r=1~5), shown in Table 5 are smaller than the corresponding ones of the "unconstrained" beam, ω_r (r=1~5), shown in Table 2, and the difference between them, $\Delta\omega_r$ =ω_r−$\bar{\omega}_r$, increases with increasing the mode number r. But the lowest five mode shapes of the "constrained" beam shown in Fig. 7 look like those of the "unconstrained" beam shown in Fig. 4. The five "identical" point masses "uniformly" distributed along the beam length should be the main reason arriving at the last results.

5. Conclusions

1. In addition to the conventional finite element method (FEM), the analytical-and-numerical-combined method (ANCM) is an alternative simple approach for the free vibration analysis of a non-uniform beam carrying "multiple" point masses if the closed-form solution for the natural frequencies and the corresponding mode shapes of the non-uniform beam are obtainable.
2. The natural frequencies and the corresponding mode shapes of the non-uniform beams with "clamped-hinged" boundary conditions are different from those with "hinged-clamped" boundary conditions, hence the transformation function including the "positive" and "negative" taper constants (α) presented in this paper will reduce half of the effort required for the free vibration analysis of the non-uniform beams.
3. The free vibration characteristics of a non-uniform beam is significantly influenced by the distributions and magnitudes of the concentrated attachments along the beam length.

References

Abrate, S. (1995), "Vibration of non-uniform rods and beams", *Journal of Sound and Vibration*, **185**(4), 703-716.
Carnahan, B., Luther, H.A. and Wilker, J.O. (1969), *Applied Numerical Methods*, New York, John Wiley.
Fertis, D.G. (1973), *Dynamics and Vibration of Structures*, John Wiley & Sons.
Fertis, D.G. (1995), *Mechanical and Structural Vibrations*, John Wiley & Sons.
Gupta, A.K. (1985), "Vibration of tapered beams", *Journal of Structural Engineering*, **111**(1), 19-36.
Gurgoze, M. (1984), "A note on the vibrations of restrained beam and rods with point masses", *Journal of Sound and Vibration*, **96**, 461-468.
Gurgoze, M. (1998), "On the alternative formulations of the frequency equations of a Bernoulli-Euler beam to which several spring-mass systems are attached inspan", *J. of Sound and Vibration*, **217**(3), 585-595.
Hamdan, M.N and Jubran, B.A. (1991), "Free and forced vibrations of a restrained uniform beam carrying an intermediate lumped mass and a rotary inertia", *Journal of Sound and Vibration*, **150**(2), 203-216.
Heidebrecht, A.C. (1967), "Vibration of non-uniform simply-supported beams", *Journal of the Engineering Mechanics Division, Proceedings of the ASCE*, **93**(EM2), 1-15.
Housner, G.W. and Keightley, W.O. (1962), "Vibrations of linearly tapered cantilever beams", *Journal of Engineering Mechanics Division*, *Proceedings of the ASCE*, **88**(EM2), 95-123.
Laura, P.A.A., Maurizi, M.J. and Pombo, J.L. (1975), "A note on the dynamic analysis of an elastically restrained-free beam with a mass at the free end", *Journal of Sound and Vibration*, **41**, 397-405.
Laura, P.A.A., Susemihl, E.A., Pombo, J.L., Luisoni, L.E. and Gelos, R. (1977), "On the dynamic behavior of structural elements carrying elastically mounted concentrated masses", *Applied Acoustic*, **10**, 121-145.
Laura, P.A.A., Filipich, C.P. and Cortinez, V.H. (1987), "Vibration of beams and plates carrying concentrated masses", *Journal of Sound and Vibration*, **112**, 177-182.
Lindberg, G.M. (1963), "Vibration of non-uniform beams", *Aeronautical Quarterly*, **14**, 387-395.
Naguleswaran, S. (1996), "Comments on 'Vibration of non-uniform rods and beams' ", *Journal of Sound and Vibration*, **195**(2), 331-337.
Meirovitch, L. (1967), *Analytical Methods in Vibrations*, New York: Macmillan.
Rossi, R.E., Laura, P.A.A., Avalos, D.R. and Larrondo, H.A. (1993), "Free vibrations of Timoshenko beams carrying elastically mounted concentrated masses", *Journal of Sound and Vibration*, **165**, 209-223.
To, C.W.S. (1979), "Higher order tapered beam finite elements for vibration analysis", *Journal of Sound and Vibration*, **63**(1), 33-50.
Wu, J.S. and Lin, T.L. (1990), "Free vibration analysis of a uniform cantilever beam with point masses by an analytical-and-numerical-combined method", *Journal of Sound and Vibration*, **136**, 201-213.

Appendix

Closed-Form Solutions for the Natural Frequencies and Normal Mode Shapes of A Non-uniform Beam with $\varphi(\xi)=(e+\xi\alpha)^2$

For the non-uniform beam with cross-sectional area $A(x)$ and moment of inertia $I(x)$ defined by Eqs. (2) and (3), the closed-form solutions for the natural frequencies and the corresponding "normal" mode shapes in some types of boundary conditions, excluding the "clamped-hinged" (CH) and "hinged-clamped" (HC) types derived in the context, are listed in this appendix.

A.1. For a clamped-free (CF) beam

The boundary conditions for a clamped-free beam are

$$w(x,t)=0, \quad \frac{\partial w(x,t)}{\partial x}=0 \quad \text{at } x=0 \tag{A.1}$$

$$\frac{\partial^2 w(x,t)}{\partial x^2}=0, \quad \frac{\partial^3 w(x,t)}{\partial x^3}=0 \quad \text{at } x=L \tag{A.2}$$

or

$$V(0)=0, \quad V'(0)=0 \tag{A.3a,b}$$

$$V''(L)=-\frac{6C^{*2}}{L^2}V(L)+\frac{4C^*}{L}V'(L) \tag{A.4a}$$

$$V'''(L)=\frac{6C^{*2}}{L^2}V'(L)-\frac{12C^{*3}}{L^3}V(L) \tag{A.4b}$$

The frequency equation is

$$S_{11r}S_{22r}-S_{12r}S_{21r}=0 \tag{A.5}$$

where

$$\begin{cases} S_{11r}=a_r^2(\cos a_r+\cosh a_r)+4C^*a_r(-\sin a_r-\sinh a_r)-6C^{*2}(\cos a_r-\cosh a_r) \\ S_{12r}=a_r^2(\sin a_r+\sinh a_r)+4C^*a_r(\cos a_r-\cosh a_r)-6C^{*2}(\sin a_r-\sinh a_r) \\ S_{21r}=a_r^3(-\sin a_r+\sinh a_r)+6C^{*2}a_r(-\sin a_r-\sinh a_r)-12C^{*3}(\cos a_r-\cosh a_r) \\ S_{22r}=a_r^3(\cos a_r+\cosh a_r)+6C^{*2}a_r(\cos a_r-\cosh a_r)-12C^{*3}(\sin a_r-\sinh a_r) \end{cases} \tag{A.6}$$

The natural frequencies are given by

$$\omega_r=a_r^2\sqrt{\frac{EI_o}{\rho A_o L^4}} \ \text{(r/s)}, \quad r=1,2,\ldots \tag{A.7}$$

and the corresponding normal mode shapes are

$$V_{3r}^*=\frac{1}{\sqrt{\rho A_o L R_{3r}}}\left(E_3\sin a_r\xi+F_3\cos a_r\xi+G_3\sinh a_r\xi+H_3\cosh a_r\xi\right) \tag{A.8}$$

where

$$E_3=-Q_{3r}, \quad F_3=1, \quad G_3=Q_{3r}, \quad H_3=-1, \quad Q_{3r}=S_{11r}/S_{12r} \tag{A.9}$$

The values of R_{3r} appearing in Eq. (A.8) may be obtained from Eq. (30b) by replacing the values E_1, F_1, G_1, H_1 of by those of E_3, F_3, G_3, H_3 defined by Eq. (A.9). Where the values of $a_r=\beta_r L$ (r=1, 2,...), are the roots of the frequency equation (A.5).

A.2. For a free-clamped (FC) beam

The boundary conditions for a free-clamped beam are

$$\frac{\partial^2 w(x,t)}{\partial x^2}=0,\ \frac{\partial^3 w(x,t)}{\partial x^3}=0 \ \text{ at } x=0 \tag{A.10}$$

$$w(x,t)=0,\ \frac{\partial w(x,t)}{\partial x}=0 \ \text{ at } x=L \tag{A.11}$$

or

$$V''(0)=-\frac{6\bar{\alpha}^2}{L^2}V(0)+\frac{4\bar{\alpha}}{L}V'(0) \tag{A.12a}$$

$$V'''(0)=\frac{6\bar{\alpha}^2}{L^2}V'(0)-\frac{12\bar{\alpha}^3}{L^3}V(0) \tag{A.12b}$$

$$V(L)=0,\ V'(L)=0 \tag{A.13a, b}$$

where

$$\bar{\alpha}=\alpha/e \tag{36}$$

The frequency equation is

$$\frac{a_r^5+12\bar{\alpha}^4 a_r}{\cosh a_r}+(a_r^5-12\bar{\alpha}^4 a_r)\cos a_r+(4\bar{\alpha}^4 a_r+12\bar{\alpha}^3 a_r^2)\sin a_r$$
$$+(4\bar{\alpha}\, a_r^4-12\bar{\alpha}^3\, a_r^2)\cos a_r\ \tanh a_r+12\bar{\alpha}^2\, a_r^3\ \sin a_r\ \tanh a_r=0 \tag{A.14}$$

The natural frequencies are given by

$$\omega_r=a_r^2\sqrt{\frac{EI_o}{\rho A_o L^4}}\ \text{(r/s)},\ r=1,2,\ldots \tag{A.7}$$

and the corresponding "normal" mode shapes are

$$V_{4r}^*=\frac{1}{\sqrt{\rho A_o L R_{4r}}}(E_4\sin a_r\xi+F_4\cos a_r\xi+G_4\sinh a_r\xi+H_4\cosh a_r\xi) \tag{A.15}$$

where

$$E_4=B_{r1}-Q_{4r}D_{r1},\ F_4=B_{r2}-Q_{4r}D_{r2},\ G_4=B_{r3}-Q_{4r}D_{r3},\ H_4=B_{r4}-Q_{4r}D_{r4} \tag{A.16a}$$

$$B_{r1}=-\frac{a_r}{\bar{\alpha}},\ B_{r2}=-\frac{1}{2}\left(\frac{a_r}{\bar{\alpha}}\right)^2+1,\ B_{r3}=-\frac{a_r}{\bar{\alpha}},\ B_{r4}=-\frac{1}{2}\left(\frac{a_r}{\bar{\alpha}}\right)^2-1 \tag{A.16b}$$

$$D_{r1}=\frac{1}{2}\left(\frac{a_r}{\bar{\alpha}}\right)^2+1,\ D_{r2}=\frac{1}{3}\left(\frac{a_r}{\bar{\alpha}}\right)^3,\ D_{r3}=\frac{1}{2}\left(\frac{a_r}{\bar{\alpha}}\right)^2-1,\ D_{r4}=\frac{1}{3}\left(\frac{a_r}{\bar{\alpha}}\right)^3 \tag{A.16c}$$

$$Q_{4r}=\frac{B_{r1}\sin a_r+B_{r2}\cos a_r+B_{r3}\sinh a_r+B_{r4}\cosh a_a}{D_{r1}\sin a_r+D_{r2}\cos a_r+D_{r3}\sinh a_r+D_{r4}\cosh a_a} \tag{A.16d}$$

The values of R_{4r} appearing in Eq. (A.15) may be obtained from Eq. (30b) by replacing the values of E_1, F_1, G_1, H_1 by those of E_4, F_4, G_4, H_4 defined by Eq. (A.16). Where the values of $a_r = \beta_r L$ (r=1, 2,...), are the roots of the frequency equation (A.14).

A.3. For a hinged-hinged (HH) beam

The boundary conditions for a hinged-hinged beam are

$$w(x, t)=0, \ \frac{\partial^2 w(x, t)}{\partial x^2}=0 \text{ at } x=0 \tag{A.17}$$

$$w(x, t)=0, \ \frac{\partial^2 w(x, t)}{\partial x^2}=0 \text{ at } x=L \tag{A.18}$$

or

$$V(0)=0, \ V''(0)=\frac{4\bar{\alpha}}{L}V'(0) \tag{A.19a, b}$$

$$V(L)=0, \ V''(L)=\frac{4C^*}{L}V'(L) \tag{A.20a, b}$$

The frequency equation is

$$2a_r C^*(\cos a_r \sinh a_r - \sin a_r \cosh a_r)+8C^*(1-\cos a_r \cosh a_r)-\frac{a_r^2}{\bar{\alpha}}(\sin a_r \sinh a_r)=0 \tag{A.21}$$

The natural frequencies are given by

$$\omega_r = a_r^2 \sqrt{\frac{EI_o}{\rho A_o L^4}} \ (\text{r/s}), \ r=1, 2, \ldots \tag{A.7}$$

and the corresponding normal mode shapes are

$$V_{5r}^* = \frac{1}{\sqrt{\rho A_o L R_{5r}}}(E_5 \sin a_r\xi + F_5 \cos a_r\xi + G_5 \sinh a_r\xi + H_5 \cosh a_r\xi) \tag{A.22}$$

where

$$E_5 = -\left(\frac{a_r}{4\bar{\alpha}}+Q_{5r}\right), \ F_5=1, \ G_5=-\frac{a_r}{4\bar{\alpha}}+Q_{5r}, \ H_5=-1, \ Q_{5r}=T_{11r} / T_{12r} \tag{A.23a}$$

$$\begin{cases} T_{11r}=(\cos a_r - \cosh a_r)-\dfrac{a_r}{4\bar{\alpha}}(\sin a_r + \sinh a_r) \\ T_{12r}=\sin a_r - \sinh a_r \end{cases} \tag{A.23b}$$

The values of R_{5r} appearing in Eq. (A.22) may be obtained from Eq. (30b) by replacing the values of E_1, F_1, G_1, H_1 by those of E_5, F_5, G_5, H_5 defined by Eq. (A.23). Where the values of $a_r = \beta_r L$ (r=1, 2,...), are the roots of the frequency equation (A.21).