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General nonlocal solution of the elastic half space loaded by a concentrated force *P* perpendicular to the boundary

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Abstract. The main purpose of this paper is to devolop the results introduced in Artan (1996) and to find a general nonlocal linear elastic solution for Boussinesq problem. The general nonlocal solution given Artan (1996) is valid only when the distance to the boundary is greater than one atomic measure. The nonlocal stress field presented in this paper is valid for the whole half plane.

Key words: nonlocal elasticity; Boussinesq problem; half plane.

1. Introduction

The problem of an elastic half space loaded by a concentrated force, is known as Boussinesq problem. The classical solution of this problem can be found in every reference book on the mathematical theory of elasticity. (For instance, see Rekach 1979). According to the classical elasticity solution of the Boussinesq problem the stresses at the application point of the force become infinite. But this solution does not display the actual situation. In other words the local and even polar continuum theories fail to apply to Boussinesq problem. To remedy this situation theories of nonlocal continua may be introduced. In the nonlocal theory the constitutive relations are nonlocal in character and the stress at a given point does not only depend on the strain at the same point, but on the strains at all points of the body. The governing equations of the nonloclal elasticity are given in Artan (1989) and Eringen (1976, 1974, 1987). Some of the early ideas for the nonlocal elastic solids were explored by Eringen, Edelen and Kunin (For a brief introduction to the subject, see Eringen *et al.* 1972 and Kunin 1968). The purpose of this paper is to present a nonlocal solution for the Boussinesq problem, and to display the stress singularities disappear at the application point of the force. The program Mathematica, Derive and Latex are used throughout.

2. The nonlocal solution of the Boussinesq problem

The classical elasticity solution of the Boussinesq problem in Cartesian coordinates are (see Rekach 1979 and Fig. 1)

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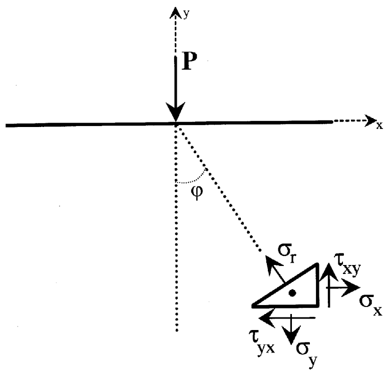

Fig. 1 Elastic half plane under the singular force

$$\sigma_x=\frac{2P}{\pi}\frac{x^2y}{(x^2+y^2)^2},\quad \sigma_y=\frac{2P}{\pi}\frac{y^3}{(x^2+y^2)^2},\quad \tau_{xy}=\frac{2P}{\pi}\frac{xy^2}{(x^2+y^2)^2} \tag{1}$$

The nonlocal stress field can be obtained as

$$t_{xx}(x,y,a)=\iint \alpha(|\mathbf{x}'-\mathbf{x}|)\sigma_{xx}(x',y')dx'\,dy' \tag{2}$$

$$t_{yy}(x,y,a)=\iint \alpha(|\mathbf{x}'-\mathbf{x}|)\sigma_{yy}(x',y')dx'\,dy' \tag{3}$$

$$t_{xy}(x,y,a)=\iint \alpha(|\mathbf{x}'-\mathbf{x}|)\tau_{xy}(x',y')dx'\,dy' \tag{4}$$

where $\alpha(|\mathbf{x}'-\mathbf{x}|)$ is called kernel function and is the mesure of the effect of the strain at the point $\mathbf{x}'$ on the stress at the point $\mathbf{x}$. (See for full detail Artan 1996 & 1997). In this article, the kernel function of the nonlocal medium will be chosen as

$$\alpha(|\mathbf{x}-\mathbf{x}'|)=\begin{cases} B\left(1-\dfrac{|\mathbf{x}-\mathbf{x}'|^2}{a^2}\right) & |\mathbf{x}-\mathbf{x}'|\le a \\ 0 & |\mathbf{x}-\mathbf{x}'|\ge a \end{cases} \tag{5}$$

where a is the atomic distance and B is a constant. In the Cartesian coordinates (5) becomes (see Fig. 2)

$$\alpha(x,y)=B\left(1-\frac{(x'-x)^2+(y'-y)^2}{a^2}\right),\quad |\mathbf{x}-\mathbf{x}'|\le a \tag{6}$$

The value of a, and B are (see for full detail Artan 1996)

$$a=4\times10^{-8}\text{cm},\ \ B=\frac{2}{\pi a^2} \tag{7}$$

When the distance to the boundary is less than one atomic measure the nonlocal stress field in the y direction is calculated as (see Fig. 4)

$$t_{yy}(x,y,a)=\int_0^{y-a}\int_{\alpha_1}^{\alpha_2}\alpha(x,y)\sigma_{yy}(x',y')dx'\,dy' \tag{8}$$

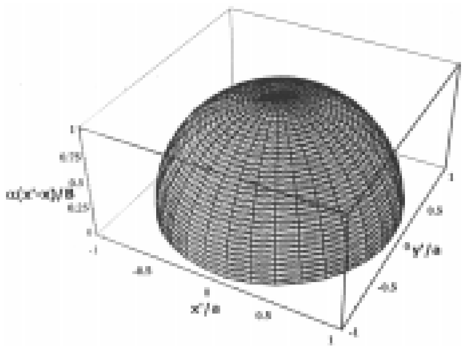

Fig. 2 Kernel function

where

$$\alpha_1 = x - \sqrt{a^2 - (y' - y)^2}, \quad \alpha_2 = x + \sqrt{a^2 - (y' - y)^2} \tag{9}$$

When the distance to the boundary is greater than one atomic measure the nonlocal stress field in the y direction is calculated as (see Fig. 3)

$$t_{yy}(x,y) = \int_{y-a}^{y+a} \int_{\alpha_1}^{\alpha_2} \alpha(x,y)\sigma_{yy}(x',y')dx'\ dy' \tag{10}$$

In the above equations the first integral over x' is calculated exactly, then the second integral over y' is calculated approximately. The nonlocal stress field becomes

$$t_{yy}(x,y,a) = \frac{4P}{\pi^2}\left\{\frac{(1+\sqrt{3})y}{a^2} - \frac{3(a-2x)y \arctan\left(\frac{\sqrt{3}a-2y}{a-2x}\right)}{4a^3} - \frac{3(a+2x)y \arctan\left(\frac{\sqrt{3}a-2y}{a+2x}\right)}{4a^3}\right.$$

$$-\frac{3xy \arctan\left(\frac{a-y}{x}\right)}{2a^3} - \frac{3xy \arctan\left(\frac{a+y}{x}\right)}{2a^3} - \frac{3(a-2x)y \ \arctan\left(\frac{\sqrt{3}a+2y}{a-2x}\right)}{4a^3} - \frac{3(a+2x)y \ \arctan\left(\frac{\sqrt{3}a+2y}{a+2x}\right)}{4a^3}$$

$$-\frac{(a^2+2x^2-y^2)\log\left((a^2+x^2-2ay+y^2)^{\frac{1}{4}}\right)}{a^3} + \frac{(a^2+2x^2-y^2)\log\left((a^2+x^2+2ay+y^2)^{\frac{1}{4}}\right)}{a^3}$$

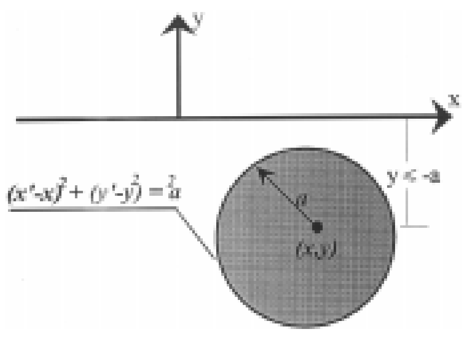

Fig. 3 Integration domain for $y \leq -a$

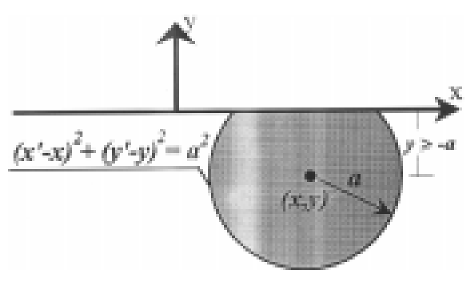

Fig. 4 Integration domain for $y \geq -a$

$$-\frac{(5a^2-8ax+4(2x^2-y^2))\log\left((a^2-ax+x^2-\sqrt{3}ay+y^2)^{\frac{1}{16}}\right)}{a^3}$$
$$-\frac{(5a^2+8ax+4(2x^2-y^2))\log\left((a^2+ax+x^2-\sqrt{3}ay+y^2)^{\frac{1}{16}}\right)}{a^3}$$
$$+\frac{(5a^2-8ax+4(2x^2-y^2))\log\left((a^2-ax+x^2+\sqrt{3}ay+y^2)^{\frac{1}{16}}\right)}{a^3}$$
$$\left.+\frac{(5a^2+8ax+4(2x^2-y^2))\log\left((a^2+ax+x^2+\sqrt{3}ay+y^2)^{\frac{1}{16}}\right)}{a^3}\right\}; \quad y\le -a \tag{11}$$

and

$$t_{yy}(x,y,a)=\frac{4P}{\pi^2}\frac{A1}{A2}, \quad -a\le y\le 0 \tag{12}$$

where

$$\begin{aligned}
A1=&(a-y)((-a+y)(8\sqrt{3}\sqrt{5a^2-2ay-3y^2}(2a^2+2x^2-3ay+y^2\\
&-x\sqrt{7a^2-6ay-y^2})(2a^2+2x^2-3ay+y^2+x\sqrt{7a^2-6ay-y^2})\\
&\times(4a^4+4x^4-4a^3y+5x^2y^2+y^4+2ay(x^2+y^2)-a^2(7x^2+3y^2))\\
&-18\sqrt{7a^2-6ay-y^2}\ (2a^2+2x^2-ay-y^2+\sqrt{3}x\sqrt{5a^2-2ay-3y^2})\\
&\times(-2a^2-2x^2+ay+y^2+\sqrt{3}x\sqrt{5a^2-2ay-3y^2})\\
&\times(4a^4+4x^4-12a^3y+5x^2y^2+y^4-6ay(x^2+y^2)+a^2(x^2+13y^2))\\
&-8(7a^2-8x^2-2ay-5y^2)(2a^2+2x^2-ay-y^2+\sqrt{3}x\sqrt{5a^2-2ay-3y^2})\\
&\times(-2a^2-2x^2+ay+y^2+\sqrt{3}x\sqrt{5a^2-2ay-3y^2})\\
&\times(-2a^2-2x^2+3ay-y^2+x\sqrt{7a^2-6ay-y^2})\quad\times(2a^2+2x^2-3ay+y^2+x\sqrt{7a^2-6ay-y^2})\\
&\times\arctan\left(\frac{-4x+\sqrt{3}\sqrt{5a^2-2ay-3y^2}}{a-y}\right)-8(7a^2-8x^2-2ay-5y^2)\\
&\times(2a^2+2x^2-ay-y^2+\sqrt{3}x\sqrt{5a^2-2ay-3y^2})\\
&\times(-2a^2-2x^2+ay+y^2+\sqrt{3}x\sqrt{5a^2-2ay-3y^2})\ \times(-2a^2-2x^2+3ay-y^2+x\sqrt{7a^2-6ay-y^2})\\
&\times(2a^2+2x^2-3ay+y^2+x\sqrt{7a^2-6ay-y^2})\\
&\times\arctan\left(\frac{4x+\sqrt{3}\sqrt{5a^2-2ay-3y^2}}{a-y}\right)-6(a^2+8x^2-6ay+5y^2)
\end{aligned}$$

$$\times\,(2a^2+2x^2-ay-y^2+\sqrt{3}x\sqrt{5a^2-2ay-3y^2})\;\;\times\,(-2a^2-2x^2+ay+y^2+\sqrt{3}x\sqrt{5a^2-2ay-3y^2})$$
$$\times\,(-2a^2-2x^2+3ay-y^2+x\sqrt{7a^2-6ay-y^2})\;\;\times\,(2a^2+2x^2-3ay+y^2+x\sqrt{7a^2-6ay-y^2})$$
$$\times\,\arctan\left(\frac{-4x+\sqrt{7a^2-6ay-y^2}}{3(a-y)}\right)+6(a^2+8x^2-6ay+5y^2)$$
$$\times\,(2a^2+2x^2-ay-y^2+\sqrt{3}x\sqrt{5a^2-2ay-3y^2})\;\;\times\,(-2a^2-2x^2+ay+y^2+\sqrt{3}x\sqrt{5a^2-2ay-3y^2})$$
$$\times\,(-2a^2-2x^2+3ay-y^2+x\sqrt{7a^2-6ay-y^2})$$
$$\times\,(2a^2+2x^2-3ay+y^2+x\sqrt{7a^2-6ay-y^2})\times\arctan\left(\frac{4x+\sqrt{7a^2-6ay-y^2}}{3a-3y}\right) \tag{13}$$

$$A2=256a^4(2a^2+2x^2-ay-y^2+\sqrt{3}x\sqrt{5a^2-2ay-3y^2})\times(-2a^2-2x^2+ay+y^2+\sqrt{3}x\sqrt{5a^2-2ay-3y^2})$$
$$\times\;(-2a^2-2x^2+3ay-y^2+x\sqrt{7a^2-6ay-y^2})\times(2a^2+2x^2-3ay+y^2+x\sqrt{7a^2-6ay-y^2}) \tag{14}$$

The nonlocal stress field in the x direction and the nonlocal shear stress field are calculated approximately by using (2), (4). For $a = 0$ the nonlocal stress field reverts to the classical stress field. That is

$$t_{yy}(x,y,0)=0.98\frac{2P}{\pi}\frac{y^3}{(x^2+y^2)^2};\quad t_{xx}(x,y,0)=1.01\frac{2P}{\pi}\frac{x^2y}{(x^2+y^2)^2}$$
$$t_{xy}(x,y,0)=1.01\frac{2P}{\pi}\frac{xy^2}{(x^2+y^2)^2} \tag{15}$$

The above stress fields are valid for all half space but the stress field in Artan (1996) is valid only when the distance to the boundary is greater than one atomic measure.

3. Conclusions

The main results are listed below

a) The nonlocal stresses are finite even at the points where local stresses are infinite (See Figs. 6-10).

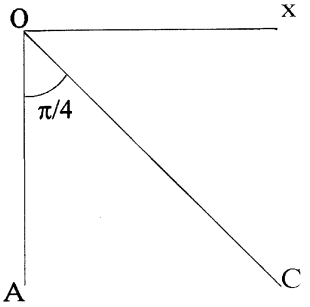

Fig. 5 The stress diagrams are given on this lines

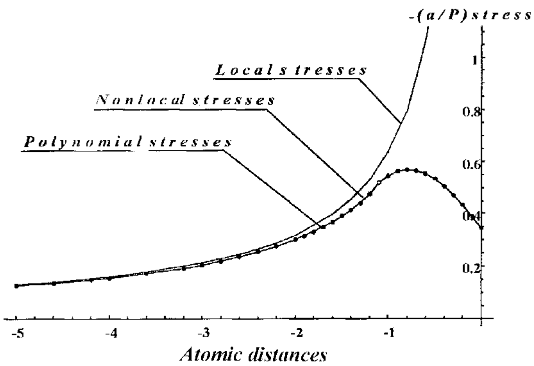

Fig. 6 Stresses in the direction y axis on the line OA

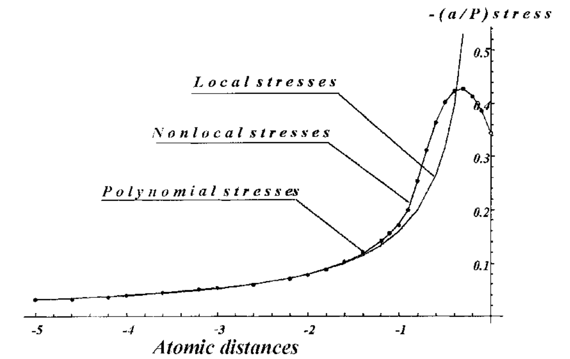

Fig. 7 Stresses in the direction y axis on the line OC

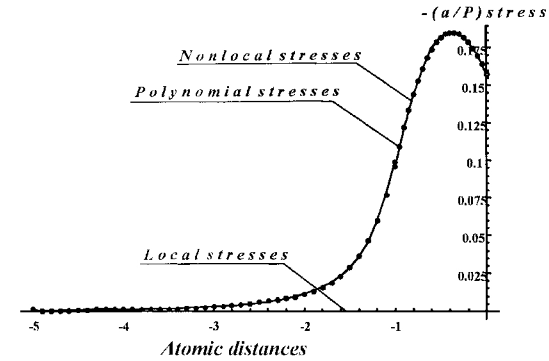

Fig. 8 Stresses in the direction x axis on the line OA

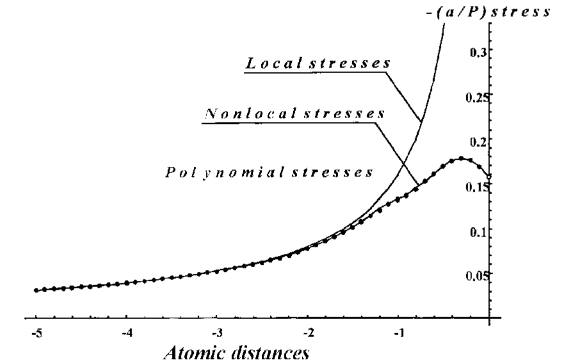

Fig. 9 Stresses in the direction x axis on the line OC

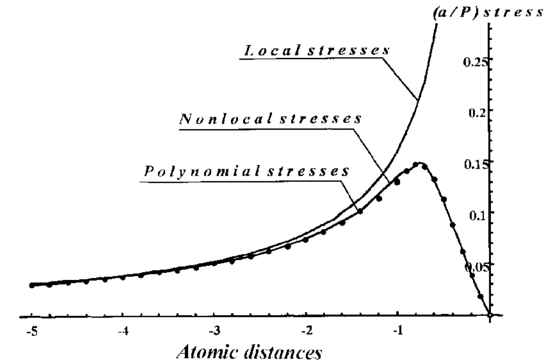

Fig. 10 Shear stresses on the line OC

b) The maximum stress does not occur at the boundary but further down. Similar results had already been obtained in some other problems (see Artan 1996 & 1997, Eringen 1979).

c) For $a = 0$ the nonlocal solution reverts to the classical solution.

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