Structural Engineering and Mechanics, Vol. 9, No. 2 (2000) 169-180
DOI: http://dx.doi.org/10.12989/sem.2000.9.2.169

Optimum design of parabolic steel box arches

Abul K. Azad† and Hani M.M. Mohdaly‡

Department of Civil Engineering, King Fahd University of Petroleum & Minerals, Dhahran 31261, Saudi Arabia

Abstract. An optimization procedure has been prescribed for the minimum weight design of symmetrical parabolic arches subjected to arbitrary loading. The cross section is assumed to be a symmetrical box section with variable depth and flange areas. The webs are unstiffened and have constant thickness. The proposed sequential, iterative search technique determines the optimum geometrical configuration of the parabolic arch which includes the optimum depth profile and the optimum lengths and areas of the required flange plates corresponding to the prescribed number of curtailments. The study shows that the optimum value of rise to span ratio (h/L) of a parabolic arch is maximum at 0.41 for uniformly distributed loading over the entire span. For any other loading, the optimum value of h/L is less than 0.41.

Key words: optimization; arches; structural design; steel; constraints; minimum weight.

1. Introduction

As the geometrical shape of an arch has a great influence on its strength, the congruent shape for a design must be searched for to obtain an economical design. Parabolic arches are frequently used in practice due to their superior aesthetic appeal and better geometrical configuration. A strong support for a parabolic shape also arises from the fact that a parabolic shape is indeed the optimal or near optimal geometrical shape of an arch for a variety of imposed loadings. For further saving in materials, and hence cost, the cross section of the arch can be varied along its length by varying the arch depth as well as the flange areas, wherever possible, to account for the variable internal forces. For aesthetic and practical reasons, the variation in depth must obviously be achieved by furnishing a smooth curved profile as depicted in Fig. 1a.

The widespread use of arches has drawn a great deal of interest of researches in seeking an optimal design through different analytical and numerical approaches, fundamentals of some of which are well covered in numerous books, such as for example, Gallagher and Zienkiewicz (1973), Kirsch (1981) and Rozvany and Karihaloo (1988). The earlier works (Budiansky *et al.* 1969, Farshad 1976 and Prager and Rozvany 1979) dealt with the optimal shape of arches using classical variational procedures but because of their inherent complexity, solutions were obtained only for simpler cases. An optimality condition for fully stressed, flexureless arches using Prager-Shield criteria is prescribed by Hill *et al.* (1979). A treatment of arches subject to axial compression and bending is given by Rozvany *et al.* (1980) and it is further followed up later by Ang *et al.* (1988) using plastic design. A numerical study of optimal design of arches of uniform depth and width but of variable plate thickness is conducted by Lipson and Haque (1980) by using complex method.

† Professor
‡ Graduate Student/Engineer

The optimal rise of an arch is the subject of study by Ermopoulos and Ioannidis (1985). The works of Yao and Choi (1989) and Zhu *et al.* (1992) can be cited as the representative sample of works in the area of optimization of arch dams. The past research, however, does not deal with the design of parabolic arches having both nonuniform depth and variable flange areas.

In this paper, an iterative procedure for optimization of a symmetrical parabolic steel box girder arch with variable depth and flange areas has been presented. The cross section is assumed to be symmetrical with uniform web thickness and identical top and bottom flange plates. The proposed method is design-oriented and is applicable to any imposed loading and support conditions. Sample results are presented to show some interesting observations pertaining to design.

2. Problem formulation

2.1. Geometrical configuration

For a typical parabolic arch shown in Fig. 1, the equation of the profile along the middepth of web is given as

$$y=\frac{4hx}{L^2}(L-x) \tag{1}$$

when h is the rise to the centreline of the arch, L is the span and x, y are the coordinates of a point on the arch.

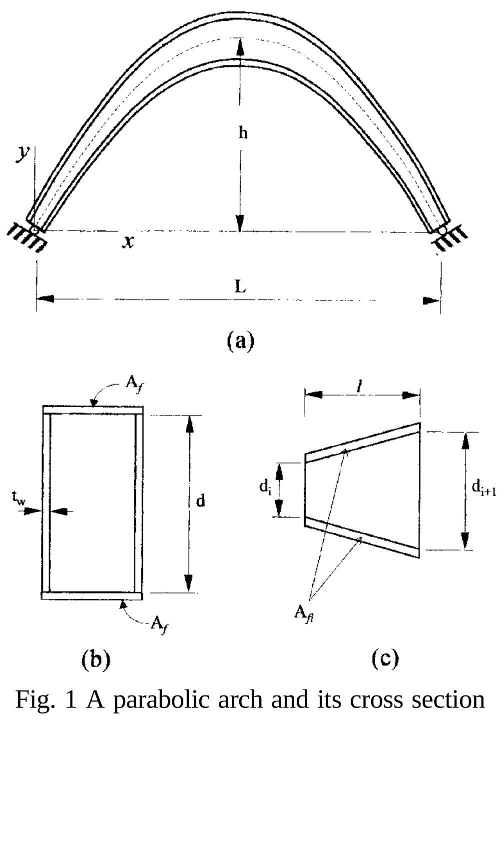

Fig. 1 A parabolic arch and its cross section

The cross section of the arch is considered as a symmetrical box section as depicted in Fig. 1b, for which the depth d of arch can vary. The web is assumed to be unstiffened, having an unchanged thickness t_w throughout the arch length. The area of flange plates A_f can discretely vary stepwise, for which the maximum number of cut-off points are prescribed apriori. The arch is considered to be symmetrical about the midspan, $x = L/2$.

2.2. Objective function

The parabolic arch is discretized into n-number of small, linear segments of equal length l with $(n+1)$ nodal points. The weight of a typical segment of a steel box arch shown in Fig. 1c is

$$W_i = 2\rho\left[\left(\frac{d_i + d_{i+1}}{2}\right)t_w + A_{fi}\right]l \tag{2}$$

where ρ = unit weight of steel, d_i = depth of web at node i, A_{fi} = area of each flange plate for the segment i and l = length of each segment.

For the entire n-segmented, symmetrical arch, the total weight is expressed as

$$W = 2\rho \sum_{i=1}^{n} (d_i t_w + A_{fi})l \tag{3}$$

The objective is to minimize Eq. (3) by finding the optimum values of the design variables (h, d_i, t_w and A_{fi}) within the design space bounded by the applicable geometrical and design constraints.

2.3. Geometrical constraints

The geometrical constraints of the arch are specified simply by prescribing a minimum and a maximum value of the arch rise h, respectively as h_L and h_u, and an acceptable minimum dimensions of d, t_w and A_f for a design. Thus, $h_L \leq h \leq h_u$, $d - d_{\min} \geq 0$, $t_w - t_{\min} \geq 0$ and $A_f - A_{f\,\min} \geq 0$. In order to provide a minimum amount of flange areas for the box section, the minimum value of A_f is taken in this study as 10% of the cross sectional area in those segments where the required value of A_f is either nonexistent or falls below this minimum. As the value of t_w must correspond to available plate thickness, t_w is not a continuous design variable and it varies only in steps conforming to those commercially available.

2.4. Strength constraints

The stress or strength constraints ensure that the strength of an arch at all nodal points is adequate to withstand the design forces. An arch section, in general, is subjected to the combined action of axial force, N, in-plane bending moment M and shearing force V. In a design, the adequacy of a section is verified by satisfying the requirements of a building code, either using allowable stress design method (ASD), load-resistance-factor design method or plastic design method. The ASD method as prescribed in the AISC specification (1989) is used in this study. The proposed methodology can, however, be pursued to adopt any other specification.

The AISC's interaction equation for the combined stress due to N and M stipulates that

$$\frac{f_a}{F_a}+\frac{f_b}{F_b}\leq 1.0 \tag{4}$$

must be satisfied at all sections. In Eq. (4), $f_a = N/A$ and $f_b = M/S$ are respectively the computed normal stress due to N and M, F_a and F_b are respectively the allowable stresses under the action of axial force and bending moment (when each force is acting alone), A is the cross sectional area and S is the elastic sectional modulus of the section. The value of F_a and F_b which are to be determined from a rational analysis taking into consideration the lateral supports to the arch, if any, are assumed to be specified as input data.

The average value of the computed shear stress, $f_V = V/(2dt_w)$, at each node must not exceed the allowable value F_V given by AISC (1989) as

$$F_V=\frac{F_y}{2.89}C_V\leq 0.40F_y \tag{5}$$

when F_y = yield stress of web steel and C_V, a parameter, is prescribed as

$$\begin{aligned} C_V &= \frac{240300}{(d/t_w)^2F_y} \quad \text{if } C_V\leq .8 \\ &= \frac{190}{d/t_w}\sqrt{\frac{5.34}{F_y}} \quad \text{if } C_V>.8 \end{aligned} \tag{6}$$

Thus, the shear requirement is satisfied by

$$\frac{f_V}{F_V}\leq 1.0 \tag{7}$$

Eqs. (4) and (7) must be satisfied at all nodes for a safe design.

As the web is considered unstiffened, the web height to thickness ratio, d/t_w, must be restricted to prevent the web from local buckling under the interaction of axial compression and bending, i.e., at each node i,

$$\frac{d_i}{t_w}\leq \alpha \tag{8}$$

where α is a numerical constant which is a function of the yield stress of web. Most building codes prescribe the maximum limit on the width-thickness ratio to prevent local buckling.

The design values of the internal forces (N, M and V) at each node i under the given loading and load combinations are calculated by a linear elastic analysis of the arch, idealizing it as an assembly of short, linear segments.

3. Optimum box section

For a typical box section, the total cross sectional area A is

$$A=2(dt_w+A_f) \tag{9}$$

Following the same procedure of Azad (1978) for finding the optimum depth of a symmetrical Ibeam for a given area A and a prescribed t_w which maximizes S, it can be shown that for a box section, the optimum depth $\bar{d}$ for maximum s is

$$\bar{d}=\frac{3A}{8t_w} \tag{10}$$

and the corresponding values of A_f and S are

$$A_f=\frac{A}{8} \tag{11a}$$

$$S=\frac{A\bar{d}}{4} \tag{11b}$$

Thus, for a given area A and web thickness t_w, the value of elastic section modulus S of a symmetrical box section reaches its maximum when the total flange area equals 25% of the total area A.

For a chosen web thickness t_w, the interaction equation, Eq. (4), can be written for a fully stressed case as

$$\frac{P}{AF_a}+\frac{M}{SF_b}=1.0 \tag{12}$$

Multiplying both sides by Ad and using the values of A_f and S from Eq. (11), the following quadratic equation in optimum depth $\bar{d}$ is obtained.

$$\frac{8t_w}{3}\bar{d}^2-\frac{P}{F_a}\bar{d}-\frac{4M}{F_b}=0 \tag{13}$$

The required value of $\bar{d}$ from Eq. (13) is then

$$\bar{d}=\frac{b+\sqrt{b^2+4ac}}{2a} \tag{14}$$

when $a=8t_w/3$, $b=P/F_a$ and $c=4M/F_b$. The value of $\bar{d}$ at each node i can be determined from Eq. (14) for a chosen value of t_w using the absolute values of N and M computed at the section.

4. Method of solution

The optimization procedure begins with an initial admissible design by using the maximum permitted value of h and the minimum acceptable web thickness t_w. The arch is idealized as an assembly of n straight segments of equal length and the maximum design values of N, M and V at each node are calculated from a linear structural analysis of the discretized arch, using applicable load combinations.

At each nodal point i, two values of web depth d_i are calculated: one from Eq. (14) to account for the interaction of N and M and the other value which satisfies the shear requirement of Eq. (7). The larger of the two values constitutes the required minimum value of d_i. The web profile thus obtained varies linearly from node to node, a shape which is unsuitable for a practical design. The irregular web must be altered to yield a smooth curved profile, both at the top and bottom of the web. For a smoothened profile, a parabolic curve has been adopted to provide a harmonious match with the mid-surface parabolic geometry of the arch.

The smoothening of web profile is achieved by a least-square curve fitting of data points at the top ($+d_i/2$) and at the bottom ($-d_i/2$) (Mohdaly 1997), using different parabolic forms and ensuring that d_i satisfies the constraint of Eq. (7). For this smooth web, the required flange areas for each segment are then calculated by satisfying Eq. (4), considering the critical node of the segment. In those segments where the depth of web dictates a small or no flange area, the minimum flange area, assumed 10% of the cross sectional area in this study, is provided.

As the flange areas would, in general, vary from segment to segment of an arch, the flange plate cut-off points must be reduced to minimize cost. The optimum locations of a prescribed number of flange cut-offs are determined by minimizing the flange weight which is expressed as

$$W_f = \rho \sum_{k=1}^{m} A_{fk} \cdot S_k \tag{15}$$

In Eq. (15), A_{fk} is the area of flange plate of length S_k and m is the preassigned number of different flange plates (number of cut-offs plus one). This minimization of W_f is achieved by a dynamic programming to determine the optimum values of S_k (Taha 1976).

The furnished design thus achieved represents the optimum solution for the chosen h, t_w and a preselected number of cut-off points. By incrementally reducing the value of h for each t_w and then repeating the search with the next higher available t_w, the global minimum weight design is achieved in the manner described below.

4.1. Iterative steps

The step-by-step procedure to find the optimal design is given below.

1. The sequential search of the optimum design begins with the specified maximum value of h and the minimum acceptable value of t_w. The initial design values of N, M and V are computed at each node of the discretized parabolic arch for the prescribed loading, assuming a hypothetical uniform cross sectional area for the entire arch.
2. For the computed design forces, the depth of web at each nodal point is determined from Eq. (7) or (14) whichever yields the larger value. The web profile at top and bottom is then smoothened by the best-fitting parabolic shapes, ensuring the requirement of Eq. (7).
3. For the smoothened web profile, the required flange area for each segment is then calculated by satisfying Eq. (4). Through a dynamic programming, the required flange areas are then reduced to the number corresponding to the specified number of curtailments.
4. Using the properties of the designed arch in step 3, the structure is reanalyzed and steps 2-3 are repeated to seek a convergence in calculation of forces. As the convergence is rapid, only two or three cycles of structural reanalysis and redesign are needed. The total weight of the arch for the final design is calculated.
5. The optimum rise is then searched iteratively by gradually decreasing the value of h in small steps and repeating steps (1-4), until the minimum weight of arch is attained. This least weight design corresponds to the chosen value of t_w.
6. For the global minimum design, the web thickness t_w is then increased to the next available plate

thickness and steps 1-5 are repeated to obtain a new design corresponding to the new web thickness. If this design yields lower weight than the previous design, the iteration is continued with increased t_w until the least weight is reached. The final design produces the optimum value of h, t_w, the web profile and the length and area of the flange corresponding to the prescribed number of curtailment.

A generalized computer program POPSAR (Mohdaly 1997) is written based on the proposed algorithm to readily obtain the optimal design of a symmetrical parabolic arch in accordance with the ASD method of the AISC (1989) specification. The algorithm yields practical design and as the search follows a unidirectional path, the minimum weight design for a chosen t_w is reached in a descending path without a local minimum.

5. Results

Mohdaly (1997) has presented solutions of several arch problems using uniformly distributed and concentrated loadings for both fixed and pin-ended arches. Three hypothetical cases of arch design are presented here to demonstrate the applicability of the proposed method and to highlight some interesting observations pertaining to design.

Case 1: *A hinged arch subjected to uniformly distributed loading of full length* (Fig. 2)

The following data is assumed: span L = 100 m, h is limited by $10 \leq h \leq 80$ m; w = 100 kN/m; web thickness, $15 \leq t_w \leq 25$ mm; modulus of elasticity of steel = 200 GPa, yield stress = 248 MPa, allowable compressive stress F_a = 100 MPa, allowable bending stress F_b = 150 MPa and the maximum width-thickness ratio, d/t_w = 100.

The weight of the arch, nondimensionalized as W_i/W_o where W_i is the weight of arch at h_i and W_o is the weight corresponding to the optimum ratio of h/L for the least weight design, is plotted against h/L in Fig. 3 for t_w = 15 mm to show the variation in the arch weight. The plot shows that the minimum weight for this case is attained at a ratio of h/L = 0.41 for t_w = 15 mm.

The arch in this example is subjected to pure compression only and, for this case, Prager and

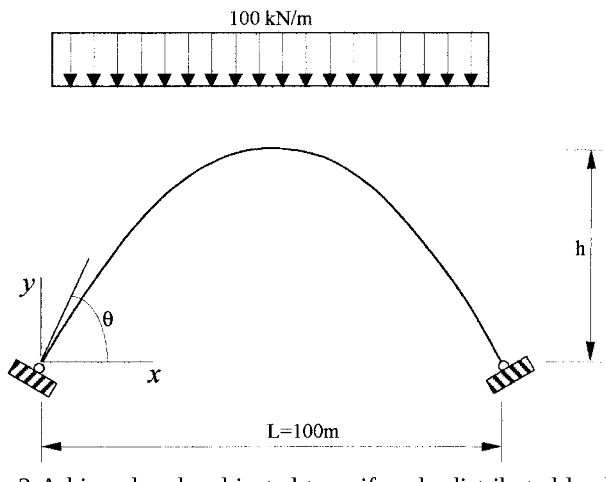

Fig. 2 A hinged arch subjected to uniformly distributed loading

Rozvany (1979) have mathematically derived an optimum ratio of h/L = 0.433. The proposed numerical technique for the optimum design yields almost the exact value of h/L. The plot also reveals that a parabolic arch having h/L ratios in the range of 0.35-0.45 is acceptable for an economical design, as the weight is insensitive in this region.

The required depth profile from Eqs. (7) and (14) in this case automatically yields a smooth curve (Fig. 3), as the arch is acted upon by only axial compression which is given as $H/\cos\,\theta$ where H is the horizontal thrust and θ is the angle between the tangent of the arch and the horizontal x-axis. However, if a parabolic depth profile is chosen instead of the exact trigonometric shape of Fig. 3, the resulting weight increases only by about 4%.

In order to verify the independency of the optimum h/L ratio on the span L, several designs were performed varying spans from 50 m to 200 m. The variations of the total weight of the arch shown in Fig. 4 show that the optimum ratio h/L is indeed a constant at around 0.41. Fig. 4 also shows that the total arch weight is relatively insensitive to h/L values within the range of 0.3-0.5 for short span arches. As the span increases, the weight function depicts a more well defined minimum weight and hence the optimum h/L.

Case 2: *A hinged arch subjected to uniformly distributed loading on the middle third span* (Fig. 5)

Using the same structural data of Case 1 and four flange plate cut-offs for this example, the plot of total weight versus h/L ratio is shown in Fig. 6. The optimum ratio of h/L in this case is 0.207 for t_w = 18 mm. The reason for this sharp decrease in optimum h/L from Case 1 is that the arch is

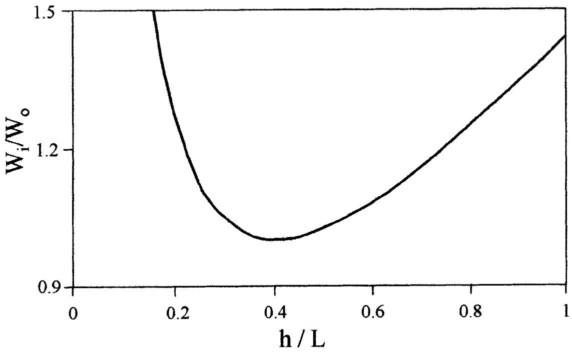

Fig. 3 Plot of arch weight versus h/L ratio for Case 1

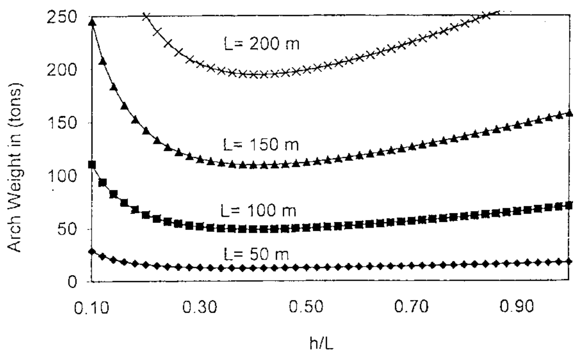

Fig. 4 Arch weight for different spans for Case 1

now subjected to bending moment in addition to axial compression. To show the variation of the optimum weight with the web thickness t_w, Fig. 7 is plotted for t_w = 8 to 30 mm. Results show that the thickness of the unstiffened web has a profound impact on the total weight and it must be searched carefully to find the optimum design.

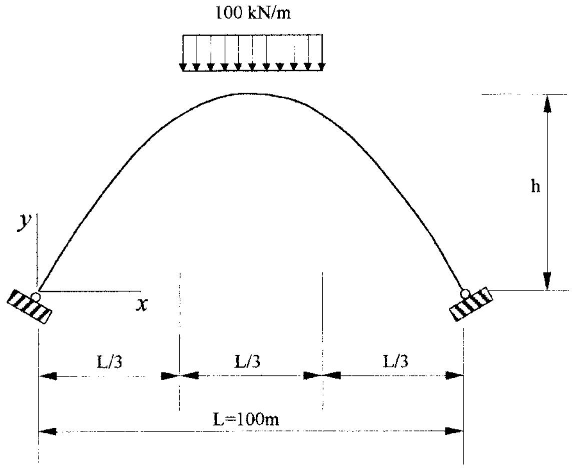

Fig. 5 A hinged arch subjected to uniformly distributed loading over the middle-third span

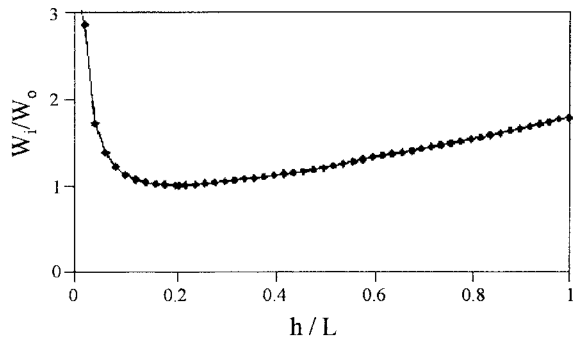

Fig. 6 Plot of arch weight versus h/L ratio for Case 2

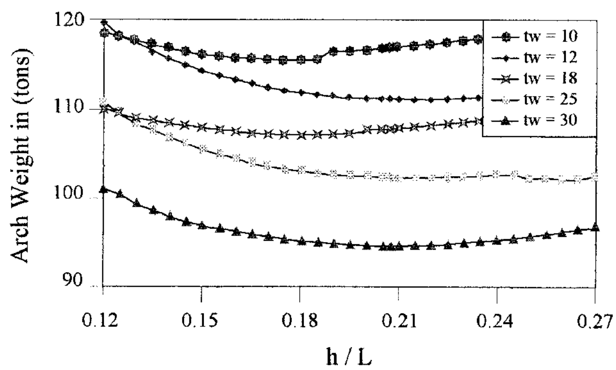

Fig. 7 Variation of arch weight for different values of t_w (t_w in mm)

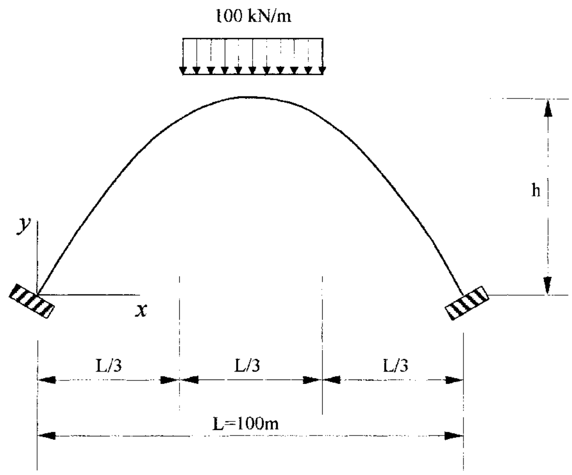

Fig. 8 A fixed arch subjected to uniformly distributed loading over the middle-third span

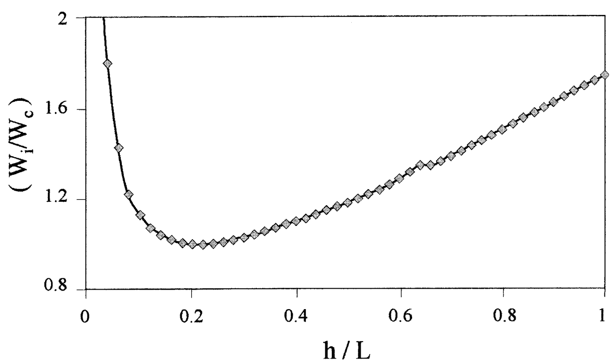

Fig. 9 Plot of arch weight versus *h*/*L* ratio for Case 3

For Case 2, different spans and different local intensities other than $w = 100$ kN/m were investigated to observe their influence on optimum *h*/*L* ratio. The results (Mohdaly 1997) show that the optimum *h*/*L* ratio remains almost unchanged, making it practically invariant with the span and the load intensity.

Case 3: *A fixed arch subjected to uniformly distributed loading on the middle third span* (Fig. 8)
Using the same data of Case 2, the nondimensionalized total weight W_i/W_o of arch is plotted as a function of *h*/*L* in Fig. 9. The optimum *h*/*L* ratio in this case equals 0.215 with t_w = 18 mm. Fig. 10 shows the variation of the arch weight with *h*/*L* ratio for three different span lengths, L = 50 m, 100 m and 150 m to show the influence of L on the optimum *h*/*L* value. Results show that unlike Case 2, the optimum *h*/*L* marginally varies and lies within the range of 0.17 to 0.215.

The optimum web depth profile captured in Fig. 11 by plotting d_i/d_c, when d_i is the depth at node i and d_c is the depth at the midspan, shows that the web has a much flatter profile. As the bending moment diagram for this case has multiple inflection points, fluctuating from positive to negative, the required depth d_i becomes highly irregular. The smoothening of the profile results in a much flatter shape than those encountered in arches with both ends hinged.

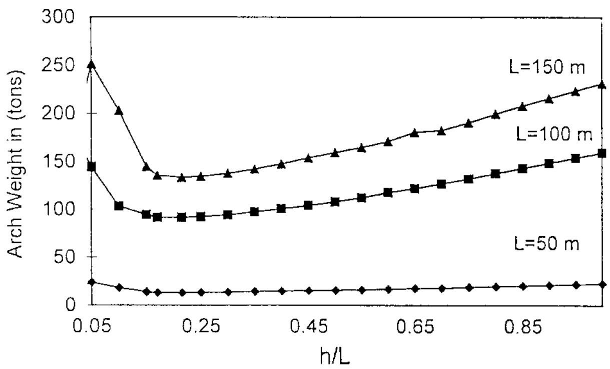

Fig. 10 Arch weight for different spans for Case 3

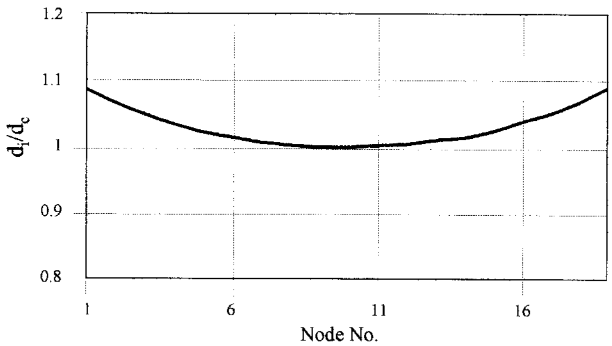

Fig. 11 Plot of optimum depth profile

The study reveals two interesting observations which are of great interest:

(a) The weight of a parabolic arch is relatively insensitive in the vicinity of the optimum *h*/*L*, which permits the use of a broader range of *h*/*L* for an economical design;

(b) As the loading on a parabolic arch departs from the case of a uniformly distributed load over the entire length *L*, it produces bending of the arch and consequently the optimum *h*/*L* is always less than 0.41. It can therefore be concluded that the case of uniformly distributed loading over *L* which produces uniform compression in the arch represents the extreme case of the optimum value of *h*/*L* which is largest at 0.41. Any other loading on the arch (partially distributed or concentrated loading) would yield the optimum *h*/*L* value less than 0.41. In other words, the optimum value of *h*/*L* decreases as the arch is subjected to bending moment.

6. Conclusions

An optimization procedure has been prescribed for the minimum weight design of symmetrical parabolic steel box arches subjected to arbitrary loading. The sequential search method determines the optimal geometrical configuration of the arch within the feasible design space by seeking a smooth parabolic, unstiffened web of variable depth and the optimum lengths and the area of flange plates corresponding to the prescribed number of flange curtailments. Based on this study, the following conclusions are drawn with regard to symmetrical box-shaped parabolic arches:

1. The design variables which significantly affect the arch weight are the ratio h/L and the web plate thickness.
2. For a parabolic arch subjected to uniformly distributed loading over the full length L (the case of pure compression in arch), the optimal h/L ratio is in the vicinity of 0.41. The weight of arch is insensitive to h/L in the neighbourhood of this optimum, and this permits the use of a broader range of h/L, say 0.35 to .50, for an economical design.
3. For any loading other than the uniformly distributed load over the whole span, the optimum h/L is less than 0.41.
4. For an arch with both ends pinned subjected to a uniformly distributed loading over a central length εL ($\varepsilon \leq 1$), the optimum h/L is invariant to the load intensity and span L. However, for a fixed-end arch, the same is not true.

Acknowledgements

The support of the department of Civil Engineering at the King Fahd University of Petroleum and Minerals for this work is acknowledged.

References

Ang, B., Teo, K. and Wang, C. (1988), “Optimal shape of arches under bending and axial compression”, *J. Eng. Mech., ASCE,* **114**(5), May, 898-905.

Azad, A.K. (1978), “Economic design of homogeneous I-beams”, *J. Struct. Div., ASCE,* **104**, 637-648.

Budiansky, B., Franenthal, J.C. and Hutchison, J.W. (1969), “On optimal arches”, *J. Appl. Mech., Am. Soc. of Mech. Engrs.,* 880-882.

Ermopoulos, I. and Ioannidis, S. (1985), “Optimum rise design of steel arch bridges”, *J. Constr. Steel Res.,* **5**, 303-310.

Farshad, M. (1976), “On optimal form of arches”, *J. Franklin Inst.,* 187-194.

Gallagher, R.H. and Zienkiewicz, O.C. (1973), *Optimum Structural Design,* John Wiley & Sons, London.

Hill, R.D., Rozvany, G., Wang, C.M. and Hwa, L.K. (1979), “Optimization, spanning capacity and cost sensitivity of fully stressed arches”, *J. Struct. Mech.,* **7**, 375-411.

Kirsch, U. (1981), *Optimum Structural Design,* McGraw Hill, New York.

Lipson, S. and Haque, M. (1980), “Optimal design of arch using complex method”, *J. Struct. Div., ASCE,* **106**, Dec., 2509-2524.

Manual of Steel Construction - Allowable Stress Design (1989), Am. Inst. of Steel Constr., Chicago, 9th Edn.

Mohdaly, H.M.M. (1997), “Optimization of parabolic steel box arches”, M.S. Thesis, Dept. of Civil Eng., King Fahd Univ. of Petroleum and Minerals, Dhahran, Jan.

Prager, W. and Rozvany, G. (1979), “A new class of structural optimization problems: Optimal archgrids”, *Comp. Methods in Appl. Mech. and Eng.,* **19**(1), 127-150.

Rozvany, G., Wang, C. M. and Dow, M. (1980), “Arch optimization using Prager-Shield criteria”, *J. Eng. Mech. Div., ASCE,* **106**(12), Dec., 1279-1286.

Rozvany, G. and Karihaloo, B. (1988), *Structural Optimization,* KLUNER Academic Publishers, Netherlands.

Taha, H.A. (1976), *Operations Research: An Introduction,* Macmillan Publishing Co. Inc., New York.

Yao, T. and Choi, K. (1989), “Shape optimal design of an arch dam”, *J. Struct. Eng., ASCE,* **115**(9), 375-411.

Zhu, B., Rao, B., Jia, J. and Li, Y. (1992), “Shape optimization of arch dams for static and dynamic loads”, *J. Struct. Eng., ASCE,* **118**, Nov., 2996-3915.