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Formulation and evaluation of incompatible but convergent rational quadrilateral membrane elements

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Abstract. This paper presents four incompatible but convergent Rational quadrilateral elements, two four-node elements (RQ4Z and RQ4B) and two five-node elements (RQ5Z and RQ5B). The difference between the so-called Rational Finite Element (Zhong and Zeng 1996) and the Free Formulation (Bergan and Nygard 1984) are discussed and compared. The importance of the mode completeness in these formulations is emphasized. Numerical results for several benchmark problems show the good performance of these elements. The two five-nodes elements RQ5Z and RQ5B, which can be viewed as complete quadratic mode elements (with seven stress modes), always give better results than the four nodes elements RQ4Z and RQ4B.

Key words: finite elements; quadrilateral; patch-tests; basic solutions of elasticity; rational elements.

1. Introduction

The first quadrilateral isoparametric plane element (Q4) appears to be the so-called Taig (Taig and Kerr 1964) element, which is a compatible displacement element. It has been proved to be overstiff because of the poor representation of shear deformation modes in the element, especially for bending dominated problems. Since 1965, many efforts have been made to improve the performance of the Taig element, using various variational principles and incompatible displacement fields. An important contribution was done by Wilson *et al*. (1973) who introduced two higher order incompatible modes to modify the Q4 element. Taylor *et al*. (1976) proposed a modification of this element in order to pass the patch-tests when the meshes are distorted. The advantages of non-conforming but convergent elements were demonstrated. The first hybrid stress element was proposed by Pian (1964). Bergan and Hanssen (1976) presented the Individual Element Test (IET) approach where the Irons (Irons and Razzaque 1972) patch-test conditions were considered to formulate the elements. Bergan and Nygard (1984) extended the IET approach which resulted in the Free Formulation (FF). Park and Stanley (1986) described and evaluated the Assumed Natural Strain (ANS) formulation.

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Tang *et al.* (1984) proposed quasi-conforming elements. Pian *et al.* (1982, 1984) presented a new hybrid stress element formulation. Simo *et al.* (1990) proposed the Enhanced Assumed Strain (EAS) elements on the basis of generalized variational principle. Chen and Cheung (1987) presented another mixed variational principle on the basis of the Pian *et al.* (1982, 1984) theory and proposed so-called refined hybrid stress elements where the approximations of stress and strain were chosen to satisfy an orthogonal condition. In this way the classical hybrid stress elements were simplified. An accurate four node element called QE2 (Piltner and Taylor 1995) was formulated using a modified Hu-Washizu formulation with bilinear displacement interpolations, seven stress/strain parameters in Cartesian coordinates satisfying the equilibrium equations and two enhanced strain modes. The element is slightly more precise than the 5 stress parameters element of Pian and Sumihara, of the mixed element of Simo and Rifai and the modified incompatible displacement element of Wilson *et al.* The element QE2 is revisited and further modified (as $\overline{\mathrm{B}}$-QE4) in a recent paper by the same authors (Piltner and Taylor 1999). The QE2 element can be considered as a hybrid-Trefftz type element (Jirousek 1978, Jirousek and Teodorescu 1982, Piltner and Taylor 1999) since the stress functions are chosen such that the homogeneous equilibrium equations are satisfied a priori. Zhao *et al*. (1997) described another generalized formulation and the identification to hybrid stress element has been proved under some specific conditions. Zhong *et al.* (1996) have recently proposed another quadrilateral element called Rational Finite Element (here named as RQ4Z). In that formulation the displacement fields *u*, *v* are approximated by a combination of eight basic solutions of plane elasticity problems instead of the classical isoparametric bilinear interpolations. With some modifications, this non-conforming quadrilateral element can pass the constant stress patch-tests and a satisfactory precision of both displacements and stresses can be observed.

In their historical paper on finite elements (1956), Turner *et al.* (1956) described a rectangular incompatible element where a quadratic displacement field (in terms of the classical eight nodal variables) is obtained by starting with a 5 parameters stress field. Equilibrium and compatibility equations of the plane stress elasticity problem are satisfied in the element which can be viewed as an hybrid Trefftz element with incompatible displacements. The elements described in this paper, RQ4Z and RQ4B, are based also on a stress field with five parameters but the elements are generalized to quadrilateral shapes and modifications are further introduced to satisfy the constant strain patch-tests.

We have observed that both the RQ4Z element and the RQ4B element do not satisfy the invariance of coordinates. In order to overcome this disadvantage, two new elements with fives nodes RQ5Z and RQ5B are formulated and tested. Numerical results show that the two elements have better performance than RQ4Z and RQ4B.

2. The rational quadrilateral element RQ4Z (Zhong and Zeng 1996)

Consider the quadrilateral element shown in Fig. 1 where *x* and *y* are the orthogonal coordinates with origin at the centroid O. We assume an isotropic material with constants Young's modulus *E* and Poisson's ratio *ν*. Then:

$$\begin{cases} A_e = \int_{A_e} dA; \quad \int_{A_e} x dA = 0; \quad \int_{A_e} y dA = 0 \\ I_x = \int_{A_e} x^2 dA; \; I_y = \int_{A_e} y^2 dA; \; I_{xy} = \int_{A_e} xy dA \end{cases} \tag{1}$$

In the formulation of the RQ4Z element, the incompatible displacement fields are approximated as:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = [N]\{\alpha_n\} \tag{2}$$

where

$$[N] = \begin{bmatrix} 1 & 0 & y & x & -\nu x & y/2 & -xy & (\nu x^2 + y^2)/2 \\ 0 & 1 & -x & -\nu y & y & x/2 & (x^2+\nu y^2)/2 & -xy \end{bmatrix} \tag{3}$$

ν is the Poisson's coefficient.

$$\langle \alpha_n \rangle = \langle \alpha_i, i = 1{,}8 \rangle \tag{4}$$

α_i, $i=1{,}8$ are the eight unknown generalized parameters.

The columns in $[N]$ are eight basic solutions of the plane stress elasticity problems in accordance with the eight basic states: three rigid body modes, three constant stress modes and two higher order in-plane bending modes for an isotropic material (i.e., $\sigma_x = E\alpha_4 - Ey\alpha_7$, $\sigma_y = E\alpha_5 - Ey\alpha_8$, $\sigma_{xy} = G\alpha_6$). As in hybrid Trefftz elements (Jirousek 1978, Jirousek and Teodorescu 1982, Piltner and Taylor 1995 and 1999) the bases functions satisfy the homogeneous (Navier) equilibrium equations for plane stress isotropic elasticity.

The membrane strain field is defined by:

$$\{\varepsilon\} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \end{Bmatrix} = [B_\alpha]\{\alpha_n\} \tag{5}$$

$$[B_\alpha] = \begin{bmatrix} 0 & 0 & 0 & 1 & -\nu & 0 & -y & \nu x \\ 0 & 0 & 0 & -\nu & 1 & 0 & \nu y & -x \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{6}$$

Substituting the coordinate values of the four nodes of the element x_i, y_i, $i=1{,}4$ into (2), the eight generalized parameters α_i, $i=1{,}8$ can be expressed by eight nodal displacement variables u_i, v_i, $i=1{,}4$.

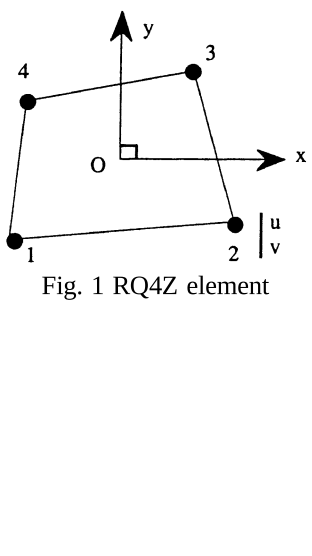

Fig. 1 RQ4Z element

$$\{u_n\} = [T]\{\alpha_n\} \tag{7}$$

$$\langle u_n \rangle = \langle u_i, v_i, i = 1,4 \rangle \tag{8}$$

The matrix $[T]$ is:

$$[T] = \begin{bmatrix} [N]_{(x_1, y_1)} \\ [N]_{(x_2, y_2)} \\ [N]_{(x_3, y_3)} \\ [N]_{(x_4, y_4)} \end{bmatrix}$$

With

$$[N]_{(x_i, y_i)} = \begin{bmatrix} 1 & 0 & y_i & x_i & -\nu x_i \dfrac{y_i}{2} & -x_i y_i & \dfrac{(\nu x_i^2 + y_i^2)}{2} \\ 0 & 1 & -x_i & -\nu y_i & y_i \dfrac{x_i}{2} & \dfrac{(x_i^2 + \nu y_i^2)}{2} & -x_i y_i \end{bmatrix}$$

If $[T]$ is not a singular matrix, we have:

$$\{\alpha_n\} = [T]^{-1}\{u_n\} \tag{9}$$

Eq. (5) can then be written as:

$$\{\varepsilon\} = [B_\alpha][T]^{-1}\{u_n\} \tag{10}$$

The plane stress field is described by:

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \end{Bmatrix} = [H]\{\varepsilon\} \tag{11}$$

with

$$[H] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} \tag{12}$$

The internal energy is given by:

$$U^e = \frac{1}{2}\int_{A_e} \langle \varepsilon \rangle [H_m]\{\varepsilon\} dA = \frac{1}{2}\langle \alpha_n \rangle [k_\alpha]\{\alpha_n\} = \frac{1}{2}\langle u_n \rangle [T]^{-T}[k_\alpha][T]^{-1}\{u_n\} \tag{13}$$

where

$$[k_\alpha] = \int_{A_e} [B_\alpha]^T [H_m][B_\alpha] dA \tag{14}$$

with:

$$[H_m] = h[H] \tag{15}$$

$[k_\alpha]$ is the generalized stiffness matrix of the element corresponding to the generalized parameters α_i, $i = 1,8$. Exact integration is performed analytically using Eq. (1).

The stiffness matrix is defined as:

$$[k]^e = [T]^{-T}[k_\alpha][T]^{-1} \tag{16}$$

Zhong *et al.* (1996) pointed out that the elements using Eq. (16) for the stiffness matrix can not pass the patch-tests. Therefore some modifications must be carried out. The three constant stress states $\sigma_x = 1$, $\sigma_y = 1$ and $\sigma_{xy} = 1$ correspond to $\alpha_4 = 1/E$, $\alpha_5 = 1/E$ and $\alpha_6 = 1/G$ respectively. The vectors of internal generalized forces associated with these three states are $\{f_{\alpha x}\}$, $\{f_{\alpha y}\}$ and $\{f_{\alpha xy}\}$:

$$\langle f_{\alpha x}\rangle = \langle\ 0\ 0\ 0\ A_e\ -\upsilon A_e\ 0\ 0\ 0\ \rangle$$

$$\langle f_{\alpha y}\rangle = \langle\ 0\ 0\ 0\ -\upsilon A_e\ A_e\ 0\ 0\ 0\ \rangle$$

$$\langle f_{\alpha xy}\rangle = \langle\ 0\ 0\ 0\ 0\ 0\ A_e\ 0\ 0\ \rangle$$

we define the two matrices $[f_\alpha]$ and $[\alpha_c]$ as:

$$[f_\alpha] = [\{f_{\alpha x}\}\ \{f_{\alpha y}\}\ \{f_{\alpha xy}\}] \tag{17}$$

$$[\alpha_c]^T = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{E} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G} & 0 & 0 \end{bmatrix} \tag{18}$$

They are such that:

$$[f_\alpha] = [k_\alpha][\alpha_c] \tag{19}$$

On the other hand, the exact internal force vectors (corresponding to $\sigma_x = 1$, $\sigma_y = 1$ and $\sigma_{xy} = 1$) defined as $\{f_x\}$, $\{f_y\}$ and $\{f_{xy}\}$ can be easily obtained by assemblage of the edge traction on the sides. We define:

$$[f_{xy}] = [\{f_x\}\ \{f_y\}\ \{f_{xy}\}] \tag{20}$$

where:

$$\langle f_x\rangle = \frac{1}{2}\langle\ y_{24}\ 0\ y_{31}\ 0\ y_{42}\ 0\ y_{13}\ 0\ \rangle \tag{21}$$

$$\langle f_y\rangle = \frac{1}{2}\langle\ 0\ x_{42}\ 0\ x_{23}\ 0\ x_{24}\ 0\ x_{31}\ \rangle \tag{22}$$

$$\langle f_{xy}\rangle = \frac{1}{2}\langle\ x_{42}\ y_{24}\ x_{13}\ y_{31}\ x_{24}\ y_{42}\ x_{31}\ y_{13}\ \rangle \tag{23}$$

with $x_{ij} = x_j - x_i$, $y_{ij} = y_j - y_i$.

In order to pass the constant patch-tests, the following condition must be verified:

$$[f_\alpha] = [T]^T[f_{xy}] \tag{24}$$

However it can be observed that this is not the case. Therefore Zhong *et al.* (1996) proposed a so-called orthogonal procedure to determine a modified matrix $[T_r]$ that satisfies Eq. (24) (see Appendix A). By this way, the stiffness matrix of the RQ4Z element can be expressed as:

$$[k]^e = [T_r]^{-T}[k_\alpha][T_r]^{-1} \tag{25}$$

Another, yet simple procedure, leading to the stiffness matrix of the RQ4B element is proposed here by considering a superposition of the constant modes and the higher order modes.
Eq. (6) can be decomposed into two following parts:

$$[B_\alpha] = [B_\alpha]_c + [B_\alpha]_h \tag{26}$$

where:

$$[B_\alpha]_c = \begin{bmatrix} 0 & 0 & 0 & 1 & -\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \tag{27}$$

$$[B_\alpha]_h = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -y & \nu x \\ 0 & 0 & 0 & 0 & 0 & 0 & \nu y & -x \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{28}$$

Substituting (26) into (14), we find that the generalized stiffness matrix $[k_\alpha]$ can be expressed in two parts:

$$[k_\alpha] = [k_\alpha]_c + [k_\alpha]_h \tag{29}$$

with

$$[k_\alpha]_c = \int_{A_e}[B_\alpha]_c^T[H_m][B_\alpha]_c dA = A_e[B_\alpha]_c^T[H_m][B_\alpha]_c \tag{30}$$

$$[k_\alpha]_h = \int_{A_e}[B_\alpha]_h^T[H_m][B_\alpha]_h dA \tag{31}$$

$$[k_\alpha]_c = h\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & EA_e & -\nu EA_e & 0 & 0 & 0 \\ 0 & 0 & 0 & -\nu EA_e & EA_e & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & GA_e & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{32}$$

$$[k_\alpha]_h = h\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & EI_x & -\nu EI_{xy} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\nu EI_{xy} & EI_y \end{bmatrix} \tag{33}$$

Considering (19) and (24), we have:

$$[K_\alpha][\alpha_c] = [T]^T[f_{xy}] \tag{34}$$

$$[K_\alpha]_c[\alpha_c] + [K_\alpha]_h[\alpha_c] = [T]^T[f_{xy}] \tag{35}$$

From (33) and (18), we can observe that:

$$[K_\alpha]_h[\alpha_c] = 0 \tag{36}$$

Therefore Eq. (35) reduces to:

$$[K_\alpha]_c[\alpha_c] = [T]^T[f_{xy}] \tag{37}$$

Using (18) and (30), we can finally obtain:

$$A_e[B_\alpha]_c^T = [T]^T[f_{xy}] \tag{38}$$

From (38), we can see that there are two possibilities to satisfy the constant stress patch tests:

- we can modify $[T]^T$ as proposed by Zhong *et al.* (1996) (element RQ4Z)
- we can modify $[B_\alpha]_c$ as proposed in this paper (element RQ4B) by replacing $[B_\alpha]_c^T$ by $1/A_e([T]^T[f_{xy}])$.

The explicit expression of the stiffness matrix of the RQ4B element is then expressed as:

$$[k]^e = \frac{1}{A_e}[f_{xy}][H_m][f_{xy}]^T + [T]^{-T}[k_\alpha]_h[T]^{-1} \tag{39}$$

The procedure is an application of the Free Formulation described by Bergan and Nygard (1984).

3. Stress recovery of RQ4Z and RQ4B

For the element RQ4Z, the strains are calculated by:

$$\{\varepsilon\} = ([B_\alpha]_c + [B_\alpha]_h)[T_r]^{-1}\{u_n\} \tag{40}$$

so, the stresses are:

$$\{\sigma\} = [H]\{\varepsilon\} = [H]([B_\alpha]_c + [B_\alpha]_h)[T_r]^{-1}\{u_n\} \tag{41}$$

For the element RQ4B, the strains and stresses are recovered by:

$$\{\varepsilon\} = \left(\frac{1}{A_e}[f_{xy}]^T + [B_\alpha]_h[T]^{-1}\right)\{u_n\} \tag{42}$$

$$\{\sigma\} = \left(\frac{1}{A_e}[H][f_{xy}]^T + [H][B_\alpha]_h[T]^{-1}\right)\{u_n\} \tag{43}$$

4. The rational quadrilateral elements of five-nodes RQ5Z and RQ5B

In the formulation of RQ4Z and RQ4B, all quadratic terms x^2, xy, y^2 appeared in the approximation of displacements (3), but they lead to a constant value of σ_{xy}. Therefore the results depend on the choice of the local coordinate system x and y. An improvement of precision and an invariance of results with respect to the orientations of x and y axes are obtained by considering a 10 term parameters displacement field. Two quadrilateral elements called RQ5Z and RQ5B with five nodes (Fig. 2) are formulated by using a complete quadratic displacement field leading to a complete linear strain/stress field (with seven terms).

For RQ5Z and RQ5B elements, the approximation of displacements is expressed as:

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = [\bar{N}]\{\alpha_n\}; \qquad \langle \alpha_n \rangle = \langle \alpha_i, i = 1, 10 \rangle \tag{44}$$

where

$$[\bar{N}] = \left[[N] \vdots \begin{matrix} xy & (-x^2+y^2)/2 \\ (x^2-y^2)/2 & xy \end{matrix} \right] \tag{45}$$

leading to

$$[\bar{B}_\alpha] = \left[[B_\alpha] \begin{matrix} \vdots & y & -x \\ \vdots & -y & x \\ \vdots & 2x & 2y \end{matrix} \right] \tag{46}$$

$[N]$ is defined by Eq. (3) and $[B_\alpha]$ by Eq. (6); α_i, $i = 1,10$ are the ten unknown generalized parameters.

The exact internal force vectors $\{\bar{f}_x\}$, $\{\bar{f}_y\}$ and $\{\bar{f}_{xy}\}$ are:

$$[\bar{f}_{xy}] = [\{\bar{f}_x\}\ \{\bar{f}_y\}\ \{\bar{f}_{xy}\}] \tag{47}$$

where:

$$\langle \bar{f}_x \rangle = \frac{1}{2} \langle\ y_{24}\ 0\ y_{31}\ 0\ y_{42}\ 0\ y_{13}\ 0\ 0\ 0\ \rangle \tag{48}$$

$$\langle \bar{f}_y \rangle = \frac{1}{2} \langle\ 0\ x_{42}\ 0\ x_{23}\ 0\ x_{24}\ 0\ x_{31}\ 0\ 0\ \rangle \tag{49}$$

$$\langle \bar{f}_{xy} \rangle = \frac{1}{2} \langle\ x_{42}\ y_{24}\ x_{13}\ y_{31}\ x_{24}\ y_{42}\ x_{31}\ y_{13}\ 0\ 0\ \rangle \tag{50}$$

The explicit expression of the stiffness matrix of the RQ5Z and RQ5B elements are derived in a

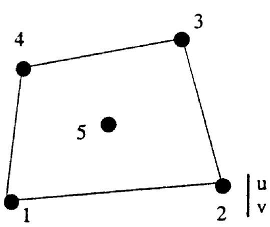

Fig. 2 RQ5Z and RQ5B elements

similar manner as for RQ4Z and RQ4B.

For RQ5Z element:

$$[k]^e = [\overline{T}_r]^{-T}[\overline{k_\alpha}][\overline{T}_r]^{-1} \tag{51}$$

$[\bar{k}_\alpha]$ is the generalized stiffness matrix of the element corresponding to the generalized parameters α_i, $i = 1,10$.

For RQ5B element:

$$[k]^e = \frac{1}{A_e}[\bar{f}_{xy}][H_m][\bar{f}_{xy}]^T + [\bar{T}]^{-T}[\bar{k}_\alpha]_h[\bar{T}]^{-1} \tag{52}$$

Where $[\bar{k}_\alpha]$, $[\bar{k}_\alpha]_h$ and $[\bar{T}]$ are given in Appendix B. Exact integration is performed using Eq. (1). The two elements RQ5Z and RQ5B can both pass the patch-tests and the results are coordinates independent.

5. Numerical results

Numerical results for classical benchmark problems are presented in this section. They are obtained using the following elements:

RQ4B : the rational quadrilateral four-node element-Batoz;
RQ4Z : the rational quadrilateral four-node element-Zhong;
RQ5Z : the rational quadrilateral five-node element-Zhong;
RQ5B : the rational quadrilateral five-node element-Batoz;
Q4 : the standard four-node isoparametric element using a 2×2 Gauss integration scheme;
Q4WT : the modified Wilson non-conforming four-node element (or QM6) (Taylor, Beresford and Wilson 1976 Batoz, and Dhatt 1992);
Q4PS : the Pian and Sumihara's four-node five-beta hybrid mixed element (or PS5β) (Pian and Sumihara 1985, Batoz, and Dhatt 1992).

5.1. The patch tests

Patch tests as described in Fig. 3 are considered using the stiffness matrices defined by Eqs. (25), (39), (51) and (52). The exact solutions for $\sigma_x = 1$, $\sigma_y = 1$ or $\sigma_{xy} = 1$ are always obtained. That is, the four incompatible elements can pass the constant stress patch tests for any distortion. The rank is obviously also correct (no spurious modes).

5.2. Cantilever beam under pure bending or end shear

The cantilever beam problem with dimension 10×2×1 shown in Fig. 4 and modelled by 1×1 regular, 1×5 regular and 1×5 irregular meshes is analysed here. The computed vertical displacement v_A at point A and the bending stress σ_{XB} are normalized and listed in Tables 1 and 2 for the two load cases. The stresses at point B are computed directly using formulas such as Eqs. (41) and (43). The aspect ratio is equal to 5 (thick beam).

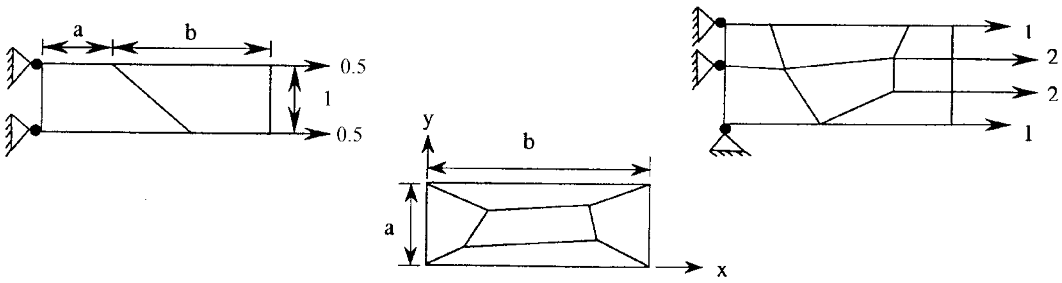

Fig. 3 Meshes for patch test

For both load cases (pure bending), we observe that for regular meshes, all elements except the isoparametric bilinear Q4 displacement model give the same results and the exact ones for the load case (1). For the non regular mesh the same results are obtained for the displacement v_A for RQ4B and RQ4Z and then for RQ5B and RQ5Z. These results are far superior to the results using Q4. The efficiency is in the order of Q4PS (best), Q4WT, RQ5B or Z, RQ4B or Z, Q4. For the evaluation of σ_{XB}, better results are obtained using the RQ4B and RQ5B compared to RQ4Z and RQ5Z. The efficiency is in the order of RQ5B = Q4PS, RQ4B, RQ5Z, Q4WT, RQ4Z, Q4 for the load case (1) and in the order of RQ4B, RQ5B = Q4PS, RQ5Z, Q4WT, RQ4Z, Q4 for the load case (2).

5.3. Two elements cantilever beam

A cantilever beam subjected to end bending is shown in Fig. 5, and modelled with two quadrilateral elements.

For the mesh with two rectangular elements ($e = 0$), we have the same remarks as for the previous example (load case (1)). The quality of results decreases with the distorsion for all elements. The

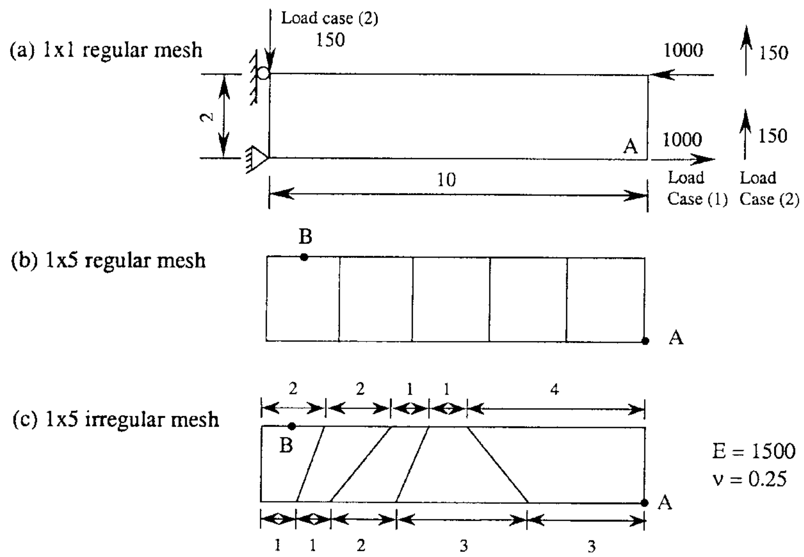

Fig. 4 Cantilever beam subjected to (1) pure bending and (2) end shear

Table 1 The results of V_A and σ_{XB} for load case (1)

Element	1×1 regular		1×5 regular		1×5 non regular	
	V_A	σ_{XB}	V_A	σ_{XB}	V_A	σ_{XB}
RQ4B(Eq. 39 and 43)	100(1.00)	-3000(1.00)	100(1.00)	-3000(1.00)	77.132(0.771)	-2962.7(0.988)
RQ4Z(Eq. 25 and 41)	100(1.00)	-3000(1.00)	100(1.00)	-3000(1.00)	77.132(0.771)	-2490.5(0.830)
RQ5B	100(1.00)	-3000(1.00)	100(1.00)	-3000(1.00)	78.575(0.786)	-3014.4(1.004)
RQ5Z	100(1.00)	-3000(1.00)	100(1.00)	-3000(1.00)	78.575(0.786)	-2538.4(0.846)
Q4WT	100(1.00)	-3000(1.00)	100(1.00)	-3000(1.00)	96.067(0.960)	-2513(0.837)
Q4PS	100(1.00)	-3000(1.00)	100(1.00)	-3000(1.00)	96.180(0.961)	-3014(1.004)
Q4	9.036(0.09)	289.2(0.096)	68.18(0.681)	-2182(0.727)	45.650(0.456)	-1762(0.587)
Theory	**100**	**-3000**	**100**	**-3000**	**100**	**-3000**

Table 2 The results of v_A and σ_{XB} for load case (2)

Element	1×1 regular		1×5 regular		1×5 non regular	
	V_A	σ_{XB}	V_A	σ_{XB}	V_A	σ_{XB}
RQ4B(Eq. 39 and 43)	77.5(0.756)	-2250(0.555)	101.5(0.989)	-4050(1.00)	78.972(0.769)	-4072.5(1.005)
RQ4Z(Eq. 25 and 41)	77.5(0.756)	-2250(0.555)	101.5(0.989)	-4050(1.00)	78.972(0.769)	-3417.5(0.844)
RQ5B	77.5(0.756)	-2250(0.555)	101.5(0.989)	-4050(1.00)	80.164(0.781)	-4134.4(1.020)
RQ5Z	77.5(0.756)	-2250(0.555)	101.5(0.989)	-4050(1.00)	80.164(0.781)	-3475.4(0.858)
Q4WT	77.5(0.756)	-2250(0.555)	101.5(0.989)	-4050(1.00)	98.120(0.981)	-3442(1.147)
Q4PS	77.5(0.756)	-2250(0.555)	101.5(0.989)	-4050(1.00)	98.188(0.957)	-4137(1.021)
Q4	9.277(0.09)	-216.9(0.053)	70.0(0.682)	-2945(0.727)	50.682(0.494)	-2448(0.604)
Theory	**102.6**	**-4050**	**102.6**	**-4050**	**102.6**	**-4050**

results using Q4 are very inaccurate. Again RQ4B and Z or RQ5B or Z give the same results. The efficiency is in the order of Q4WT, Q4PS, RQ5B or Z, RQ4B or Z, and Q4.

5.4. Tapered panel under end shear

A tapered panel of unit thickness with one edge subjected to a distributed shear load and with the other edge fully clamped ($u = v = 0$) is shown in Fig. 6. The panel is analysed using 2×2 and 4x4 meshes. The normalized vertical deflection v_C, maximum principal stress $\sigma_{A(\max)}$ at point A and minimum principal stress $\sigma_{B(\min)}$ are presented in table 4. Principal stresses at point A and B are

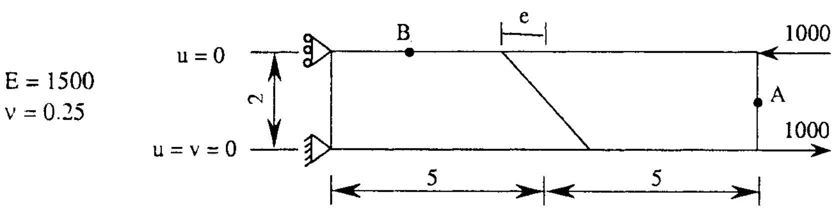

Fig. 5 Cantilever beam subjected to end bending (u = 0 at the sliding node)

Table 3 The deflection v_A and the stress at point B (σ_{XB})

Element	$e=0$		$e=1$		$e=2$		$e=3$	
	V_A	σ_{XB}	V_A	σ_{XB}	V_A	σ_{XB}	V_A	σ_{XB}
RQ4B (Eq. 39 and 43)	100 (1.00)	-3000 (1.00)	54.137 (0.541)	-1750.7 (0.584)	38.641 (0.386)	-1695.8 (0.565)	27.454 0.274)	-1464.9 (0.488)
RQ4 Z(Eq. 25 and 41)	100 (1.00)	-3000 (1.00)	54.137 (0.541)	-1750.7 (0.584)	38.641 (0.386)	-1695.8 (0.565)	27.454 (0.274)	-1464.9 (0.488)
RQ5B	100 (1.00)	-3000 (1.00)	60.839 (0.608)	-1928.4 (0.643)	44.990 (0.449)	-1841.7 (0.614)	30.054 (0.300)	-1558.7 (0.519)
RQ5Z	100 (1.00)	-3000 (1.00)	60.839 (0.608)	-1928.4 (0.643)	44.990 (0.449)	-1841.7 (0.614)	30.054 (0.300)	-1558.7 (0.519)
Q4WT	100 (1.00)	-3000 (1.00)	67.287 (0.673)	-2447 (0.815)	62.418 (0.624)	-3088 (1.029)	65.657 (0.656)	-4554 (1.518)
Q4PS	100 (1.00)	-3000 (1.00)	65.186 (0.651)	-2132 (0.710)	59.035 (0.590)	-2450 (0.816)	60.928 (0.609)	-2596 (0.865)
Q4	28.037 (0.280)	-897.2 (0.299)	14.286 (0.142)	-566.10 (0.188)	9.763 (0.097)	-644.5 (0.214)	8.302 (0.083)	-818.5 (0.272)
Theory	**100**	**-3000**	**100**	**-3000**	**100**	**-3000**	**100**	**-3000**

evaluated based on the averaged stress components of the elements sharing nodes A and B, respectively.

We observe that similar results are obtained for RQ4B and *Z*, or for RQ5B and *Z* with a superiority of RQ5 compared to RQ4. Better results are obtained using Q4WT and Q4PS for the displacement but not always for the stresses. The efficiency for the 4x4 mesh is in the order Q4WT, Q4PS, RQ5B or *Z*, RQ4B or *Z*, Q4 for the displacement, and in the order of RQ5B, RQ5Z, Q4PS, Q4WT, RQ4B, RQ4Z, Q4 for $\sigma_{A\max}$. It is in order of Q4PS, RQ5B, RQ5Z, Q4WT, RQ4Z , RQ4B, Q4 for $\sigma_{B\min}$.

5. Conclusions

The formulation of new quadrilateral plane (membrane) elasticity elements have been presented. These incompatible displacement elements have four or five nodes with two dof per node. They can be viewed as a generalization of the first rectangular element of Turner *et al.* (1956). Quadratic displacement fields are considered leading to an internal equilibrium stress field for isotropic

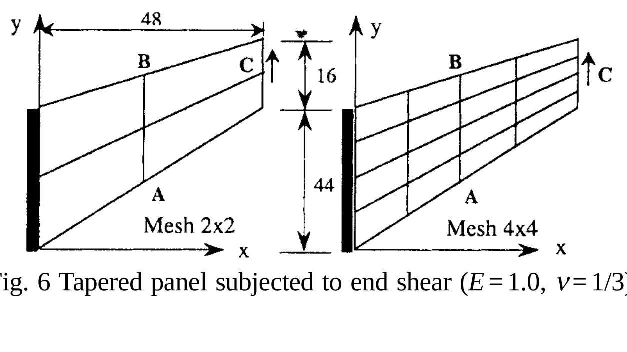

Fig. 6 Tapered panel subjected to end shear ($E = 1.0$, $\nu = 1/3$)

Table 4 The results of displacement and stresses for tapered panel under end shear

Element	2×2			4×4		
	V_C	$\sigma_{A\max}$	$\sigma_{B\min}$	V_C	$\sigma_{A\max}$	$\sigma_{B\min}$
RQ4B(Eq. 39 & 43)	14.016(0.586)	0.1270(0.583)	-0.1217(0.605)	19.944(0.835)	0.2225(0.943)	-0.1585(0.788)
RQ4Z(Eq. 25 & 41)	14.01 (0.586)	0.1253(0.531)	-0.1271(0.632)	19.944(0.835)	0.2193(0.929)	-0.1610(0.801)
RQ5B	20.460(0.856)	0.1927(0.816)	-0.1461(0.727)	22.971(0.961)	0.2265(0.960)	-0.1869(0.930)
RQ5Z	20.460(0.856)	0.1932(0.818)	-0.1435(0.714)	22.971(0.961)	0.2260(0.957)	-0.1863(0.927)
Q4PS	21.129(0.884)	0.1850(0.783)	-0.1550(0.771)	23.022(0.963)	0.2240(0.949)	-0.1940(0.965)
Q4WT	21.050(0.880)	0.177 (0.750)	-0.169 (0.841)	23.016(0.963)	0.2225(0.942)	-0.1855(0.923)
Q4	11.85 (0.496)	0.1281(0.543)	-0.0916(0.456)	18.30 (0.766)	0.1905(0.807)	-0.1510(0.751)
Value ref.	**23.90**	**0.2360**	**-0.2010**	**23.90**	**0.2360**	**-0.2010**

materials. The so-called higher order strain modes are (energy) orthogonal to the constant strain modes. Modifications are introduced to satisfy the constant strain patch-tests. The stiffness matrices can be obtained explicitly. Four elements have been presented. The five nodes version RQ5 is based on complete quadratic displacement field leading to complete linear strains. These elements are coordinate independent. The *B* version and the *Z* version differ by the procedure to satisfy the constant strain patch-tests. The *Z* version presented by Zhong and Zeng (1996) consists in a modification of the transformation matrix [*T*] between the generalized parameters of the displacement field and the nodal variables. The *B* version proposed in this paper consists in a modification of the constant strain matrix in the spirit of the Individual Element Test and Free Formulation of Bergan *et al.* (1975, 1984). The present elements RQ4B and RQ5B can be viewed as a combination of hybrid-Trefftz and Free Formulation elements. Several classical benchmark problems have been considered where the results of the new elements RQ4B, RQ4Z, RQ5B and RQ5Z are compared with the results of other displacements or hybrid-stress elements (Q4, Q4WT and Q4PS).

As expected the five node elements (RQ5B or *Z*) give in general better results than the four nodes one (RQ4B or *Z*), and the *B* version appears simple to implement and give better results for the stresses than the *Z* version. This study shows that the incompatible Q4WT and hybrid Q4PS elements remain good performers, mainly for displacements but the RQ5B element can compete for stresses estimations.

This study is not an end. Based on a similar formulation we are deriving new quadrilateral plate bending elements based on the Discrete Kirchhoff Technique (Batoz, Hammadi, Zheng and Zhong 1998).

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References

Batoz, J.L. and Dhatt. G. (1992), *Modélisation des Structures par Éléments Finis.* Poutres et Plaques, Hermès, Paris.

Batoz, J.L., Hammadi, F. , Zheng, C.L. and Zhong, W.X. (1998), "On the linear analysis of plates and shells using a new sixteen dof flat shell element", *Advances in Finite Element Procedures and Techniques*, ed by B.H.V. Topping, Civil-Comp press, Edinburgh, Scotland, UK, 31-41., also to be published in *Computers and Structures.*

Bergan, P.G. and Hanssen, L. (1975), "A new approach for deriving good finite elements", *MAFELAP II Conference*, Brunel University (1976), *The Mathematics of Finite Elements and Applications,* **II**, ed by J.R. Whiteman, Academic Press, London, 483-497.

Bergan, P.G. and Nygard, M.K. (1984), "Finite elements with increased freedom in closing shape functions", *International Journal for Numerical Methods in Engineering*, **20**, 643-664.

Chen, W. and Cheung, Y.K. (1987), "A new approach for the hybrid element method", *International Journal for Numerical Methods in Engineering*, **24**, 1697-1709.

Irons, B.M. and Razzaque, A. (1972), "Experiences with the patch test for convergence of finite elements", *The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations*, Edition A.K. Aziz, Academic press, NewYork, 551-587.

Jirousek, J. (1978), "Basis for development of large finite element locally satisfying all field equations", *Computers Methods in Applied Mechanics and Engineering*, **14**, 65-92.

Jirousek, J. and Teodorescu, P. (1982), "Large finite elements method for the solution of problems in the theory of elasticity", *Computers and Structures*, **15**, 575-587.

Park, K.C. and Stanley, G.M. (1986), "A curved C° shell element based on assumed natural-coordinate strains", *Journal of Applied Mecanics*, **53**, 51-54.

Pian, T.H.H. (1964), "Derivation of element stiffness matrices by assumed stress distributions", *AIAA J-2*, 1333-1376.

Pian, T.H.H. and Chen, D.P. (1982), "Alternative ways for formulation of hybrid stress elements", *International Journal for Numerical Methods in Engineering*, **18**, 1679-1684.

Pian, T.H.H. and Sumihara, K. (1984), "Rational approach for assumed stress finite element", *International Journal for Numerical Methods in Engineering*, **20**, 1685-1695.

Piltner, R. and Taylor, R. (1995), "A quadrilateral mixed finite element with two enhanced strain modes", *International Journal for Numerical Methods in Engineering*, **38**, 1783-1808.

Piltner, R. and Taylor, R. (1999), "A systematic construction of B-bar functions for linear and non-linear mixed-enhanced finite elements for plane elasticity problems", *International Journal for Numerical Methods in Engineering*, **44**, 615-639.

Simo, J.C. and Rifai, M.S. (1990), "A class of mixed assumed strain methods and the method of incompatible modes", *International Journal for Numerical Methods in Engineering,* **29**, 1595-1638.

Taig, I.C. and Kerr, R. (1964), "Some problems in the discrete element representation of aircraft structure", *Matrix Method of the Structural Analysis*, Pergamon Press, London, 267-315.

Tang, L.M., Chen, W.J. and Liu, Y.X. (1984), "Formulation of quasi-conforming element and Hu-Washizu principle", *Computers and Structures*, **19**, 247-250.

Taylor, R.L., Beresford, P.J. and E.L. Wilson, (1976), "A non-conforming element for stress analysis", *International Journal for Numerical Methods in Engineering,* **10**, 1211-1219.

Turner, M., Clough, R., Martin, H. and Topp, L. (1956), "Stiffness and deflection analysis of complex structures", *J. Aeronaut. Sci.*, **23**, 1805-1823.

Wilson, E.L., Taylor, R.L., Doherty, W.P. and Ghaboussi, J. (1973), "Incompatible displacement models", *Numerical and Computer Methods in Structural Mechanics*, Academic Press, New York, 43-57.

Zhao, P., Pian, T.H.H. and Sheng, T. (1997), "A new formulation of isoparametric finite elements and the relationship between hybrid stress element and incompatible element", *International Journal for Numerical Methods in Engineering*, **40**, 15-27.

Zhong, W.X. and Zeng. J. (1996), "Rational finite elements", *Journal of Computational Structural Mechanics and Application*, in Chinese, **13**, 1-8.

Appendix A

In order to pass the constant stress patch-tests the following equation should be verified by the $[B_\alpha]$ matrix:

$$[T]^{-1}\int_{A^e}([B_\alpha]_c^T + [B_\alpha]_h^T)dA = [f_{sy}] \tag{A.1}$$

were $[f_{xy}]$ is given by Eq. (20).
Since $[B_\alpha]_c$ is constant and $[B_\alpha]_h$ energy orthogonal to $[B_\alpha]_c$, the above relation leads to:

$$A^e[B_\alpha]_c^T = [T]^T[f_{xy}] \tag{A.2}$$

Using Eqs. (7), (8) and (20)~(30), we have :

$$[T]^T[f_{xy}] = \begin{bmatrix} 0 & 0 & 0 & A & -\nu A & 0 & \langle v_7\rangle\{f_x\} & \langle v_8\rangle\{f_x\} \\ 0 & 0 & 0 & -\nu A & A & 0 & \langle v_7\rangle\{f_y\} & \langle v_8\rangle\{f_y\} \\ 0 & 0 & 0 & 0 & 0 & A & \langle v_7\rangle\{f_{xy}\} & \langle v_8\rangle\{f_{xy}\} \end{bmatrix} \tag{A.3}$$

were $\{v_7\}$ are $\{v_8\}$ the seventh column and eighth column of the matrix $[T]$ respectively.
From Eq. (27) and (A.3), we can see that (A.2) is not verified. The main idea of modification of the matrix $[T]$ is to substitute $\{v_7\}$ and $\{v_8\}$ by $\{\bar{v}_7\}$ and $\{\bar{v}_8\}$, such that :

$$\{\bar{v}_7\} = \{v_7\} + b_1\{v_4\} + b_2\{v_5\} + b_3\{v_6\} \tag{A.4}$$

$$\{\bar{v}_8\} = \{v_8\} + c_1\{v_4\} + c_2\{v_5\} + c_3\{v_6\} \tag{A.5}$$

were b_1, b_2, b_3 and c_1, c_2, c_3 are constants to be determined by the following orthogonal conditions:

$$\begin{cases} \langle\bar{v}_7\rangle\{f_x\} = 0 \\ \langle\bar{v}_7\rangle\{f_y\} = 0 \\ \langle\bar{v}_7\rangle\{f_{xy}\} = 0 \end{cases} \tag{A.6}$$

$$\begin{cases} \langle\bar{v}_8\rangle\{f_x\} = 0 \\ \langle\bar{v}_8\rangle\{f_y\} = 0 \\ \langle\bar{v}_8\rangle\{f_{xy}\} = 0 \end{cases} \tag{A.7}$$

For example, using (A.4) and Eq. (8), we have for (A.6):

$$\begin{aligned} \beta_1 A - \nu\beta_2 A &= -\langle v_7\rangle\{f_x\} \\ -\nu\beta_1 A + \beta_2 A &= -\langle v_7\rangle\{f_y\} \\ A\beta_3 &= -\langle v_7\rangle\{f_{xy}\} \end{aligned} \tag{A.8}$$

which can be solved to obtain β_1, β_2 and β_3.
Substituting β_1, β_2 and β_3 in (A.4), we obtain $\langle\bar{v}_7\rangle$. $\langle\bar{v}_8\rangle$ can be obtained by applying the same type of operations.
A modified $[T_r]$ matrix is then obtained by replacing $\{v_7\}$ and $\{v_8\}$ by $\langle\bar{v}_7\rangle$ and $\langle\bar{v}_8\rangle$ in the matrix $[T]$ Eq. (8). This procedure proposed by Zhong and Zeng will lead to element RQ4Z, passing successfully the patch-tests.

Appendix B

$$[\bar{k}_\alpha]_c = h \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & EA_e & -\nu EA_e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\nu EA_e & EA_e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & GA_e & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[\bar{k}_\alpha]_c = h \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & EA_e & -\nu EA_e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\nu EA_e & EA_e & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & GA_e & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & EI_x & -\nu EI_{xy} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\nu EI_{xy} & EI_y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G(I_x + I_y) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & G(I_x + I_y) \end{bmatrix}$$

$$[\overline{T}\,] = \begin{bmatrix} [\overline{N}]_{(x_1,\,y_1)} \\ [\overline{N}]_{(x_2,\,y_2)} \\ [\overline{N}]_{(x_3,\,y_3)} \\ [\overline{N}]_{(x_4,\,y_4)} \\ [\overline{N}]_{(x_5,\,y_5)} \end{bmatrix}$$