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Effective modeling of beams with shear deformations on elastic foundation

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Abstract. Being a significant mode of deformation, shear effect in addition to the other modes of stretching and bending have been considered to develop two finite element models for the analysis of beams on elastic foundation. The first beam model is developed utilizing the differential-equation approach; in which the complex variables obtained from the solution of the differential equations are used as interpolation functions for the displacement field in this beam element. A single element is sufficient to exactly represent a continuous part of a beam on Winkler foundation for cases involving end-loadings, thus providing a benchmark solution to validate the other model developed. The second beam model is developed utilizing the hybrid-mixed formulation, i.e., Hellinger-Reissner variational principle; in which both displacement and stress fields for the beam as well as the foundation are approxmated separately in order to eliminate the well-known phenomenon of shear locking, as well as the newly-identified problem of "foundation-locking" that can arise in cases involving foundations with extreme rigidities. This latter model is versatile and indented for utilization in general applications; i.e., for thin-thick beams, general loadings, and a wide variation of the underlying foundation rigidity with respect to beam stiffness. A set of numerical examples are given to demonstrate and assess the performance of the developed beam models in practical applications involving shear deformation effect.

Key words: beams; elastic foundation; hybrid/mixed formulation; shear deformat; foundation rocking.

1. Introduction

Beams on elastic foundation are widely used in many civil, mechanical and aerospace engineering applications. For instance, they provide typical idealization in soil structure interaction problems, and are also utilized to simulate the behavior of some engineering problems such as distortion of box-type bridges, the dowel action effect for shear transfer in concrete after cracks take place, cylindrical and spherical bearings, railroad tracks, etc. Traditionally, a straight beam supported by elastic medium has been modeled on the basis of classical Euler-Bernoulli beam theory for undeformed cross section, resulting in the so called C^1-continuous beam elements, with

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cubic interpolation (shape) functions for transverse displacements. Several approaches are used to include the foundation effect in conjunction with this conventional beam element. For instance, Tong and Rossettos (1977) utilized the same cubic function for the C^1-element to idealize the Winkler foundation; Bowles (1974, 1977) combined the conventional beam element with discrete springs at the ends of the beam; while Cheung and Nag (1968) treated the foundation as an isotropic elastic half space and utilized the Flamant equation (Timoshenko and Goodier 1970) to derive the flexibility matrix for the foundation which can be added after "inverting" the conventional beam stiffness. An alternative approach is to develop a stiffness matrix based on the solution of the differential equation of beam on elastic foundation. Several researchers have followed this technique by using initial parameters to express the solution of the fourth order differential equation in terms of four special functions; i.e., the deflection, slope, moment, and shearing force (e.g., Miranda 1966, Ting 1982, Eisenberger *et al.* 1985, Ting and Mockuy 1984).

However, more recent developments in structural applications (e.g., Bathe 1982, Cook 1982, Noor and Peters 1981, Stolarski *et al.* 1983, Gendy *et al.* 1992) emphasized the use of shear flexible modeling approach; thus extending the range of applicability to thick beams where the coupled flexural-shear deformations become significant, which is in fact more likely to be the case in practice (e.g., the height-to-span ratio of foundation beams is greater than that in the superstructure beams). This is the approach adopted in the present work.

In particular, two shear-flexible-beam- on-elastic-foundation models have been developed utilizing two different approaches. The first model is called DEBF2 element which is a differential equation-based model whose interpolation functions are of the complex- variables type. These latter functions are obtained from the solution of the differential equation governing the behavior of a beam with shear deformation that is supported on elastic medium and subjected to edge loadings. Few DEBF2 elements are thus sufficient to provide an "exact" solution for a typical foundation problem; whereas a single element can exactly represent a continuous part of a foundation beam with edge loadings.

The second two-noded model is formulated utilizing the hybrid-mixed approach (Gendy *et al.* 1992, Pian *et al.* 1982, Pian 1985), and is designated as HMBF2 element. In this C^0-beam model, the Hellinger-Reissner variational principle, with independent discretizations for displacements, generalized internal stresses, and subgrade reaction ("foundation pressure") fields, are utilized. This allows us to use more conventional (i.e., a low-order polynomial) interpolation functions and in the meanwhile avoid the shear locking phenomenon (both due to large shear-flexural beam stiffness ratios, as well as large relative foundation-to-beam-stiffness ratios) which are typically exhibited by alternative models associated with the element developed utilizing the customary potential-energy/displacement-bases forms. The versatility of this HMBF2 model provides for general applicability.

A selected number of numerical examples are given to assess the effectiveness of the two developed models in practical applications. A special emphasis is given to show the effect of shear deformations on the deflections and internal forces of beams on elastic foundation, and to demonstrate the successful treatment of associated phenomena of flexural-shear and foundation locking.

2. The differential equation-based beam model

2.1. Differential equations of beam with shear deformation on elastic foundation

The stiffness matrix of a planer frame element supported on infinite number of closely spaced separated springs (i.e., Winkler foundation) shown in Fig. 1 has been developed using the differential equation approach. The shear deformations are accounted for by considering the rotation, θ, and deflection, v, as independent; i.e., plane section remaining plane but is no longer restricted to be perpendicular to the beam center line after deformations. The differential equations for the deflection and the rotation of a beam on elastic foundation shown in Fig. 1 can be expressed as

$$GsA\left(\frac{d^2v}{dx^2}-\frac{d\theta}{dx}\right)-kv+p=0 \tag{1}$$

$$EI\,\frac{d^2\theta}{dx^2}+GsA\left(\frac{dv}{dx}-\theta\right)=0 \tag{2}$$

where E is the Young's modulus; G is the shear modulus; I is the moment of inertia of cross section about z-axis; A is the cross sectional area; s is the shear correction factor; k is the subgrade reaction modulus which is defined as the distributed reaction per unit length of the beam due to unit deflection ($k=k_0\times b$, where b is the beam width and k_0 is the subgrade reaction per unit area); and p is the transverse distributed load as shown in Fig. 1.

Differentiating Eq. (1) twice with respect to (w.r.t.) beam axis x, and Eq. (2) once w.r.t. variable x, one gets

$$\frac{d^3\theta}{dx^3}=\frac{d^4v}{dx^4}+\frac{1}{GsA}\left(-k\frac{d^2v}{dx^2}+\frac{d^2p}{dx^2}\right) \tag{3}$$

and

$$EI\,\frac{d^3\theta}{dx^3}+GsA\left(\frac{d^2v}{dx^2}-\frac{d\theta}{dx}\right)=0 \tag{4}$$

Substituting from Eq. (3) into Eq. (4) and utilizing Eq. (1), one can write a differential equation of fourth-order for a beam (with shear deformations) on elastic foundation in terms of the deflection, v, as follows

$$EI\,\frac{d^4v}{dx^4}-\frac{EI}{GsA}\left(k\frac{d^2v}{dx^2}\right)+kv=p-\frac{EI}{GsA}\,\frac{d^2p}{dx^2} \tag{5}$$

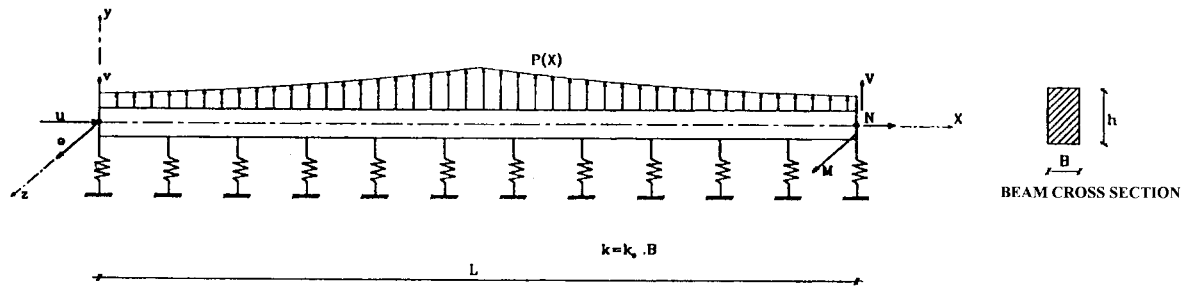

Fig. 1 Generalized forces and kinematic degrees of freedom for a typical beam on elastic foundation

The rotation can be obtained in terms of the deflection by differentiating Eq. (1) with respect to x and substituting into Eq. (2) as

$$\theta = \frac{EI}{GsA}\left[\frac{d^3v}{dx^3} + \frac{1}{GsA}\left(-k\frac{dv}{dx} + \frac{dp}{dx}\right)\right] + \frac{dv}{dx} \tag{6}$$

Assuming that the beam has no distributed load, i.e., p=0, Eqs. (5) and (6) can be rewritten as follows; with a=0.5k/(GsA) of dimension 1/(length)^2

$$\frac{d^4v}{dx^4} - 2a\frac{d^2v}{dx^2} + \frac{k}{EI}v = 0 \tag{7}$$

$$\theta = \frac{EI}{GsA}\left[\frac{d^3v}{dx^3} + \left(-2a + \frac{GsA}{EI}\right)\frac{dv}{dx}\right] \tag{8}$$

It is of particular interest here to note the two parenthetical terms in Eq. (8); the first "a" gives the relative rigidity of foundation-to-beam-shear-stiffness, and the second (GsA/EI) is a measure of the shear-to-flexure stiffness of the beam.

Eq. (7) is the governing differential equation of the beam with shear deformations on elastic foundation. Once the deflection expression is obtained by solving Eq. (7), the rotation can be determined by back substituting in Eq. (8).

The solution of the differential Eq. (7) can be written in the form

$$v(x) = \sum_{i=1}^{4} c_i\, e^{\alpha_i x} \tag{9}$$

where

$$\alpha_{1,2} = \pm\sqrt{a + ib}\,, \qquad \alpha_{3,4} = \pm\sqrt{a - ib} \tag{10a}$$

$$b = \sqrt{\frac{k}{EI} - a^3}, \qquad c_i = \text{generalized coordinates} \tag{10b}$$

2.2. Stiffness matrix

The displacement field in Eq. (9) can be rewritten in the matrix form as

$$v(x) = \underset{\sim}{\phi}\,\underset{\sim}{C} \tag{11}$$

where

$$\underset{\sim}{\phi} = [e^{\alpha_1 x}\; e^{\alpha_2 x}\; e^{\alpha_3 x}\; e^{\alpha_4 x}], \qquad \underset{\sim}{C} = [c_1,\, c_2,\, c_3,\, c_4]^T \tag{12}$$

The generalized coordinates can be expressed in terms of the unknown element nodal point displacements v_1, θ_1, v_2, and θ_2 by substituting in Eqs. (8) and (11) with the following:

$$\begin{array}{llll} \text{at } x=0 & v=v_1 & \text{and} & \theta=\theta_1 \\ \text{at } x=L & v=v_2 & \text{and} & \theta=\theta_2 \end{array} \tag{13}$$

This leads to

$$\underset{\sim}{q} = \underset{\sim}{A}\,\underset{\sim}{C} \tag{14}$$

where

$$\underset{\sim}{q} = [v_1,\ \theta_1,\ v_1,\ \theta_2]^T \tag{15a}$$

$$\underset{\sim}{A} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ e^{\alpha_1 L} & e^{\alpha_2 L} & e^{\alpha_3 L} & e^{\alpha_4 L} \\ \lambda_1 e^{\alpha_1 L} & \lambda_2 e^{\alpha_2 L} & \lambda_3 e^{\alpha_3 L} & \lambda_4 e^{\alpha_4 L} \end{bmatrix} \tag{15b}$$

with

$$\lambda_i = \frac{EI}{GsA}\left[\alpha_i^3 + \left(\frac{-k}{GsA} + \frac{GsA}{EI}\right)\alpha_i\right] \tag{15c}$$

Solving Eq. (14) for $\underset{\sim}{C}$ and substituting into Eq. (11), one gets

$$v(x) = \underset{\sim}{N}\,\underset{\sim}{q} \tag{16}$$

where $\underset{\sim}{N}$ is the one-row matrix of interpolation functions corresponding to the four nodal degrees of freedom and can be expressed as

$$\underset{\sim}{N} = \underset{\sim}{\phi}\,\underset{\sim}{A}^{-1} \tag{17}$$

The transverse shear strain, γ_{xy}, and bending curvature, ϕ_z, which are the two components of the generalized strain vector $\underset{\sim}{\varepsilon}$ for the beam can be expressed in terms of nodal displacements as

$$\underset{\sim}{\varepsilon} = [\gamma_{xy},\ \phi_z]^T = \left[\left(\frac{dv}{dx} - \theta\right),\ \frac{d\theta}{dx}\right]^T = \underset{\sim}{E}\,\underset{\sim}{A}^{-1}\,\underset{\sim}{q} \tag{18}$$

where

$$\underset{\sim}{E} = \begin{bmatrix} (\alpha_1 - \lambda_1)\,e^{\alpha_1 x} & (\alpha_2 - \lambda_2)\,e^{\alpha_2 x} & (\alpha_3 - \lambda_3)\,e^{\alpha_3 x} & (\alpha_4 - \lambda_4)\,e^{\alpha_4 x} \\ \lambda_1\alpha_1 e^{\alpha_1 x} & \lambda_2\alpha_2 e^{\alpha_2 x} & \lambda_3\alpha_3 e^{\alpha_3 x} & \lambda_4\alpha_4 e^{\alpha_4 x} \end{bmatrix} \tag{19}$$

The stress-resultant vector for the beam, $\underset{\sim}{\sigma}$, whose components include the shearing force, V, and bending moment, M, can be expressed in terms of the generalized strain vector, ε, through the constitutive relation as

$$\underset{\sim}{\sigma} = \text{Diag.}\,[GsA,\ EI]\,\underset{\sim}{\varepsilon} \tag{20}$$

where Diag.[ ] is a diagonal matrix. Using Eqs. (18) and (20), the nodal force vector, $\underset{\sim}{Q}$, that corresponds to the four degrees of freedom are

$$\underset{\sim}{Q} = \underset{\sim}{D}_m \begin{bmatrix} \underset{\sim}{E}\big|_{x=0} \\ \hline \underset{\sim}{E}\big|_{x=L} \end{bmatrix} \underset{\sim}{A}^{-1}\,\underset{\sim}{q} = \underset{\sim}{K}\,\underset{\sim}{q} \tag{21}$$

where

$$\underset{\sim}{D}_m = \text{Diag.}\ [-GsA\,, -EI, GsA\,, EI\,] \tag{22}$$

and the symbols $.|_{x=0}$ and $.|_{x=1}$ indicate evaluations at locations x=0, and x=L, respectively, of 2×4 array in Eq. (19). It can be noticed that since the roots of the differential equation contain imaginary terms, the coefficients of $\underset{\sim}{A}$ and $\underset{\sim}{E}$ matrices should be treated as complex numbers. The stiffness matrix, K, in Eq. (21) can be extended to account for the axial deformation by adding the conventional term EA/L with positive or negative sign to be associated with the axial displacements; resulting in (6×6) stiffness matrix.

3. Hybrid/mixed beam model

3.1. Two-field form variational principle

The Hellinger-Reissner, two-field, variational principle in which stresses and displacements are assumed independently (Gendy *et al.* 1992, Pian and Chen 1982, Pian 1985, Gendy *et al.* 1994, Washizu 1982) is utilized. For a typical beam element on elastic foundation whose longitudinal axis is x, and y and z are principal centroidal axes, the functional π_{HR} can be written in the form

$$\pi_{HR} = \int_L \left\{ \left[-\frac{1}{2}\, \underset{\sim}{\sigma}_R^T\, \underset{\sim}{D}^{-1}\, \underset{\sim}{\sigma}_R + \underset{\sim}{\sigma}_R^T\, \underset{\sim}{\bar{\varepsilon}}_R \right] - \left[-\frac{1}{2}\, F\, D_s^{-1} F + F\, v \right] \right\} dx - W \tag{23}$$

The first two terms in the above equation are the internal strain energy for the beam element utilizing the two-field principle; while the following two terms are the foundation effect; and the term W denotes collectively the work of the prescribed external forces and moments. In Eq. (23), L is the element length, $\underset{\sim}{\sigma}_R$ is the independently assumed generalized stress-resultant vector which can conveniently be written in terms of extensional/shear/bending actions as

$$\underset{\sim}{\sigma}_R = [N, V, M]^T \tag{24}$$

where N is the normal force, V is the shear force in the y direction, M is the bending moment; F is the independently assumed subgrade force (force/unit length); $\underset{\sim}{\bar{\varepsilon}}_R$ is the generalized geometric strain vector (derived from displacements) and defined similarly

$$\underset{\sim}{\bar{\varepsilon}}_R = [\bar{\varepsilon}_0,\ \bar{\gamma}_{xy},\ \bar{\phi}_z]^T \tag{25}$$

where $\bar{\varepsilon}_o$ is the axial stretch, $\bar{\gamma}_{xy}$ the (average) transverse shear strain due to flexure, and $\bar{\phi}_z$ the bending curvature; $\underset{\sim}{D}^{-1}$ is the inverse of the material stiffness (compliance matrix), i.e.,

$$\underset{\sim}{D} = \text{Diag.}\ [EA\,, GsA\,, EI\,] \tag{26}$$

The D^{-1}_s is the inverse of the subgrade modulus (i.e., $-1/k$) such that

$$F = -k\ v \tag{27}$$

Utilizing the strain-displacement relations [e.g., Gendy *et al.* 1992], the three generalized strain components in the present one-dimensional beam model (i.e., Eq. 25) can be written as

$$\underset{\sim}{\bar{\varepsilon}}_R = [u', (v' - \theta), \theta']^T \tag{28}$$

The prime indicates differentiation w.r.t. the axial coordinate x.

3.2. Finite element formulation

In the hybrid/mixed finite element approximation, the displacement components of a reference point on the beam cross section $\underset{\sim}{u}$, $\underset{\sim}{u} = [u, v, \theta]^T$, stress resultants $\underset{\sim}{\sigma}_R$, and subgrade force F, are interpolated in terms of nodal displacement $\underset{\sim}{q}$, stress parameters $\underset{\sim}{\beta}$ and subgrade parameters $\underset{\sim}{\beta}_s$, as

$$\underset{\sim}{u} = \underset{\sim}{N}\,\underset{\sim}{q}, \qquad \underset{\sim}{\sigma}_R = \underset{\sim}{P}\,\underset{\sim}{\beta}, \qquad F = \underset{\sim}{P}_s\,\underset{\sim}{\beta}_s \tag{29}$$

where $\underset{\sim}{N}, \underset{\sim}{P}$ and $\underset{\sim}{P}_s$ are the matrices of interpolation functions for element displacements, stress resultants, and subgrade force, respectively. They are polynomial functions of the coordinate x. Some rational comments concerning the choice of these interpolation functions are given next.

First, with regard to the displacement interpolation assumptions, only the compatible displacement functions are used and they are the familiar Lagrange polynomial interpolation functions for one-dimensional elements (Bathe 1982, Cook 1982). That is with natural coordinate, r=($2x/L$-1), the interpolation functions for two-noded element are

$$N_1 = \frac{1}{2}(1 - r), \quad N_2 = \frac{1}{2}(1 + r) \tag{30}$$

Second, considering the beam's internal force field, the stress parameters may be chosen in such a way that the homogeneous parts of the stress equilibrium conditions are satisfied pointwise within the element. The resulting element is then referred to as an "equilibrium" hybrid element (Pian 1982). This has been the most popular approach in the early developments of hybrid elements. However, Pian *et al.* (1982) developed a hybrid formulation in which the stress equilibrium conditions are not enforced initially, but are brought in (if required), in an "approximate" (integral or variational) sense, through the use of the functional form in Eq. (23) itself. Here, for our study, the equilibrium requirements are completely relaxed.

There are two important issues that must be taken into considerations in order to select the stress parameters properly: (1) avoiding all kinematic deformation modes; (2) ability of the resulting element to handle constrained problems. For the latter, we imply herein the thin element, possibly exhibiting flexural shear locking, or beams on excessively rigid foundations relative to the beam shear stiffness.

With regard to the first issue, suppression of kinematic deformation modes is the most important requirement in the hybrid-mixed formulation. The necessary condition for the stiffness matrix to be of sufficient rank is that the number of internal stress parameters, $\underset{\sim}{\beta}$, should be greater or at least equal to the difference between the total number of all kinematic degrees of freedom of the element and the rigid-body modes (Gendy, Saleeb and Chang 1992). Based on the scheme proposed by Pian and Chen (1983) for choosing the proper set of stress terms, a single β-term should be choosen corresponding to each of the terms in the strain expressions obtained from strain-displacement relations, thus resulting in the least ("optimal") number of parameters. In the context of the present beam-on-elastic-foundation formulation, this condition can be easily achieved by using the following specific forms for $\underset{\sim}{P}$, and $\underset{\sim}{P}_s$ of Eq. (29); i.e.,

$$\underset{\sim}{P} = \underset{\sim}{I}, \qquad \underset{\sim}{P}_s = 1 \tag{31}$$

where $\underset{\sim}{I}$ is a diagonal (3×3) unit matrix, and the scalar $\underset{\sim}{P}_s$ is a (1×1) unit matrix. It is noticed that a constant (one internal stress parameter) is chosen for each of the stress components and that the choice of a constant field for the foundation/subgrade reaction field is crucial for the success of the element in applications involving extreme rigidity of the foundation (relative to the beam stiffness); i.e., large values of aL^2; see Eq. (8).

The second issue is to check the element behavior so that any potential locking problem is precluded, for example in its applications to "thin" beam structures and/or rigid foundation. A convenient way to examine this behavior is to use the method of constraint counting, often termed as "constraint index" as suggested in Hughes *et al.* (1978), and Malkus *et al.* (1978). The constraint index, CI, is defined as the difference between the number of kinematic degrees of freedom brought by an element, when added to an existing finite element mesh, and the number of independent constraints per element when it is used in the limiting case (e.g., thin beam). A favorable value of CI is equal to or greater than one which implies that the element is locking free. For illustration, consider the two-noded hybrid-mixed beam element, HMBF2, under flexural action only (this is the most critical situation). In the limit as the beam thickness approaches to zero, the shear deformations should vanish. Correspondingly, the stress parameter β_2 associated with the shearing force approaches to zero for such a case. This will impose one constraint condition. Thus, when an HMBF2 element is added to existing mesh, the resulting CI is (2-1)=1 (with two pertinent kinematic degrees of freedom associated with flexural response), hence no locking is expected in this case. On the other hand, the corresponding calculations for the more conventional counterpart displacement-based model (see Eq. 38), with "exact" integration for the associated component stiffness arrays (i.e., two Gauss integration points), CI is equal to (2-2)=0; this indicates complete failure due to locking in thin beam limit. This conclusion has indeed been supported by numerical results given in the next section.

A similar locking phenomenon (that is peculiar to the present case of beams on elastic foundations) pertains to extreme relative values of aL^2 in Eq. (8) approaching infinity, with finite GsA values. In this case, the pertinent deformation parameter to be suppressed corresponds to β_s, thus resulting in one constraint, while the number of free degrees of freedom is still two, giving CI=2-1=1; i.e., no locking. Here, again, alternative displacement-based models will yield CI=0 and exhibit severe locking in these cases.

Substituting Eq. (29) into Eq. (23) and following standard arguments (Pian and Chen 1982, Pian 1985, Gendy and Saleeb 1994, Washizu 1982, Pian 1982), the functional π_{HR} can be written as

$$\pi_{HR} = -\frac{1}{2}\underset{\sim}{\beta}^T \underset{\sim}{H}\underset{\sim}{\beta} + \underset{\sim}{\beta}^T \underset{\sim}{G}\underset{\sim}{q} + \frac{1}{2}\underset{\sim}{\beta}_s^T \underset{\sim}{H}_s \underset{\sim}{\beta}_s - \underset{\sim}{\beta}_s^T \underset{\sim}{G}_s \underset{\sim}{q}_v - \underset{\sim}{Q}^T \underset{\sim}{q} \tag{32}$$

where $\underset{\sim}{H}$ and $\underset{\sim}{H}_s$, the flexibility matrices, are defined as

$$\underset{\sim}{H} = \int_L \underset{\sim}{P}^T \underset{\sim}{D}^{-1} \underset{\sim}{P}\, dx, \quad \underset{\sim}{H}_s = \int_L \underset{\sim}{P}_s^T \underset{\sim}{D}_s^{-1} \underset{\sim}{P}_s\, dx \tag{33}$$

The matrices $\underset{\sim}{G}$ and $\underset{\sim}{G}_s$ are expressed in terms of strain-displacement matrix $\underset{\sim}{B}$ and interpolation function for transverse displacement $\underset{\sim}{N}_v$, respectively, as

$$\underset{\sim}{G} = \int_L \underset{\sim}{P}^T \underset{\sim}{B}\, dx, \quad \underset{\sim}{G}_s = \int_L \underset{\sim}{P}_s^T \underset{\sim}{N}_v\, dx \tag{34}$$

the $\underset{\sim}{q}_v$ is the transverse nodal displacement, and $\underset{\sim}{Q}$ is the equivalent nodal force vector, such that $\underset{\sim}{Q}^T \underset{\sim}{q}$ = term in W in Eq. (23).

Invoking the stationary condition of Eq. (32) with respect to the independent variation of stresses yields $\underset{\sim}{\beta}$ and $\underset{\sim}{\beta}_s$ in terms of $\underset{\sim}{q}$ and $\underset{\sim}{q}_v$,

$$\underset{\sim}{\beta}=\underset{\sim}{H}^{-1}\underset{\sim}{G}\underset{\sim}{q}, \qquad \underset{\sim}{\beta}_s=\underset{\sim}{H}_s^T\underset{\sim}{G}_s\underset{\sim}{q}_v \tag{35}$$

thus allowing for the elimination of stress parameters $\underset{\sim}{\beta}$ and $\underset{\sim}{\beta}_s$ on the element level. Substitution of Eq. (35) into Eq. (32) yields

$$\pi_{HR}=\frac{1}{2}\underset{\sim}{q}^T\underset{\sim}{K}\underset{\sim}{q}-\frac{1}{2}\underset{\sim}{q}_v^T\underset{\sim}{K}_s\underset{\sim}{q}_v-\underset{\sim}{Q}^T\underset{\sim}{q} \tag{36}$$

from which the stiffness matrices for the hybrid-mixed element on elastic foundation, HMBF2, are given by

$$\underset{\sim}{K}=\underset{\sim}{G}^T\underset{\sim}{H}^{-1}\underset{\sim}{G}, \ \ \underset{\sim}{K}_s=\underset{\sim}{G}_s^T\underset{\sim}{H}_s^{-1}\underset{\sim}{G}_s \tag{37}$$

For the purpose of comparison, the stiffness matrix for the displacement-based beam element on elastic foundation with two nodes, designated here as DBF2, has been obtained from the familiar expression e.g., (Bathe 1982, Cook 1982) of the principle of minimum potential energy. Thus, for the present case accounting for shear deformation, we use linear shape functions for both the transverse deflection and rotational components of the single (displacement) field involved. A straightforward driviation will then lead to the final stiffness equations for the DBF2 element:

$$\underset{\sim}{K}_D=\int_L\underset{\sim}{B}^T\underset{\sim}{D}\underset{\sim}{B}\,dx-\int_L\underset{\sim}{N}_v D_s\underset{\sim}{N}_v\,dx \tag{38}$$

In the above, the first term is the contribution from the internal strain energy, while the second term is the foundation effect. Note that the foundation pressure field (calculated from displacements in this case) is also linear.

4. Numerical examples

A number of test problems are considered in this section to assess the performance of the developed elements. As before, we use the following designations throughout: DEBF2 for the differential equation-based element; and HMBF2 for the hybrid-mixed-based element. For the purpose of comparisons, results for some of the test problems are also obtained using the "more-conventional" displacement-based element DBF2.

4.1. Mesh convergence

A simply supported beam on elastic foundation, shown in Fig. 2, has been analyzed using a different meshes of DEBF2, HMBF2, and DBF2 elements. The beam length is 2.0 m., width is 1.0 m., and depth is 0.10 m; with the material and foundation constants given in Fig. 2. By making use of symmetry, only one half of the beam is analyzed. The normalized deflection under the load, and the rotation at the left end are plotted in Figs. 3 and 4, respectively. The deflection and rotation converged to -0.0841 m. and -0.1060 rad., respectively, using one DEBF2 element, when the shear deformations are taken into considerations. However, the DEBF2 element gives -0.0837m. and -0.1064 rad. for the deflection and rotation, respectively, when the shear deformations are neglected (i.e., by considering "large" values for shear rigidity *GsA*). These former values are identical to those obtained in Eisenberger and Yankelevsky (1985) using stiffness matrix developed based on the solution of the differential equation of beam on elastic foundation without shear effects; and also agree with the "exact" solutions given in Timoshenko (1956). The results

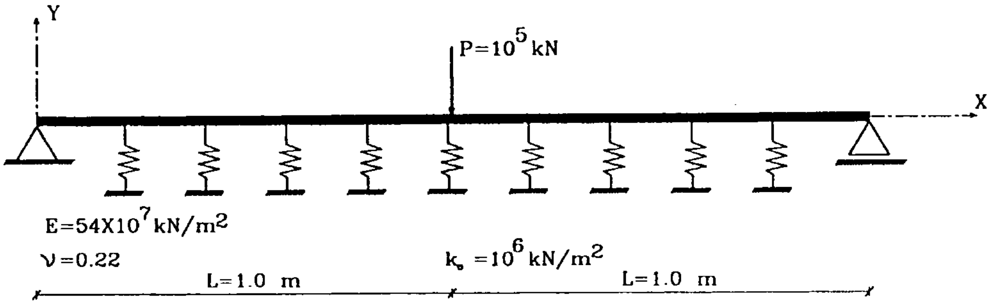

Fig. 2 The problem of a simply supported beam on elastic foundation

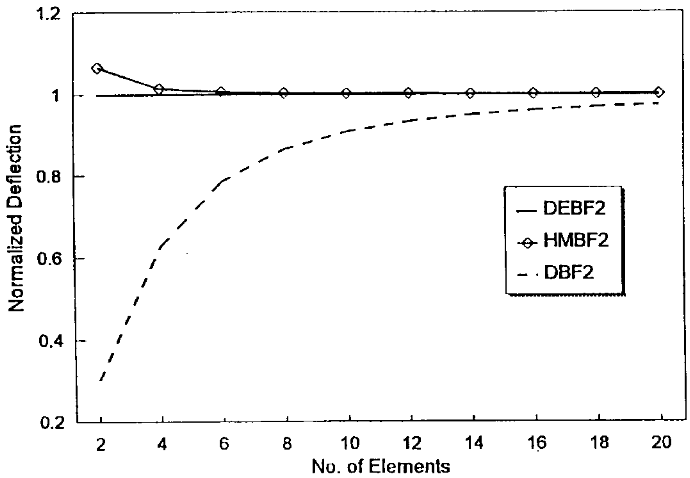

Fig. 3 Convergence study for the mid-span deflection of a simply supported beam on elastic foundation

obtained with the HMBF2 and DBF2 elements are normalized with respect to those obtained by the DEBF2 in Figs. 3 and 4. As shown from these figures, solutions using HMBF2 element give good results with about 1% error using 4 elements. On the other hand, due to the mild degree of flexural shear locking, the convergence is very slow using DBF2 element. For instance, the deflection exhibits an error of 37% with a mesh of 4 DBF2 elements; and even with a mesh of 20 elements the deflection still exhibits an error of approximately 3%.

4.2. Shear locking phenomenon

The effect of shear locking is investigated by considering again the simply supported beam on elastic foundation shown in Fig. 2. Here, the beam is analyzed using different values of aspect ratio L/h (length to height ratio). Four elements are used to idealize one half of the beam. The vertical deflection under the load (i.e., in the middle of beam span) using DEBF2, HMBF2, and DBF2 elements are shown in Fig. 5. In fact, the solution obtained using HMBF2 elements are almost identical to those obtained by the DEBF2 elements for the entire range of aspect ratio L/h. On the other hand, the results obtained with the DBF2 element exhibit a shear locking of this element. That is, for a limiting case of thin beam, i.e., L/h becomes very large, the predicted values for the deflection approaches to zero, obviously a worthless result contradicting the

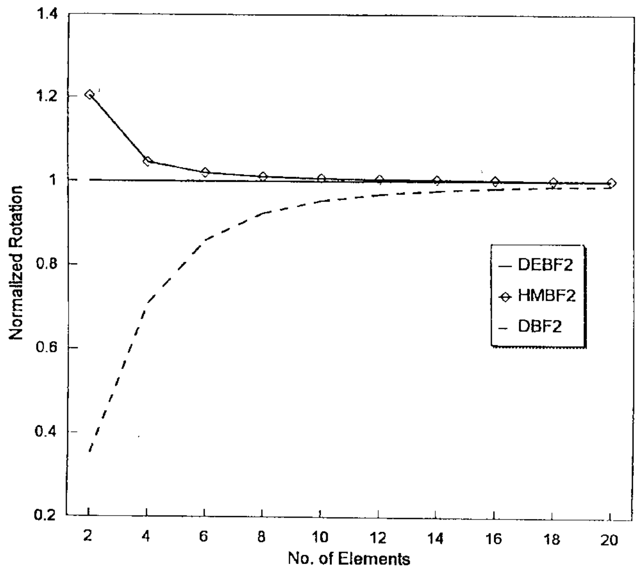

Fig. 4 Convergence study for the rotation of a simply supported beam on elastic foundation

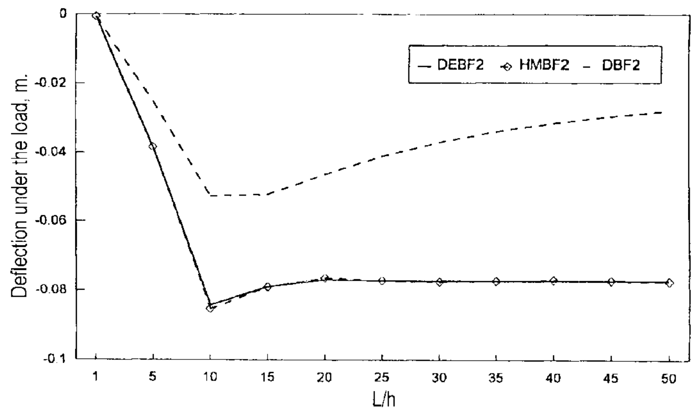

Fig. 5 Mid-span deflections for beams with various height-to-span ratios supported on elastic foundation

physical problem considered. For instance, for L/h=50 the solution with DBF2 element exhibits an error of 64% when it is compared to that obtained with the DEBF2 element.

4.3. Rigid foundation locking phenomenon

We consider the same beam solved previously in Fig. 2, with a mesh of ten elements in one symmetric half and for a particular value L/h=2; i.e., a case of relatively thick beam so that flexural-shear locking is not an issue (both HMFB2 and DEBF2 behave well for this case for k= 10^6). Rather, a wide range of values for the foundation stiffness constant k (10^6 to 10^{14}) will be considered to investigate the phenomenon of foundation locking; see pertinent discussion in Sec. 3.2.

For convenience, results are reported in Table 1 in a normalized format for the mid-span deflection, v, under the load; i.e., $2kv/Pg$, versus the foundation stiffness k, where $g=[k/4EI]^{1/4}$, P is the central load and L is the total length of the beam. As seen from Table 1, for "moderate" values of k, both the displacement and mixed models give equally good results (e.g., compare the top two rows in the table, where the first row pertains to the case considered in Figs. 3 and 4, and Sec. 4.2 above). However, results from the two models start to deviate with increasing k. In particular, very significant differences are seen for the extreme higher values of k; e.g., the deflection obtained from DBF2 is nearly one-order-of-magnitude less than that produced by HMBF2 for the case corresponding to the last row in the table. This clearly signifies the severe "foundation" locking exhibited by DBF2 in such cases. On the other, the mixed model HMBF2 has maintained excellent performance for the entire range of k, indeed yielding identical results to the "exact" model DEBF2 (not shown in Table 1).

4.4. Effect of shear deformations

The effect of shear deformations is investigated by analyzing the same simply supported beam on elastic foundation shown in Fig. 2. The beam is analyzed for different aspect ratios L/h; and with and without taking into considerations the shear deformations. The vertical deflection obtained using the DEBF2 and HMBF2 elements are given in Table 2. As shown from this table, the results obtained with the HMBF2 element are in good agreement with those obtained by DEBF2 element. Shear deformations are significant for small values of aspect ratio L/h. For instance, the deflection under the load is increased by 60% for $L/h=1$ when the shear deformations are accounted for.

Table 1 Normalized deflection demonstrating the rigid-foundation locking phenomenon

k	Normalized Deflection	
	DBF2	HMBF2
10^6	0.04626	0.04636
10^8	0.97718	0.97910
10^{10}	2.6836	2.7433
10^{12}	3.6266	8.4618
10^{14}	1.2674	13.3459

Table 2 Effect of shear deformations on the mid-span deflection

L/h	DEBF2*		HMBF2*	
	no shear	with shear	no shear	with shear
1	-0.0370	-0.0590	-0.0365	-0.0587
2	-0.2946	-0.3386	-0.2901	-0.3341
3	-0.9717	-1.0352	-0.9578	-1.0215
4	-2.1707	-2.2477	-2.1454	-2.2228
5	-3.7819	-3.8621	-3.7553	-3.8367

*all values are in cm.

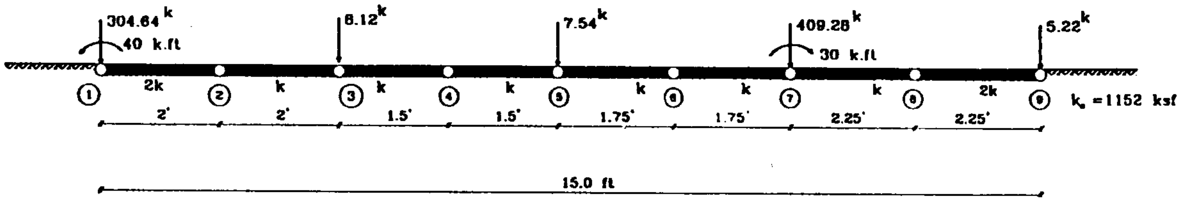

Fig. 6 The problem of unrestrained beam on elastic foundation

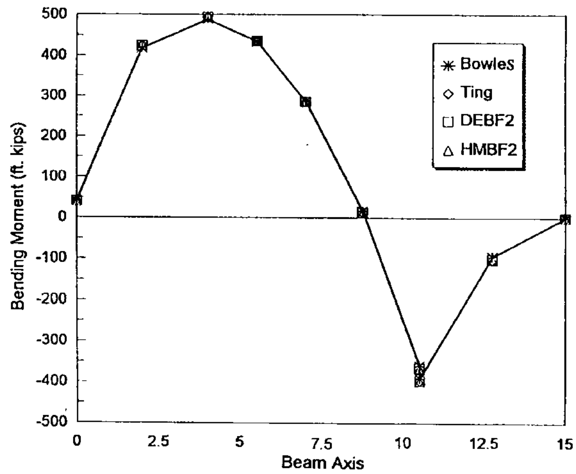

Fig. 7 Bending moment distribution of unrestrained beam on elastic foundation

4.5. Unrestrained beam on elastic foundation

Consider the beam on elastic foundation given by Bowles (1977) and shown in Fig. 6. The beam length is 15 ft; width is 8 ft; and depth is 2 ft. The beam is made of material with Young's modulus E=452 700 ksf; and supported on a soil with subgrade reaction modulus k_o=1152 ksf. The subgrade modulus of the two end elements is double that of the other as shown in Fig. 6. The beam is idealized using 8 elements for both DEBF2 and HMBF2 models. This is the same mesh size utilized in the reference solution of Timoshenko and Goodier (1977). The nodal deflections obtained with the DEBF2 and HMBF2 models are listed in Table 3 along with those obtained by Bowles (1977) and by Ting and Mockry (1984). As is evident from this table, predictions of nodal deflections obtained by DEBF2 and HMBF2 elements are almost identical to those given in Bowles (1977) and Ting and Mockry (1984).

The distribution of the bending moment along the beam using DEBF2 and HMBF2 elements are depicted in Fig. 7 along with the results reported by Bowles (1977) and by Ting and Mockry (1984). Again the results given by DEBF2 and HMBF2 elements compare favorably with those given in Bowles (1977) and Ting and Mockry (1984).

4.6. Two-bay two-story frame on elastic foundation

A two-bay two-story frame supported on a continuous beam on elastic foundation (see Fig. 8)

Table 3 Nodal deflections of a beam on elastic foundation

Node	DEBF2*	HMBF2	Bowles (1977)	Ting *et al.* (1982)
1	-0.05091	-0.05097	-0.05015	-0.05075
2	-0.04487	-0.04500	-0.04452	-0.04490
3	-0.03963	-0.03975	-0.03951	-0.03972
4	-0.03627	-0.03636	-0.03625	-0.03636
5	-0.03334	-0.03340	-0.03339	-0.03339
6	-0.03029	-0.03028	-0.03036	-0.03025
7	-0.02729	-0.02718	-0.02732	-0.02712
8	-0.02262	-0.02246	-0.02292	-0.02260
9	-0.01778	-0.01750	-0.01825	-0.01784

*all values are in feet; 1 ft=0.305 m.

is chosen as an example of practical application of superstructure-soil interaction. The frame is made of concrete with Young's modulus of 25×10^6 KN/m^2, Poisson's ratio of 0.22, and subgrade modulus for soil of 5×10^5 KN/m^2. The two girders of the frame are of T-cross section, the columns are of rectangular cross section, and the foundation beam is of (inverted tee) $\perp$-cross section. The frame cross sections have the following dimensions: (1) for girders with T-section: total depth is 0.6 m., flange width and thickness are 1.0 and 0.12 m., respectively, web width is 0.2 m.; (2) for columns in the first story: the depth and width are 0.5 and 0.25 m., respectively; for columns in the second story: the depth and width are 0.4 and 0.25 m., respectively; (3) for the foundation beam: total depth is 1.5 m., flange width and thickness are 1.5 and 0.4 m., respectively, and web width is 0.75 m. The structure is idealized using DEBF2 as well as HMBF2 elements. In order to demonstrate the effects of shear deformations, the analysis has been carried out with and without taking the shear deformations into considerations. The deflections as well as the forces obtained by using DEBF2 and HMBF2 are identical. The deflection curves of the foundation beam with as well as without shear effect are depicted in Fig. 9. It can be noted from this figure that the maximum deflection is located under the maximum loaded point, i.e., middle column, with a value of 0.4785 mm. when shear effect is considered, and 0.3550 mm. when shear effect is

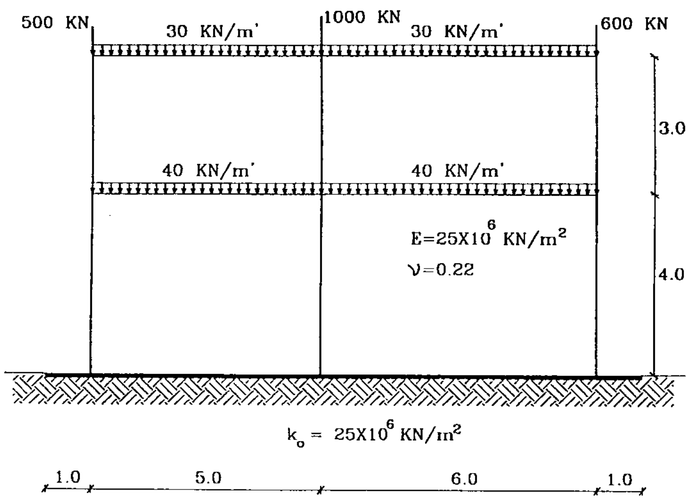

Fig. 8 The problem of two-bay two-story frame on elastic foundation

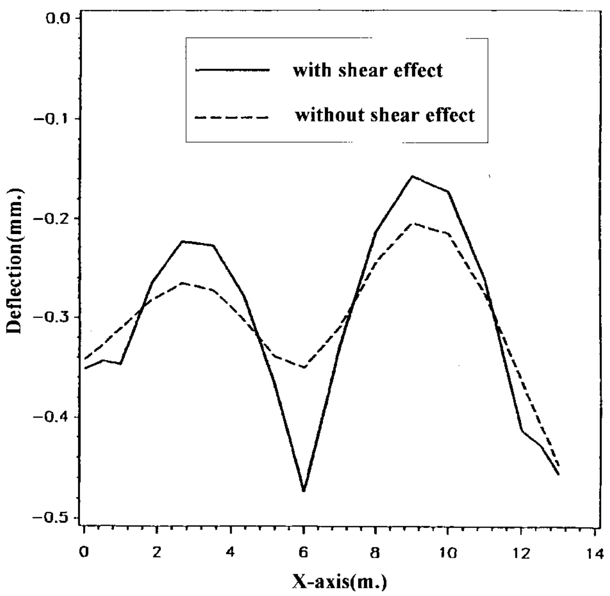

Fig. 9 Deflecion of foundation beam of two-bay two-story frame

neglected (i.e., relative difference of 25.8%). Similarly, the maximum bending moments for these two cases are located just to the left of the middle column with values of 639 and 710 KN.m, respectively; i.e., the relative difference is nearly 11%.

5. Conclusions

Two models have been developed to represent shear-flexible beams on elastic foundations. The first is formulated utilizing the differential equation-based approach, thus facilitating comparisons with benchmark, "exact", solution for cases of end loads. The second is developed for general applications, using the hybrid-mixed variational principle approach. The effectiveness of this latter model is demonstrated in a fair set of numerical examples. In particular, these models were shown to be free from the well-known shear locking, as well as the special type of locking phenomenon resulting from the extreme foundation rigidity relative to the supported beam's stiffness. They exhibited accurate displacement- and stress-prediction capabilities. This is in sharp contrast to the counterpart element developed utilizing the conventional, "pure", displacement formulation. In several cases, the shear deformations are shown to be very significant, and thus must be accounted for in both deflection and internal stress calculations. This is especially true when the ratio of the beam height to its span increases, which is typically the case in practical foundation beams.

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