Structural Engineering and Mechanics, Vol. 8, No. 4 (1999) 421-438
DOI: http://dx.doi.org/10.12989/sem.1999.8.4.421

Practical estimation of the plastic collapse limit of curved pipes subjected to complex loading

A.M. Yan† and D.H. Nguyen‡

LTAS-University of Liege, Rue Ernest Solvay 21, 4000 Liege, Belgium

Ph. Gilles†‡

Framatome, Tour Fiat 92084, la Défense Paris, France

Abstract. In this paper a practical limit load estimating procedure is proposed for general pipe-elbow structures subjected to complex loading (in-plane and out-of-plane bending, internal pressure and axial force). The explicit calculating formulae are presented on the basis of theoretical analysis combined with numerical simulation. Von Mises' yield criterion is adopted in both analytical and numerical calculation. The finite element examination shows that the method provides a simple but satisfactory prediction of pipe structures in engineering plastic analysis.

Key words: limit analysis; limit load; bending of curved pipe; pipe structures; elbow.

1. Introduction

In petrol-chemical and power producing plants, some failure accidents are caused by local fracture or excessive deformation of a pipeline (especially curved pipes). The analysis of a pipe bend is very interesting in a bending situation where, as has been known for many years, its flexibility is significantly greater than that of a straight pipe. Such flexibility may be useful in reducing thermal expanding reactions. Besides, the pipe bends are also commonly employed as internal components in various reactors. Usually they are subjected to important and complex stress and strain arising from differential movements and also due to complex geometry. Despite the large amount of literature on linear elastic behaviour of bends under various loads, little information is available to guide the designer to assess the plastic collapse characteristics of pipe bends.

Failure in curved pipes may be initiated by various causes such as wrinkling, corrosion, creep, and so on. However, in the present paper, only the plastic limit analysis is studied where the loads are supposed to be monotonic and proportional. The alternative situation concerning shakedown analysis (where the loads vary arbitrarily or cyclically) was discussed in a separate report, Yan *et al.* (1997a, b). Although in a practical case the loading is always variable, the present analysis still

† Ph.D. Research Engineer
‡ Professor
†‡ Senior Engineer

has a significant role. On one hand, the limit load as a principal parameter is used not only in plastic design but also in the fracture toughness estimation of the structure with defects; e.g., Joch *et al.* (1993), and Gilles *et al.* (1996). On the other hand, limit analysis may be a particular case of shakedown analysis (Konig 1987). In fact, in some practical cases without thermal loading, the limit factor may be an upper bound approximation of the incremental plasticity factor of shakedown analysis, while the alternating plasticity factor (plastic fatigue) could be estimated by elastic analysis, Gokhfeld & Cherniavsky (1980). In classical analysis, the material is supposed to be elastic (or rigid) - perfectly plastic and obey von Mises' yield criterion. The small displacement model is also adopted. Although these hypotheses are never strictly satisfied in practical engineering, they lead to unique (theoretical) and generally safe estimation of real limit load for pipe structures. When more accurate analysis is necessary, the two hypotheses should be removed as discussed by some authors in the work of Mroz *et al.* (1995). Hence, it should be understood that the obtained limit load solution in this paper, instead of a real load-bearing limit, actually concerns a starting point of unlimited plastic flow of structure. The curved pipe has been considered to be of circular cross section and constant thickness. This is not an unreasonable assumption since hot forged pipe elbows can be manufactured to close dimensional tolerances. Finally, it may be necessary to note that the present work is concerned with a general pipe elbow assemblage rather than a complete pipe structure.

Finding the exact limit solution is generally very difficult due to the complexity of curved pipes. Two fundamental theorems could be used in classical limit analysis, Martin (1975), and Chen & Han (1988):

1) The lower bound theorem: Any stress distribution throughout a structure which is internally and externally in equilibrium and does not violate the yield condition anywhere, corresponds to a lower bound of limit load. This approach lays in finding an optimal static stress field.

2) The upper bound theorem: Any kinematically admissible strain distribution may correspond to an upper bound of limit load. The solution is obtained by equating the internal plastic dissipation to the external loading power in a postulated compatible mechanism of deformation. This approach lays in finding an optimal kinematic velocity field.

Obviously, if the lower bound and upper bound of solution coincide, which usually happens as the optimal fields are really found, the exact solution is obtained. Based on these theorems, some analytical and experimental approaches were studied on pipe structures, among which are Marcal *et al.* (1961, 1967), Bolt & Greenstreet (1971), Spence & Findlay (1973), Calladine (1974), Griffiths (1979), Touboul *et al.* (1988), Kussmaul *et al.* (1995) and so on. A finite element computing code ELSA (Elbow & Structure Limit and Shakedown Analysis) has been developed in our laboratory using special pipe-elbow finite elements and a mathematical programming technique. This computing code is based on the kinematical method connecting an optimization procedure. A large number of numerical tests for various structures shows that its solution is very close to the exact one. A complete description of the ELSA code and its numerical verification are presented in Yan (1997a).

This paper does not give a detailed theoretical description but only provides a simple and effective means for limit analysis of pipe structures. Some available solutions are used for comparison. Based on both analytical and numerical work, the explicit formulae of computing limit load are presented in a separate loading case. Then a general solution for pipe-elbow assemblage under complex loading (in-plane bending, out-plane bending, axial loading and internal pressure) is also derived. The proposed solutions are verified by finite element calculations. By their precision and simplicity, these formulae may constitute useful tools for engineers because elastic plastic finite element analysis is generally troublesome due to the complexity of curved pipe geometry and loading.

2. Limit moment of elbow (without end constraint) under in-plane bending

A pipe bend (elbow) subjected to complex loading is shown in Fig. 1. In this section we consider only bending moment M_I in the symmetric plane. We will call this problem in-plane elbow. A geometric coefficient λ is used to characterise the curve extent of the elbow:

$$\lambda = Rh/r^2 \tag{1}$$

Generally $\lambda < 0.5$ corresponds to a highly curved pipe, while $\lambda \rightarrow \propto$ corresponds to a straight pipe.

Limit load indicates theoretically the maximum load-bearing capacity of structure beyond which the plastic collapse happens. Here we represent the limit bending moment by a limit factor α:

$$\alpha = M_I/M_I^b \tag{2a}$$

where M_I is the in-plane limit moment of the curved pipe; M_I^b is Bernoulli's straight pipe solution which does not include the effect of ovalization and warping:

$$M_I^b = 4h\left(r^2 + \frac{h^2}{12}\right)\sigma_y \tag{2b}$$

where σ_y is the yield limit of material. For a thin-walled pipe we have approximately

$$M_I^b = 4hr^2\sigma_y \tag{2c}$$

2.1. Calladine - Proposed solution

By using the elastic solution of Clark and Reissner (1951) and the yield criterion of Hodge (1961) as Eq. (3) (a kind of sandwich approximation of Mises' criterion for cylindrical shell), Calladine (1974) proposed a static solution as in Eq. (4a) for a highly-curved pipe. This solution is considered in literature to be very close to the exact one. Our finite element examination shows

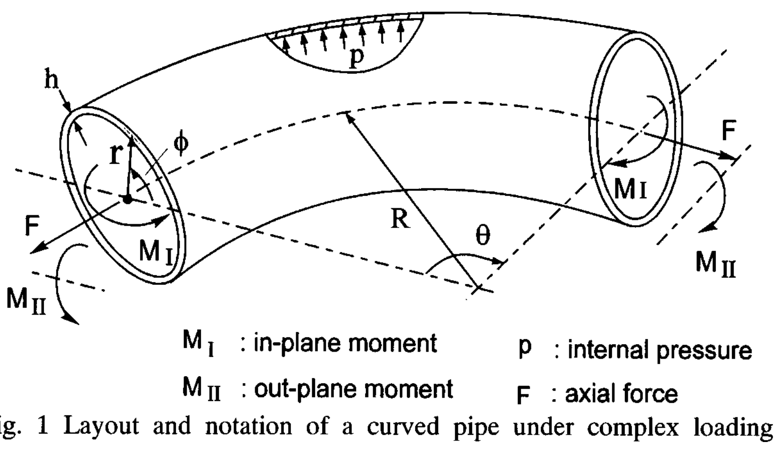

Fig. 1 Layout and notation of a curved pipe under complex loading

that it is valid as $\lambda < 0.7$ (Calladine gives this solution as $\lambda \leq 0.5$). For a slightly-curved pipe ($\lambda \geq 0.7$), Yan (1997a) proposed an approximate solution (4b) validated by numerical analysis. Hence, Eq. (4a, b) may construct a complete formula of in-plane elbow without end constraint (torus).

$$n_\theta^2 + \frac{3}{4} m_\varphi^2 = 1 \tag{3}$$

where n_θ and m_φ represent nondimensional traction and bending moment respectively.

$$\alpha_t = 0.934\lambda^{2/3} \qquad \lambda < 0.7 \tag{4a}$$

$$\alpha_t = \cos\left(\frac{\pi}{6\lambda}\right) \qquad \lambda \geq 0.7 \tag{4b}$$

We note that by using a static method as Calladine did, Desquines *et al.* (1997) recently deduced a lower bound solution. With yield criterion (3), their solution was presented as (5):

$$\alpha_D = \frac{1}{\sqrt{1 + \frac{0.3015}{\lambda^2}}} \tag{5}$$

It is interesting to make a comparison between Eqs. (4b) and (5). To this end, we take the series expansion of (4b) and (5) respectively:

$$\cos\left(\frac{A}{\lambda}\right) = 1 - \frac{A^2}{2!\lambda^2} + \frac{A^4}{8!\lambda^4} - \cdots \tag{6}$$

$$\frac{1}{\sqrt{1 + \left(\frac{B}{\lambda}\right)^2}} = 1 - \frac{B^2}{2\lambda^2} + \frac{1 \cdot 3B^4}{2 \cdot 4\lambda^4} - \cdots \tag{7}$$

where $A=\pi/6$ and $B=0.549$. Indeed when λ is large enough such as $\lambda > 0.8$, the difference between (4b) and (5) is very small, Table 1. However the solution (5) seems to overestimate limit moment for a highly-curved elbow in comparing with finite element solutions in Fig. 2.

2.2. Spence & Findlay's solution

Two theoretical approaches are employed and developed by Spence and Findlay (1973). The first one concerns the use of a linear elastic analysis in the space of stress resultant combining to the corresponding yield criterion; another is to manipulate the results of a creep analysis of bends to give an approximate bound. The obtained numerical results were fit into the following formula (8):

$$\begin{aligned} \alpha_{SF} &= 0.8\lambda^{0.6} && \lambda < 1.45 \\ \alpha_{SF} &= 1 && \lambda \geq 1.45 \end{aligned} \tag{8}$$

Above analytic formulae are examined by comparing with the finite element solution of ELSA. In Fig. 2 and Table 1, we see the Calladine-proposed solution (4) is in excellent agreement with the numerical solution for all λ values. It appears that the solution of Spence and Findlay (8) would generally constitute a lower bound.

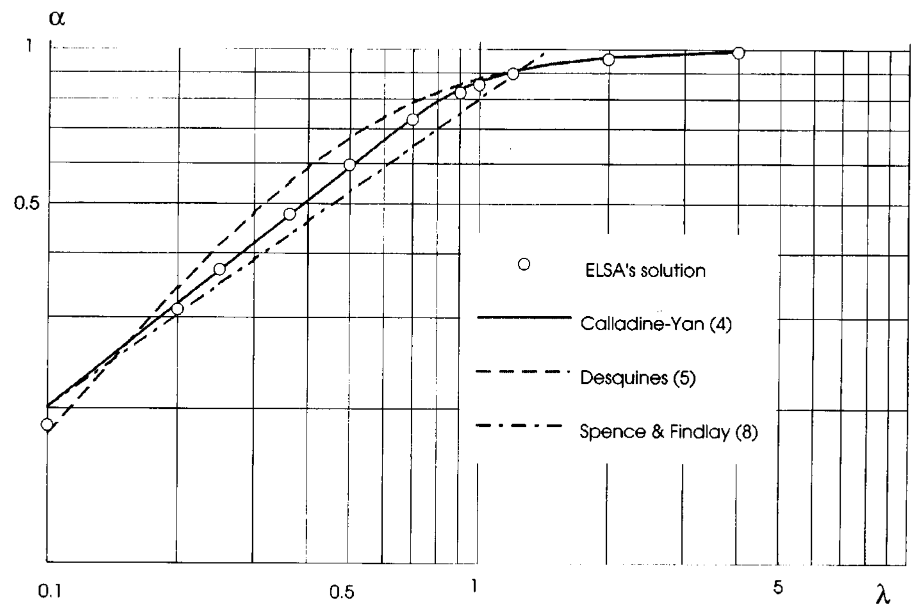

Fig. 2 Limit in-plane bending moment of elbow (torus); $\alpha_t = M/M_l^b$

Table 1 Limit factor of in-plane bending of elbow: $\alpha_t = M_l/M_l^b$

λ (Rh/r^2)	ELSA	Calladine Eq. (4a)	Yan Eq. (4b)	Desquines Eq. (5)	Spence & Findlay (8)
0.100	**0.185**	**0.2013**		0.1791	0.2001
0.200	**0.311**	**0.3196**		0.3422	0.3046
0.250	**0.371**	**0.3709**		0.4144	0.3482
0.300	-	**0.4188**		0.4794	0.3885
0.363	**0.476**	**0.4756**		0.5515	0.4355
0.400	-	**0.5074**	-	0.5888	0.4617
0.500	**0.595**	**0.5887**	(0.5000)	0.6732	0.5278
0.600	-	**0.6648**	(0.6428)	0.7377	0.5888
0.650	-	**0.7013**	(0.6927)	0.7639	0.6178
0.700	**0.729**	**0.7368**	**0.7330**	0.7868	0.6458
0.750	-	(0.7715)	**0.7660**	0.8069	0.6732
0.800	-	(0.8054)	**0.7933**	0.8244	0.6998
0.903	**0.821**	(0.8731)	**0.8365**	0.8544	0.7525
1.000	**0.852**	(0.9346)	**0.8660**	0.8765	0.8000
1.200	**0.895**	-	**0.9063**	0.9093	0.8925
1.450	-		**0.9355**	0.9352	0.9997
2.000	**0.954**		**0.9659**	0.9643	1.0
4.000	**0.985**		**0.9914**	0.9907	1.0

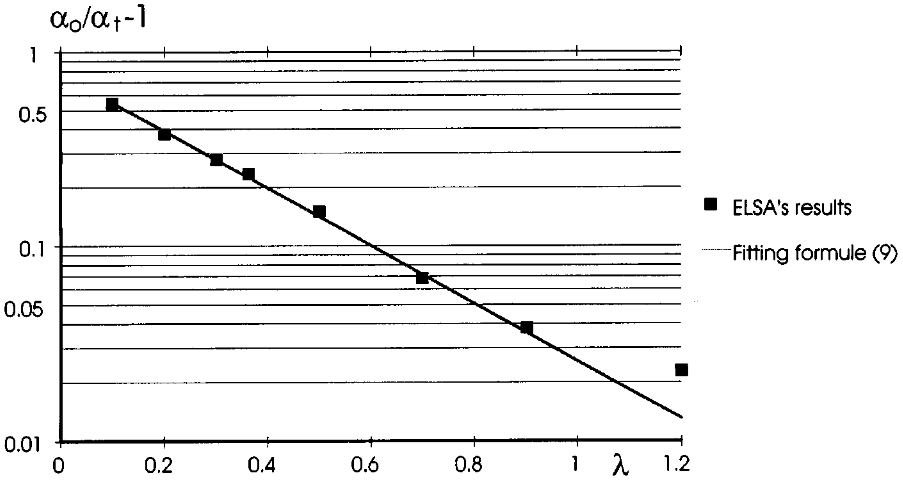

Fig. 3 Comparison of limit moment between pipe-elbow and torus

3. Limit moment for in-plane elbow connected straight pipes at its two ends

It should be pointed out that the above solutions are independent of the open angle θ of elbow if there is no constraint at the ends of elbow to reduce ovalization and warping. Actually it corresponds to a torus. For an in-plane elbow connected with straight pipes at its two ends which we will call simply pipe-elbow, the limit bending moment increases due to the stiffening effect of the straight pipes. Hence a correction seems to be necessary. First we consider respectively an elbow and a 180° pipe-elbow, the results of which are presented in Fig. 3. It is shown that the difference between an elbow and a pipe-elbow increases with λ decreasing. The following correcting formula as Eq. (9) may be used:

$$\alpha_0/\alpha_t = 1 + 0.77 \times 0.028^{0.95\lambda} \tag{9}$$

where α_0 is the solution of elbow joined to straight pipe at ends (pipe-elbow), α_t the solution of torus (by Eq. 4).

For a general pipe-elbow with the open angle θ less than 180°, the limit moment increases as the open angle of elbow decreases. Simply, we may use a linear interpolation between the solution of a 180° pipe-elbow and a general pipe-elbow:

$$\alpha_I = \max\left[\alpha_0,\ 1 + (\alpha_0 - 1)\frac{\theta}{\theta_0}\right] \tag{10}$$

where the reference angle θ_0=135° is based on the fact that the stress field of pipe-elbow with $\theta_0 >$ 135° changes little in comparing with the pipe-elbow of θ_0=135°, Gilles *et al.* (1996). The numerical examination of Eq. (10) is presented in Fig. 4.

It is seen that the predicted results by (10) may be overestimated in comparison with finite element's results somewhere such as for pipe-elbow of λ=0.903, because in this slightly-curved pipe the θ_0 of Eq. (10) may be smaller than 135°. Therefore, Gilles *et al.* (1996) suggested that the solution of Spence and Findlay (8) and interpolation formula, Eq. (10) might be combined to give a lower bound estimation of in-plane pipe elbow:

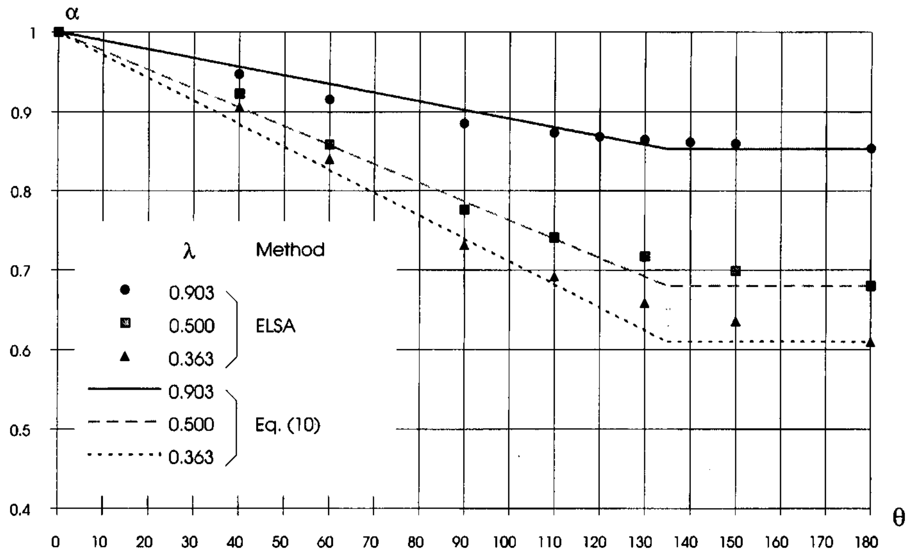

Fig. 4 Limit moment of pipe-elbow under in-plane bending, θ: open angle of elbow ($\alpha_I = M_I/M_I^b$)

$$\alpha_I = \max\left[\alpha_{SF},\ 1 + \frac{\theta}{135}(\alpha_{SF} - 1)\right] \tag{11}$$

So the exact solution can be generally enveloped by a quasi upper bound (using Eqs. 4, 9 and 10) and a lower bound (using Eqs. 8 and 11).

4. Limit moment for a 90° elbow under out-of-plane bending

In contrast with the above in-plane elbow, little information is available for an elbow subjected to out-of-plane bending, ref. Fig. 1. We will call this kind of problem as out-plane elbow in comparison with in-plane elbow. The internal resultant for in-plane elbow is bending alone while it becomes the one of a mixed mode of bending and torsion for out-plane elbow.

It is known that the plastic limit solution of the beam-bending model is different from that of the axis-torsion model. For a thin-walled straight pipe, one has:

Beam-bending limit:

$$M_I^b = 4hr^2\sigma_y \tag{2c}$$

Axis-torsion limit:

$$M_{II}^b = \frac{2}{\sqrt{3}}\pi h r^2 \sigma_y \tag{12}$$

Their ratio is

$$\frac{M_{II}^b}{M_I^b} = \frac{\pi}{2\sqrt{3}} \approx 0.9060 \tag{13}$$

This factor shows implicitly the decreasing tendency of limit moment for out-plane elbow in comparison with in-plane elbow.

Another point is that the largest ovalization happens in the centric section of in-plane elbow while it deviates little from centric section to loading end for out-plane elbow. It seems that the ovalization is more sensitive to bending than to torsion. This factor will lead to an increasing tendency of limit moment for out-plane elbow in comparison with in-plane elbow, especially for a highly-curved elbow.

Due to the interaction of these two factors, the plastic collapse of out-plane elbow does not happen always at centric section of the elbow as in-plane elbow does. It happens at nonloading end for slightly-curved pipe (e.g. $\lambda > 1$) while it passes toward another end (where the bending moment is applied) as λ decreases. So there is not a consistently proportional relation of limit bending moment between in-plane elbow and out-plane elbow. Generally, for a slightly-curved pipe (λ is large) the out-plane elbow has the limit moment lower than that of in-plane elbow. However, it will be contrary when λ is small because the effect of ovalization is now more important. So it is reasonable that we use two empirical formulae to present the limit moment of out-plane elbow respectively for $\lambda < 0.5$ and $\lambda \geq 0.5$ as Eq. (15). The numerical results are presented in Table 2 and Fig. 5. Here we use the axis - torsion solution (12) as reference because it is actually an asymptotic solution of out-plane bending as $\lambda \to \infty$.

- Defining

$$\alpha_{II} = M_{II}/M_{II}^{b} \tag{14}$$

where M_{II}^{b} is Bernoulli's axis-torsion solution as (12).

- Proposed solution:

$$\alpha_{II} = 1.1\lambda^{0.6} \quad \lambda < 0.5 \tag{15a}$$

$$\alpha_{II} = 0.9\lambda^{1/3} \quad 1.4 \geq \lambda \geq 0.5 \tag{15b}$$

Table 2 Limit factor of out-plane elbow (90°)

λ (Rh/r^2)	**ELSA**	Proposed solution (15a)	Proposed solution (15b)
0.100	0.281	0.2763	
0.200	0.431	0.4188	
0.250	0.480	0.4788	-
0.363	0.615	0.5989	(0.642)
0.500	0.713	(0.7257)	0.7143
0.550	-	(0.768)	0.7374
0.600	0.763	(0.809)	0.7591
0.700	0.806	(0.888)	0.7991
0.800	-	-	0.8355
0.903	0.873		0.8699
1.00	0.899		0.9000
1.20	0.939		0.9564
1.40	-		1.0
1.50	0.980		-
2.00	1.000		

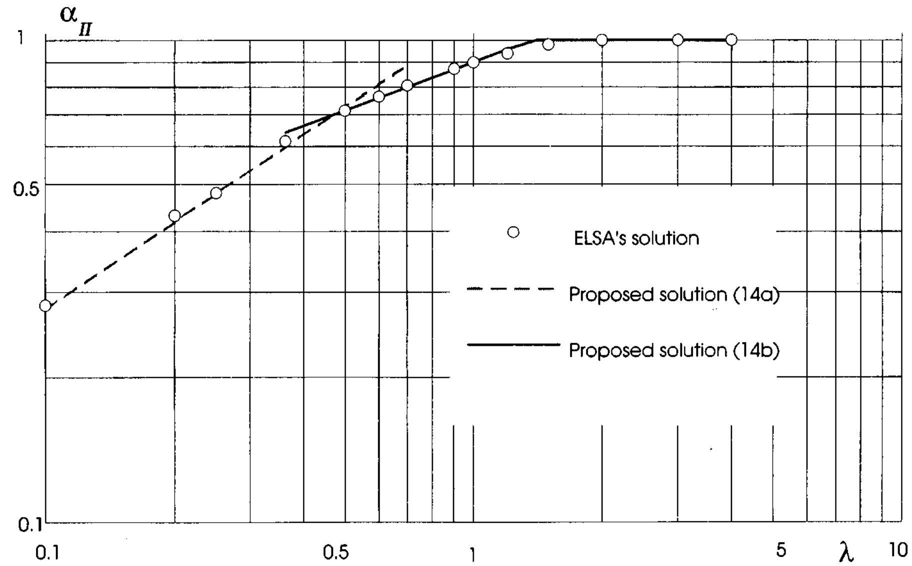

Fig. 5 Limit factor of out-plane elbow (90°); $\alpha_{II}=M/M_{II}^{b}$

Eq. (15) is proposed on the basis of qualitative analysis and an empirical fitting of numerical results of ELSA. Because the solution of ELSA has been proved to be very accurate at least for single straight pipe or single elbow, the proposed solution can be applied with a good engineering precision. The out-plane elbow connected to straight pipes at two ends of the elbow were studied by Save *et al.* (1995), Yan (1997a) and other authors. According to their results, it is suggested that Eq. (9) may be used to estimate the stiffening effect of straight pipe for out-plane-elbow if no data of higher precise are available. The obtained results appear conservative. More investigation seems necessary.

It is interesting to compare two types of elbows (in-plane bending and out-of-plane bending). For the sake of simplicity, we consider a 90° elbow without the end-constraint effect, subjected respectively to in-plane bending and out-of-plane bending. The results of the comparison are presented in Fig. 6. It is shown that in-plane bending is more dangerous than out-plane bending for a highly-curved elbow while it is the opposite for a slightly-curved pipe. They are approximately equivalent when λ is about 0.7. This result has a practical meaning. It is well known that the in-plane elbow is much easier to analyse than the out-plane elbow due to the symmetry. The deformation mode of the latter is more complicated and the whole structure must be discretized in numerical calculation. So from the point of view of practical application, it is simple and generally safe to apply the results of in-plane bending to out-of-plane bending for a highly curved elbow (e.g., $\lambda \leq 0.5$).

5. Limit internal pressure of curved pipe

The limit pressure of a pipe bend is reduced in comparison with a straight pipe due to the nonuniform circular stress. By solving Laplace's equation of a curved pipe, one could easily obtain an approximate analytic solution (16), which has been already recommended in the ASME code:

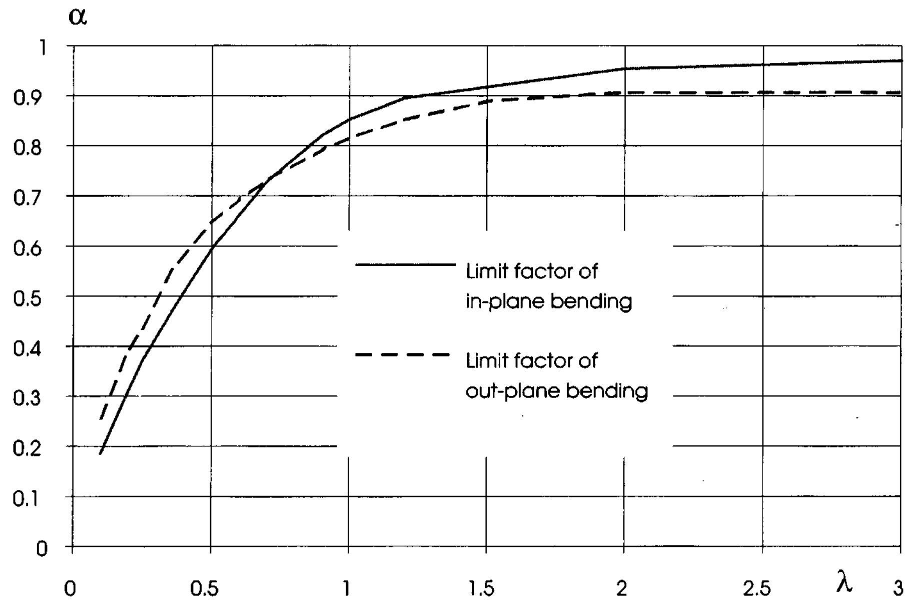

Fig. 6 Comparison of limit moments between in-plane and out-plane bending ($\alpha = M/M_l^b$)

Table 3. Plastic limit pressure of curved pipe $\alpha_l^p = p_l/p_l^*$

r/R	λ (Rh/r^2)	Numerical solution	Analytical solution, Eq. (16)
0.221	0.903	0.881	0.875
0.4	0.5	0.751	0.750
0.551	0.363	0.609	0.620

Note: mean radius r=30, thickness h=6 mm

$$\alpha_l^p = p_l/p_l^* \qquad p_l^* = \frac{1-r/R}{1-r/2R}\,\frac{\sigma_y h}{r} \tag{16}$$

For numerical examination, we consider a curved pipe as a torus under internal pressure. The results of plastic limit pressure are presented in Table 3. We note that neither analytical nor numerical solutions include the effect of axial force. However, in the case of considering the axial force effect of internal pressure, the numerical examination suggests that Eq. (16) is still usable as an approximation.

6. Pipe-elbow structure under complex loading

We consider now a pipe-elbow system subjected to a general bending moment M, internal pressure p and axial force F. If F is in self-equilibrium to internal pressure (as with an end-closed pipe), no bending moment is caused by the axial force. Otherwise, the added moment by axial force F should be taken into account, as will be shown later. Clearly we can always decompose a general bending moment into its two components as in-plane bending M_I and out-plane bending M_{II}:

$$M = \sqrt{M_I^2 + M_{II}^2} \tag{17}$$

Its collapse limit is

$$M^o \subset (M_I^o, M_{II}^o)$$

where M_I^o and M_{II}^o are the limit bending moments of in-plane and out-plane elbows, respectively, without pressure and axial force. For in-plane bending, $M^o = M_I^o$; for out-plane bending $M^o = M_{II}^o$. We propose using the following relation (18) to investigate the effect of internal pressure and axial force on limit bending moment.

$$\bar{\alpha} = \sqrt{\left(\frac{M_I}{M_I^o}\right)^2 + \left(\frac{M_{II}}{M_{II}^o}\right)^2} = \bar{\alpha}(p,\ F) \tag{18}$$

Obviously, if p=0 and F=0, we should have $\bar{\alpha}$=1.

$$\left(\frac{M_I}{M_I^o}\right)^2 + \left(\frac{M_{II}}{M_{II}^o}\right)^2 = 1 \tag{19}$$

This shows that the interaction of in-plane bending and out-plane-bending is a cycle. This relation can be taken as an equivalent exchange between in-plane and out-plane bending. A comparison with finite element calculation is presented in Fig. 7.

In order to investigate $\bar{\alpha}(p, F)$ in Eq. (18), i.e., the effect of internal pressure and axial force on bending limit, we first consider a static model, as shown in Fig. 8, which has been used for straight pipe analysis by Larson *et al.* (1975), Yan (1997a). For the present pipe elbow problem, the moment of torsion need not be considered since it has been included in out-plane bending.

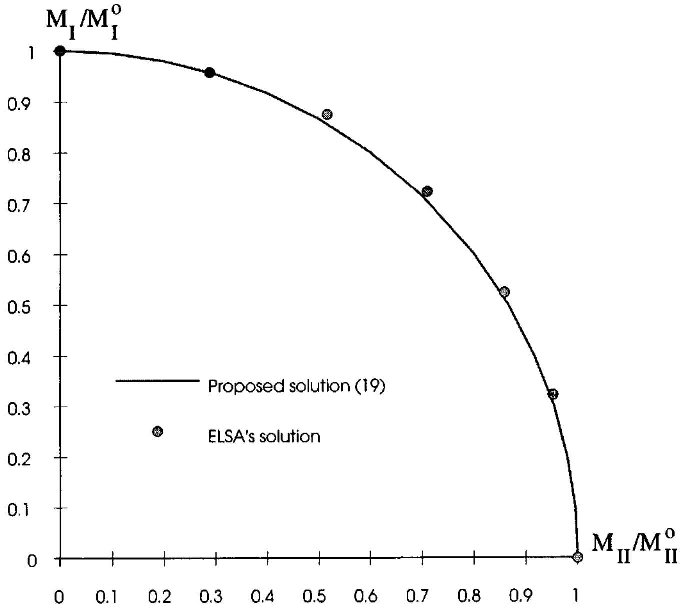

Fig. 7 Interaction of in-plane and out-plane bending (λ=0.5)

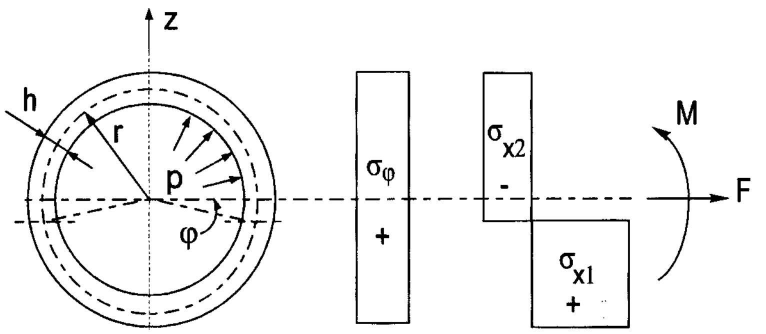

Fig. 8 Static stress distribution of a straight pipe model

We define:

$$n_x = \sigma_x / \sigma_y = F / F_l, \quad F_l = 2\pi r h \sigma_y \tag{20}$$

$$n_\varphi = \sigma_\varphi / \sigma_y = p / p_l, \quad p_l = \sigma_y h / r \tag{21}$$

where σ_y is the yield limit of material, σ_x and σ_φ are respectively axial stress and circumferential stress. F_l and p_l are respectively collapse limit of axial force and internal pressure of straight pipe. Mises' criterion becomes:

$$n_x^2 + n_\varphi^2 - n_x n_\varphi = 1 \tag{22}$$

Its solution is

$$n_{x1} = \frac{n_\varphi + \sqrt{4 - 3n_\varphi^2}}{2}, \quad n_{x2} = \frac{n_\varphi - \sqrt{4 - 3n_\varphi^2}}{2} \tag{23}$$

However, this static model is not directly useable for curved pipe because the circumferential stress field σ_φ is not uniform due to the ovalization and warping of the cross section, and also due to the difference of areas between the intrados part and the extrados part of a curved pipe, according to Yan (1997a). Besides, the circular stress distribution due to internal pressure is not uniform and it leads to different behaviours between closing-elbow bending (M_l in the direction of closing the elbow) and opening-elbow bending (M_l in the direction of opening the elbow). The situation is indeed quite complicated.

As an approximation, we still apply this static model but in the meanwhile introduce a coefficient C_0 that represents the effect of ovalization and warping of a curved pipe. Besides, a modification is necessary owing to the fact that the plastic limit pressure of a curved pipe is reduced from (21) to (16). To take this effect into consideration we replace n_φ in (22-23) by

$$n_\varphi^* = \frac{\sigma_{\varphi\max}}{\sigma_y} = \frac{p}{p_l^*}, \quad p_l^* = \sigma_y \frac{h}{r} \frac{1 - r/R}{1 - r/2R} \tag{24}$$

Now we can approximately determine the limit moment of a curved pipe by a simple integration:

$$\begin{aligned} M &= C_0 \int \sigma_\varphi z \, dS \\ &= 4 C_0 r^2 h \sigma_y \frac{\sqrt{4 - 3n_\varphi^{*2}}}{2} \cos\left[\frac{n_\varphi^* - 2n_x}{\sqrt{4 - 3n_\varphi^{*2}}} \frac{\pi}{2} \right] \end{aligned} \tag{25}$$

Since C_0 is numerically equal to the reduced factor α in (4) of a curved pipe relative to straight pipe, one can write:

$$\bar{\alpha} = M/M_I^o, \quad M_I^o = 4C_0 r^2 h\, \sigma_y \tag{26}$$

where M_I^o is the limit moment of a curved pipe under in-plane bending. The modification for a general pipe-elbow has been discussed in §8.3 and §8.4. Considering (18) we have a general solution as follows:

$$\bar{\alpha} = \frac{M}{M^o} = \sqrt{\left(\frac{M_I}{M_I^o}\right)^2 + \left(\frac{M_{II}}{M_{II}^o}\right)^2} = \frac{\sqrt{4-3n_\varphi^{*2}}}{2} \cos\left[\frac{n_\varphi^* - 2n_x}{\sqrt{4-3n_\varphi^{*2}}} \ \frac{\pi}{2}\right] \tag{27}$$

On the other hand, by taking $n_\varphi^* = n_\varphi$, n_φ being defined by Eq. (21), we obtain immediately the limit solution of thin-walled straight pipe, which has a formula completely similar to Eq. (27).

For a very thin-walled close-ended pipe elbow: $n_\varphi^* \approx 2n_x$, Eq. (27) can be simplified further:

$$\bar{\alpha} = \frac{\sqrt{4-3n_\varphi^{*2}}}{2} \tag{28}$$

Another analytic solution concerning the effect of internal pressure on the limit bending moment of an in-plane elbow was proposed by Goodall (1978). He adopted the two-limits yield surface (named limited interaction) assuming that there is no interaction between bending and stretching. The solution was obtained for highly-curved pipes.

$$c^* = \frac{2c}{(3\lambda)^{2/3}} = \left(\frac{1-n_\varphi^*}{2}\right)^{1/3} \tag{29}$$

where n_φ^* is defined in Eq. (24) above. By comparing qualitatively (29) with (27) or (28), one may take c^* as equivalent to $\bar{\alpha}$. This formula means the maximum reduction of limit bending moment due to internal pressure is about 21%.

7. Numerical examination

We consider a pipe elbow structure under complex loading as shown in Fig. 9. Different combinations of loading are dealt with. The in-plane bending moment is in the direction of closing the elbow. The numerical results of finite elements are compared with the analytic solution (28) in Fig. 10 and good agreement is observed. It seems that the yield criterion when used by Goodall is too simple and the solution is deviates from present solutions by using Mises' criterion when $p/p_l^* > 0.6$.

The proposed method is also examined in a 40° pipe-elbow, as shown in Fig. 11, subjected to in-plane moment M_I, internal pressure p and axial force F. Four loading conditions are considered:

1) $p=F=0$, $M_I = ?$
2) $p=15$ Mpa, $F=\pi r^2 p$, $M_I = ?$
3) $p=15$ Mpa, $F=1.5\pi r^2 p$, $M_I = ?$
4) $p=15$ Mpa, $F=2\pi r^2 p$, $M_I = ?$

Case 1 and 2 are simple because the internal pressure and axial force are in equilibrium, and we

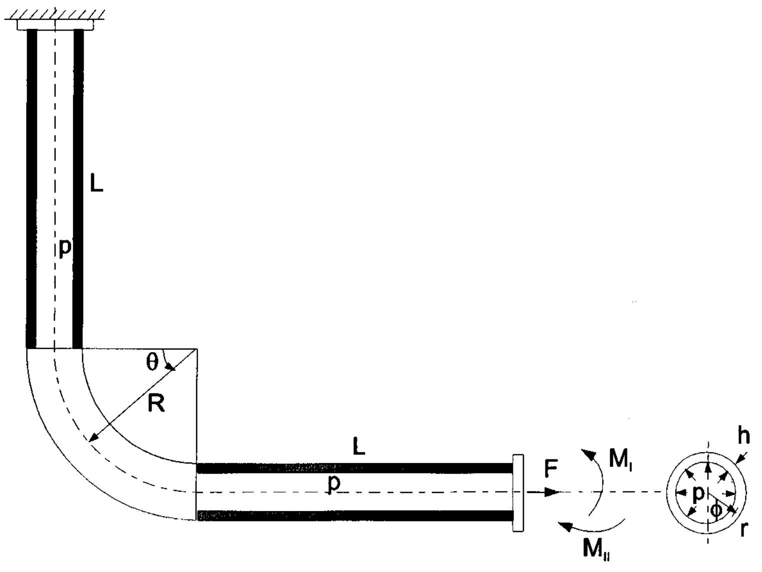

Fig. 9 Elbow with prolongation under combined loading (R=750, h=60, r=300, L=2000 mm, σ_y=160 MPa)

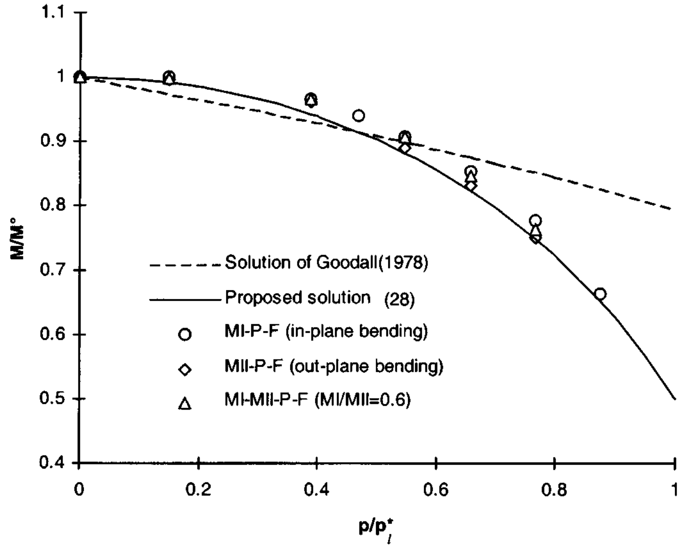

Fig. 10 Interaction solution of general bending and internal pressure of close-ended pipe elbow (λ=0.5)

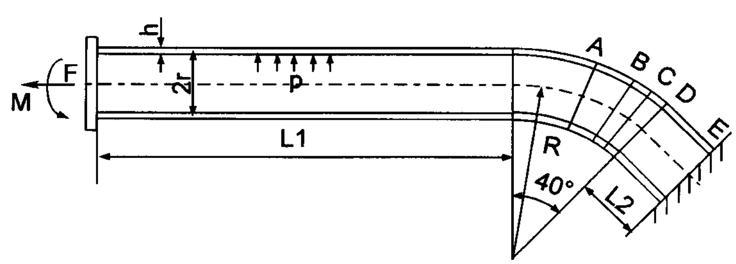

Fig. 11 Geometry and calculating model (length in mm) (R=1354, h=62.5, r_i=393.7, L_1=3330, L_2=300, λ=0.469, θ=40°, σ_y=160 MPa)

Table 4. The effect of pressure and axial force on limit bending moment (p=15 MPa)

No. of case Axial force	1 F=0, p=0	2 $F=\pi r^2 p$	3 $F=1.5\pi r^2 p$	4a $F=2\pi r^2 p$	4b $F=2\pi r^2 p$	4c $F=2\pi r^2 p$
Numerical solution	**M_l=6505**	**M_l=5203**	**M_l=4336**	**M_l=2504**	**M_l=2504**	**M_l=2504**
Prediction Eq. (2, 4, 9, 10, 31)	**M_l=6490.7**	**M_l=4630**	**M_l=4108**	**M_l=2860**	**M_l=2901**	**M_l=2483***
Relative error	-0.2%	-12.3%	-5.1%			-0.8%
Failure section	A (centre of elbow)	A (centre of elbow)	B (1/4 elbow)	C (1/8 elbow)	D (exit of elbow)	E (fixed end of pipe)

*The stiffening effect of fixed end is estimated as 8%

know that the plastic collapse is initiated at the centre of the elbow. For Cases 3 and 4, surplus axial force will cause the bending moment at the right part of the structure, but we do not know exactly where the plastic collapse commences. In this situation, we can select a series of possible mechanisms (locations) for the prediction. According to the upper bound limit theorem, the minimum of the obtained limit load is the best approximate solution. The predicted results and the finite element solutions of ELSA are presented in Table 4. Here we take Cases 3 and 4c as examples to illustrate the predicting method.

- Case 3: $F=1.5\pi r^2 p$

For Case 1 without p and F, we obtain limit bending moment M_l^*=6490.7 KNm by using Eqs. (2, 4, 9, 10). Now for Case 3 one needs only to consider the effect of pressure and axial force. Assuming the plastic collapse is initiated at section B(θ=30°), one has

$$F = \pi r_i^2 p(1+0.5\cos\theta),\ n_x = F/F_l = 0.392$$
$$p=15\text{ MPa},\ n_\varphi^* = p/p_l^* = 0.783$$

Substituting n_x, n_φ^* into Eq. (27):

$$M_l^l = \bar{\alpha} M_l^* = 0.753\times 6490.7 = 4770.7\text{ kNm}$$

The bending moment due to supplemental axial force at section B is

$$M_l^F = 0.5\times \pi r_i^2 p\times R(1-\cos\theta) = 662.5\text{ kNm}$$

So the limit bending moment is finally found to be:

$$M_l = M_l^l - M_l^F = 4108\text{ kNm}$$

- Case 4c: $F=2\pi r_i^2 p$

We assume that the failure mechanism forms near section E where the bending moment attains the maximum. At first the stiffening effect of the fixed end is not considered, so it becomes a straight pipe problem. Its bending limit is given by (2b)

$$M_l^b = 7236\text{ kNm}$$

Now we consider the effect of internal pressure and axial force:

$$F=\pi r_i^2 p(1+\cos 40),\ n_x=F/F_l=0.483$$
$$p=15\ \mathrm{MPa},\ n_\varphi=p/p_l^0=0.6374$$
$$\bar{\alpha}=\frac{\sqrt{4-3n_\varphi^2}}{2}\cos\left[\frac{n_\varphi-2n_x}{\sqrt{4-3n_\varphi^2}}\frac{\pi}{2}\right]=0.794$$
$$M_l^0=\bar{\alpha}M_l^b=\ 5746\ \mathrm{kNm}$$

Considering the stiffening effect of the fixed end (about 8% according to numerical analysis) the limit bending moment near section E may be estimated as:

$$M_l^l\approx\ 1.08\times M_l^0=6205\ \mathrm{kNm}$$

The bending moment due to surplus axial force at section E is

$$M_l^F=\pi r^2 p[R(1-\cos 40)+L_2\sin 40]=3722\ \mathrm{kNm}$$

So the limit bending moment is finally predited to be:

$$M_l=M_l^l-M_l^F=2483\ \mathrm{kNm}$$

By comparing the limit bending moments corresponding to different mechanisms (see cases 4a, 4b in Table 4), we know the collapse happens very near the fixed end. The predicted results are in good agreement with the finite element solution, although the predicting calculation is relatively simple and sketchy.

This encouragingly shows that the above predicting method can be applied to estimate the collapse limit of a general pipe elbow subjected to complex loading with good precision in the condition of selecting well the collapse mechanism (here the mechanism means a location where the plastic collapse happens initially). This can be realized by a simple qualitative analysis. For a complex problem, one needs to select some different mechanism to find the minimum limit value. We note also that the effect of a traversing load (shear stress in cross section) and radial stress are not included in the above analytic calculation, which is generally considered small. However, for a short pipe with a large diameter, a correction to consider the effect of shear stress is necessary.

8. Conclusions

We have presented a predicting method for the limit loading estimation of a general pipe-elbow system under complex loading. Some explicit predicting formulae are proposed on the basis of theoretical and numerical analysis. They can be easily applied to a general pipe-elbow structure with good engineering precision. The upper bound and lower bound of limit bending moment are suggested to satisfy different situations. The numerical examples show the application of the proposed method which is useful for engineers who wish to estimate the plastic collapse limit of pipe structures by a simple calculation.

However, it should be pointed out that the present study is carried out under some idealised conditions. The limit solution may not represent real sustainable load for real pipe structures. The negligee of the strain hardening effect of material should be introduced somewhat conservatively. Nevertheless, the experimental results of Griffiths (1979) and Kussmaul *et al.* (1995) showed that this safe margin due to material model is not important for pipe structures. In fact, despite the displacement in the plastic state, the limit bending stress of elbows could be well characterised by the initial yield limit or $\sigma_{0.2}$ of material. However, the large deformation of a pipe section may have definitive influence on the ultimate load-bearing capacity. For example, the experimental results show that the loading level of the opening moment may overpass largely the analytic prediction, while only the in-plane closing mode has a real collapse limit. This is due to the different ovalization modes in different bending modes. The AMSE code (1989) has recommended a "twice slope of linear response" method to define the limit load. The limit solution given in this paper may be used as a theoretical parameter to characterize the limit state or to control excess deformation of pipe structures.

References

ASME Boiler and Pressure Vessel Code, Section III, Nuclear Power Plant Components, Subsection NB, Class 1 Components, 1989.

Bolt, S.E. and Greenstreet, W.L. (1971), "Experimental determination of plastic collapse loads for pipe elbows", *ASME paper 71-PVP-37.*

Calladine, C.R. (1974), "Limit analysis of curved tubes", *Journal Mechanical Engineering Science,* **16**(2), 85-87.

Chen, W.F. and Han, D.J. (1988), *Plasticity of Structural Engineers*, Springer-Verlag, N.Y.

Clark, R.A. and Reissner, E. (1951) "Bending of curved tubes", *Advances in Applied Mechanics,* **2**, 93-122.

Corona, E. and Kyriakides, S. (1988), "On the collapse of inelastic tubes under combined bending and pressure", *Int. J. Solids Structures,* **24**(5), 505-535.

Desquines, J. Plancq, D. and Wielgosz, Ch. (1997), "In-plane limit moment for an elbow lower bound analytical solution and finite element processing by elastic compensation method", *Int J. Ves. Piping,* **71**, 29-34.

Edmunds, H.G. and Beer, F.J., (1961), "Notes on incremental collapse in pressure vessels", *Journal of Mechanical Engineering Science,* **3**(3), 187-199.

Gilles, Ph., Bois, C. and Nguyen, D.H. (1996), "J estimation scheme for surface cracked pipes under complex loading: Part I Theoretical basis, Part II Complex shaped elbow solution", *Proceedings of ECF11 Conference*, Poitiers, France.

Gokhfeld, D.A. and Cherniavsky, O.F. (1980), *Limit Analysis of Structures at Thermal Cycling,* Sijthoff and Noordhoff, The Netherlands,

Goodall, I.W. (1978), "Lower bound limit analysis of curved tubes loaded by combined internal pressure and in-plane bending moment", CEGB-RD/B/N4360, UK.

Griffiths, J.E. (1979), "The effect of cracks on the limit load of pipe bends under in-plane bending: experimental study", *Int. J. Mech. Sci.,* **21**, 119-130.

Hodge, P.G. Jr. (1961), "The Mises yield condition for rotationally symmetric shells", *Quarterly of Applied Mechanics,* **18**(4), 305-311.

Joch, J., Ainsworth, R.A. and Hyde, T.H. (1993), "Limit load and J-estimates for idealised problems of deeply cracked welded joints in plane-strain bending and tension", *Fatigue Fract. Engng Mater. Struct.,* **16**(10), 1061-1079.

Konig, J.A. (1987), *Shakedown of Elastic-Plastic Structure,* Elsevier and PWN-Polish Scientific Publishers.

Kussmaul, K., Diem, H.K. and Uhlmann, D. Kobes, E. (1995), "Pipe bend behaviour at load levels beyond design", *Transactions of the 13th Int. Conf. On Structural Mechanics in Reactor Technology (SMiRT 13)*, Porto Alegre, Brazil, August, 187-198.

Marcal, P.V. and Turner, C.E. (1961), "Elastic solution in the limit analysis of shells of revolution with special reference to expansion bellows", *Journal of Mechanical Engineering Science,* **3**(3), 252-257.

Marcal, P.V. (1967), "Elastic-plastic behaviours of pipe bends with in-plane bending", *Journal Strain*

Analysis, **2**(1), 84-90.
Martin, J.B. (1975), *Plasticity*, MIT Press, Cambridge.
Mroz, Z. Weichert, D. and Dorosz, S. (1995), "Inelastic behaviour of structures under variable loading", *Proceedings of EURMECH Colloquium 298*, Kluwer Academic Publishers.
Spence, J. and Findlay, G.E. (1973), "Limit loads of pipe bends under in-plane bending", *Proc. 2nd Int. Conf. on Pressure Vessel Technology,* **1**, ASME, 393-399.
Touboul, F., Ben Djedidia, M., Acker, D. (1988), "Stress index method of design: application to in-plane closing collapse of elbows", *Int. J. of Pres. Ves. and Piping,* **33**, 153-164.
Save, M. Jospin, R.J. and Nguyen, D.H. (1995), "Limit loads of pipe elbows", *Nuclear Science and Technology*, Report EUR 15696 EN, European Commission.
Yan, A.M. (1997a), "Contributions to the direct limit state analysis of plastified and cracked structures", PhD Thesis, University of Liege.
Yan, A.M. and Nguyen, D.H. (1997b), "Application of shakedown methods to piping", *Report of Project RAI-0235UK*, European Commission.
Yan, A.M. and Nguyen, D.H. (1997c), "Plastic collapse of curved pipes and shells by finite element method and mathematical programming technique", *Proceedings of IASS International Colloquium on Computation of Shell and Spatial Structures*, Taipai, 5-7 November.
Yan, A.M., Nguyen, D.H. and Jospin, R.J. (1997d), "Internal pressure effects on the limit load of elbows", *Proceedings of International Symposium on Shell and Spatial Structures, Singapore*, 10-14 November, 911-920.