Structural Engineering and Mechanics, Vol. 8, No. 3 (1999) 257-272
DOI: http://dx.doi.org/10.12989/sem.1999.8.3.257

Out of plane vibrations of thin-walled curved beams considering shear flexibility

V.H. Cortínez† and M.T. Piovan‡

Mechanical Systems Analysis Group, Universidad Tecnológica Nacional, 11 de Abril 461, 8000, Bahía Blanca, Argentina

R.E. Rossi‡†

Department of Engineering, Universidad Nacional del Sur, 8000, Bahía Blanca, Argentina

Abstract. In this paper a simple finite element is proposed for analyzing out of plane vibration of thin walled curved beams, with both open and closed sections, considering shear flexibility. The present element is obtained from a variational formulation governing the dynamics of a three-dimensional elastic body in which the stress tensor as well as the displacements are variationally independent. The element has two nodes with four degrees of freedom in each. Numerical examples for the first six frequencies are performed in order to assess the accuracy of the finite element formulation and to show the influence of the shear flexibility on the dynamics of the member.

Key words: beam; bridges; finite element; shear flexibility; thin walled; vibration.

1. Introduction

Thin walled curved beams are frequently used in several fields of engineering, in particular in highway bridge structures. Consequently, extensive research has been performed concerning the behavior of such members.

Much of the available publications dealing with the static and dynamic behavior of horizontally curved beams are summarized in survey articles (Mc Manus *et al.* 1969, Task Committee 1978, Chidamparam and Leissa 1993). Some of the relevant papers considering the out of plane dynamics of thin walled curved beams are cited below.

Culver (1967), Christiano and Culver (1969), and Shore and Chadhuri (1972) have determined the free vibration frequencies of horizontally curved beams by means of analytical solutions of the equations of motion. The dynamic response of a single span curved beam to moving loads was considered by Tan and Shore (1968). Chaudhuri and Shore (1977) and Yoo and Feherenbach (1981) analyzed the free vibration of horizontally curved beams using the finite element method. In this last paper the case of shear center and centroid not coincident was considered.

† Research Director
‡ Graduate Student
‡† Professor

The finite difference method for determining natural frequencies of box girder bridges was used by Heins and Sahin (1979). Recently Kang *et al.* (1996) proposed the differential quadrature method for determining natural frequencies of circularly curved thin-walled beams.

Free vibrations of continuously horizontally curved beams were studied by Snyder and Wilson (1992) by means of a closed form solution of the equations of motion.

The previously mentioned works have utilized the theoretical model developed by Vlasov (1961). This theory does not consider the influence of the shear deformability. However this effect is of importance when higher modes have to be analyzed or even in lower modes if the beam is deep.

For the case of straight thin walled beams, few papers are available dealing with the shear effect (Capuani, Savoia and Laudiero 1992, Cortínez and Rossi 1996, 1998). Even fewer works consider the out of plane vibration of shear deformable thin walled curved beams. Kawakami *et al.* (1995) studied the in plane and out of plane vibration of curved thin walled beams considering shear effects associated to the lateral motion although neglecting the warping shear deformability. Inversely, Fu and Hsu (1995) have analyzed the statics of thin walled beams taking into account the warping shear effect but not the bending shear deformability.

According to the authors' knowledge the only one study dealing with vibrations of curved thin walled beams considering the shear flexibility in a full form is that of Gendy and Saleeb (1994). However in their excellent paper the main objective was the development of a mixed finite element but no parametric analyses were done.

In this paper a curvilinear finite element for analyzing the out of plane dynamics of shear deformable thin walled curved beams, with both open and closed sections, is developed. This element may be considered a generalization of the straight beam element presented in an earlier paper by the authors (Cortínez and Rossi 1998). A Vlasov finite element is obtained as a particular case of the present one.

Using the current model the first six frequencies of vibration of curved beams with several boundary conditions are calculated and compared with the corresponding results given by the Vlasov theory in order to show the influence of the shear deformability on the dynamics of the beam.

2. Theory

2.1. Variational formulation

The present theoretical development closely follows to that performed by the authors in (Cortínez and Rossi 1998).

The dynamic behavior of a vibrating elastic body with imposed displacements on Su may be described by means of the variational equations

$$\int_V \sigma_{ij}\,\delta\varepsilon_{ij}\,dV + \int_V \rho\,\frac{\partial^2 u_i}{\partial t^2}\,\delta u_i\,dV = 0 \tag{1a}$$

$$\int_V \left(\varepsilon_{ij} - \frac{\partial E_c}{\partial \sigma_{ij}}\right)\delta\sigma_{ij}\,dV = 0 \tag{1b}$$

with i, j=x, y, z, and

$$\varepsilon_{xx}=\left(\frac{\partial u_x}{\partial x}+\frac{u_z}{R}\right)\frac{R}{R+\bar{z}},\qquad \varepsilon_{yy}=\frac{\partial u_y}{\partial y},\qquad \varepsilon_{zz}=\frac{\partial u_z}{\partial z}$$

$$\varepsilon_{xy}=\frac{1}{2}\left(\frac{\partial u_y}{\partial x}\frac{R}{R+\bar{z}}+\frac{\partial u_x}{\partial y}\right),\qquad \varepsilon_{yz}=\frac{1}{2}\left(\frac{\partial u_z}{\partial y}+\frac{\partial u_y}{\partial z}\right),$$

$$\varepsilon_{xz}=\frac{1}{2}\left[\left(\frac{\partial u_z}{\partial x}-\frac{u_x}{R}\right)\frac{R}{R+\bar{z}}+\frac{\partial u_x}{\partial z}\right]$$

$$E_C(\sigma_{ij})=\frac{-\mu}{2E}(\sigma_{ii})^2+\frac{1}{4G}\sigma_{ij}\sigma_{ij} \tag{2a-g}$$

where $\bar{y}$ and $\bar{z}$ are the principal axes, y and z are parallel to the first ones but having their origin at the shear center SC, x is tangent to the curved axes of the member as shown in Fig. 1, t: temporal coordinate, E_C: volumetric complementary energy, ε_{ij}: strain components in curvilinear coordinates, σ_{ij}: stress components, ρ: material density, E: Young modulus, G: transverse elasticity modulus, μ: Poisson ratio and R: curvature radius.

The Eq. (1a) corresponds to the D'Alembert principle of virtual works and Eq. (1b) constitutes the Hooke's law expressed in a variational form. In Eq. (1) σ_{ij} and u_i are variationally independent. The displacement field is subjected to the restraint

$$\delta u_i=0,\quad (u_i=\bar{u}_i)\quad \text{on } Su \tag{3}$$

2.2. Thin walled curved beam model

In order to obtain a one dimensional dynamical theory for the member shown in Fig. 1, it is necessary to propose certain distributions of displacements and stresses over the cross section of the beam. To do this, the following assumptions are made: (a) the cross section is rigid in its own plane; (b) the stress tensor results of the composition of two states: the first one is the Saint Venant pure torsion state and the second one is a membranal state; (c) the variation of curvature through the thickness is neglected.

From the first assumption, it is possible to write the displacement field for the out of plane

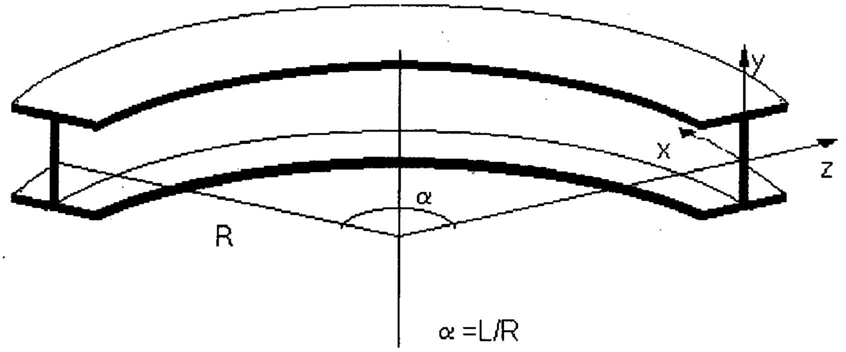


Fig. 1 Thin-walled curved beam

motion as

$$u_x = -\theta_z(x,t)(y-y_0) + \omega\left(\theta(x,t) + \frac{\theta_z(x,t)}{R}\right) + [\xi(y,z) - \omega(s)]\left(\frac{\partial\phi(x,t)}{\partial x} + \frac{\theta_z(x,t)}{R}\right) \quad \text{(4a)}$$

$$u_y = v_S(x,t) - \phi(x,t)z, \qquad u_z = y\,\phi(x,t) \quad \text{(4b-c)}$$

where v_S is the transverse displacement of the shear center, ϕ is the torsion rotation, θ_z is the flexural rotation around the centroidal axis $\bar{z}$, $\omega(s)$ is the sectorial coordinate defined according to Vlasov, s is the curvilinear coordinate shown in Fig. 2, θ is a measure of warping along the beam , ξ is the Saint Venant warping function.

Expressions (4) coincide with those of Vlasov (Yang and Kuo 1986) when there are verified the following internal restraints

$$\theta_z = \frac{\partial v_S}{\partial x}, \qquad \theta = \frac{\partial\phi}{\partial x} \quad \text{(5)}$$

However these restraints are not considered in the present paper. Accordingly, as shown by Cortínez and Rossi (1998), the components of shearing strain in the middle-line are not neglected here.

On the other hand, from assumptions (b) and (c) one may express, the non vanishing components of the stress tensor for thin walled open section beams, according to Vlasov, as

$$\sigma_{xx} = \frac{M_z(x,t)}{I_z}\bar{y} + \frac{B(x,t)}{C_W}\,\omega(s) \quad \text{(6a)}$$

$$\sigma_{xs} = \frac{T_W(x,t)}{C_W}\lambda_W(s) - \frac{Q_y(x,t)}{I_z}\lambda_z(s) - 2\frac{T_{SV}(x,t)}{J}n \quad \text{(6b)}$$

with

$$\lambda_W(s) = \int_0^s \omega(s)\,ds; \; \lambda_z(s) = \int_0^s \bar{Y}(s)\,ds \quad \text{(7)}$$

where n is the curvilinear coordinate shown in Fig. 2, $\bar{Y}(s)$ corresponds to the middle line of the cross section, M_z is the bending moment about $\bar{z}$ axis, B is the bimoment, T_W is the flexural torsional torque, Q_y is the shear force pointing to y direction, T_{SV} is the Saint Venant torsion torque, I_z is the inertia moment respect to $\bar{z}$, C_W is the warping constant and J is the torsion constant.

It must be noted that these expressions coincide with those of a straight beam according to assumption (c). The present variational formulation requires that the given stress resultants are considered variationally independent functions.

Substitution of expressions (4) and (6) in (1) allows, after integrating respect to y and z, to obtain new variational equations in a one dimensional form, whose unknowns are given by θ_z, θ, v_S, ϕ, M_z, B, Q_y, T_W and T_{SV}.

2.3. One dimensional variational equation

Replacing (4) and (6) in (1b), as explained above, integrating respect to y and z, and neglecting terms of higher order according to assumption (c), one arrives to

$$M_z = -EI_z\left(\frac{\partial\theta_z}{\partial x} - \frac{\phi}{R}\right), \qquad B = EC_W\left(\frac{\partial\theta}{\partial x} + \frac{1}{R}\frac{\partial\theta_z}{\partial x}\right),$$

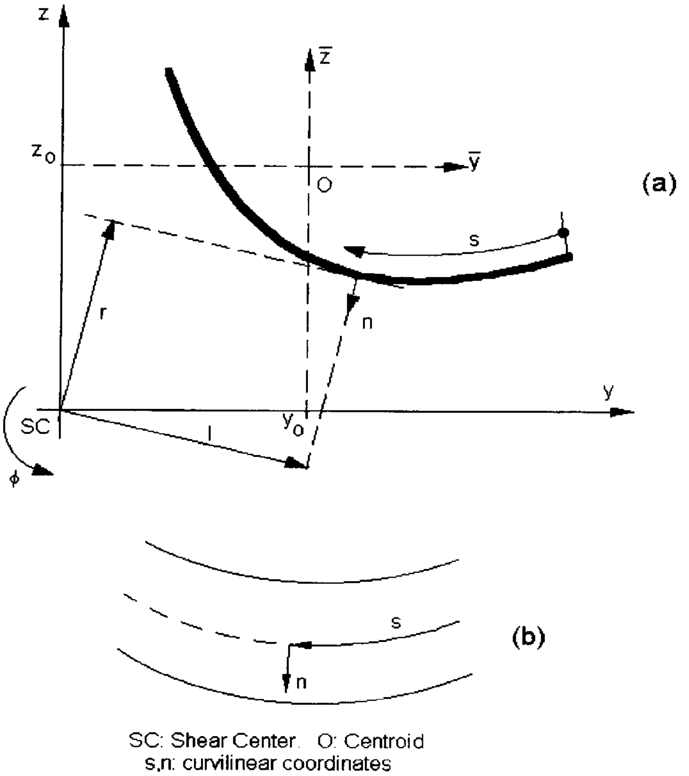


Fig. 2 General cross sectional geometry

$$Q_y = GAk_z\left(\frac{\partial v_s}{\partial x} - \theta_z\right), \qquad T_W = GI_R k_W\left(\frac{\partial \phi}{\partial x} - \theta\right), \qquad T_{SV} = GJ\left(\frac{\partial \phi}{\partial x} + \frac{\theta_z}{R}\right) \tag{8a-e}$$

where

$$A = \int_0^m e\, ds, \qquad I_R = \int_0^m e\left(\frac{d\omega}{ds}\right)^2 ds,$$
$$k_z = \frac{I_z^2}{Ae\int_0^m \lambda_z^2 ds}, \qquad k_W = \frac{C_W^2}{I_R e\int_0^m \lambda_W^2 ds} \tag{9a-d}$$

Here e and A are the thickness and the area of the section, respectively. It must be noted that k_z is a correction factor due to flexure shear and k_W is a correction factor due to warping shear (Cortínez and Rossi 1998). These coefficients are inherent of the present formulation.

On the other hand, substituting (4) and (6) in (1a) the following variational equation is obtained

$$L_K + L_M = 0 \tag{10}$$

where

$$L_K = \int_0^L \left[-M_z\,\delta\left(\frac{\partial \theta_z}{\partial x} - \frac{\phi}{R}\right) + B\,\delta\left(\frac{\partial \theta}{\partial x} + \frac{1}{R}\frac{\partial \theta_z}{\partial x}\right) + Q_y\,\delta\left(\frac{\partial v_s}{\partial x} - \theta_z\right)\right.$$

$$+T_W \delta\left(\frac{\partial \phi}{\partial x}-\theta\right)+T_{SV} \delta\left(\frac{\partial \phi}{\partial x}+\frac{\theta_z}{R}\right)\Bigg] dx \tag{11a}$$

$$L_M=\int_0^L \rho\left[I_z \frac{\partial^2 \theta_z}{\partial t^2} \delta \theta_z+C_W\left(\frac{\partial^2 \theta}{\partial t^2}+\frac{1}{R} \frac{\partial^2 \theta_z}{\partial t^2}\right) \delta \theta\right.$$

$$\left.+A \frac{\partial^2}{\partial t^2}\left(v_S-z_0 \phi\right) \delta v_s+\frac{\partial^2}{\partial t^2}\left(-A z_0 v_s+I_S \phi\right) \delta \phi\right] dx \tag{11b}$$

where y_0, z_0 are the centroidal coordinates, I_S is the polar moment respect to the shear center, L_K and L_M constitute the virtual work done by the elastic and inertial forces respectively.

In deriving the above equations it has been supposed, according to assumption (c), that $1 \pm c/R \simeq 1$, where c is a characteristic length of the cross section.

The curvature radius arising in Eqs. (10) and (11) corresponds to the centroidal line. However in the framework of assumption (c), and following the Vlasov reasoning, it is also possible to refer the radius R to the line of shear centers with an insignificant error. On the other hand in deriving (10) and (11b) it has been supposed that y_0=0, but in view of the assumption (c), according to Vlasov (1961) and Yoo and Feherenbach (1981), these equations are applicable to beams with asymmetric sections while the curvature is not large. The present formulation was developed for open cross sections. However, it is also valid for closed sections provided the proper choice of the cross sectional properties (Krenk and Gunneskov 1981). On the other hand it is applicable to members with variable cross section and radius of curvature.

3. Finite element formulation

To determine natural frequencies of free vibrations, a finite element based on the present theoretical model is developed. A uniform two-node element, of length l_e, with four degrees of freedom per node, corresponding to the generalized displacements, is considered. Accordingly the nodal displacement vector may be written as

$$p=[v_s^{(1)}, \theta_z^{(1)}, \phi^{(1)}, \theta^{(1)}, v_s^{(2)}, \theta_z^{(2)}, \phi^{(2)}, \theta^{(2)}]^T \tag{12}$$

The displacement field in the element is assumed to be given by

$$v_s=b_0+b_1 \bar{x}+b_2 \bar{x}^2+b_3 \bar{x}^3, \qquad \theta_z=b_1+\frac{\chi_1 b_3}{2}+2 b_2 \bar{x}+3 b_3 \bar{x}^2,$$

$$\phi=d_0+d_1 \bar{x}+d_2 \bar{x}^2+d_3 \bar{x}^3, \qquad \theta=d_1+\frac{\chi_2 d_3}{2}+2 d_2 \bar{x}+3 d_3 \bar{x}^2 \tag{13}$$

where the parameters b_i's and d_i's are indeterminate functions of the time coordinate, whereas

$$\bar{x}=\frac{x}{l_e}, \qquad \chi_1=\frac{12 E I_z}{G A k_y l_e^2}, \qquad \chi_2=\frac{12 E C_W}{G I_R k_W l_e^2} \tag{14}$$

The present interpolation yields

$$\frac{\partial v_S}{\partial x} - \theta_z = -\frac{\chi_1 b_3}{2}, \qquad \frac{\partial \phi}{\partial x} - \theta = -\frac{\chi_2 d_3}{2} \tag{15}$$

It may be seen that when χ_i (i=1, 2) is negligible, in fact, when the beam is very slender, expressions (15) coincide with those of Vlasov given by (5). Therefore the Vlasov model is obtained as a limiting case, thus avoiding the shear-locking phenomenon.

The coefficients b_i's and d_i's are expressed as functions of the nodal displacements, obtaining

$$v_s = \boldsymbol{N}_1 \boldsymbol{p}, \qquad \theta_z = \boldsymbol{N}_2 \boldsymbol{p}, \qquad \phi = \boldsymbol{N}_3 \boldsymbol{p}, \qquad \theta = \boldsymbol{N}_4 \boldsymbol{p} \tag{16}$$

where N_i are matrices of shape functions of 1×8 order.

Substituting expressions (16) in Eq. (10) and assembling in the usual way, one arrives to

$$\boldsymbol{M}\frac{d^2\boldsymbol{P}}{dt^2} + \boldsymbol{K}\boldsymbol{P} = 0 \tag{17}$$

where $\boldsymbol{K}$ and $\boldsymbol{M}$ are the stiffness and mass matrices respectively and $\boldsymbol{P}$ is the global vector of displacements. Gaussian quadrature formulae have been used for performing the required integrations. Details are not given because they are well known.

A Vlasov element, such as that of El-Amin and Brotton (1976), is a particular case of the present one, when very large values for the shear rigidities are taken (k_w=k_y=10^{10}) in the stiffness matrix and I_z=C_w=0 in the mass matrix. If it is only neglected the shear effect associated to flexure (k_y=10^{10}) the Fu and Hsu (1995) element is obtained.

On the other hand the straight thin walled beam element developed by Cortínez and Rossi (1998) is obtained from the present one for very large values of the radius R (i.e., R=10^{15}).

It is assumed for the free vibration problem that

$$\boldsymbol{P} = \boldsymbol{P}^* e^{i\Omega t} \tag{18}$$

where the natural frequency (in Hz) is obtained from f=$\Omega/2\pi$.

Substitution of expression (18) in (17) leads to the well known eigenvalue problem

$$[\boldsymbol{K} - \Omega^2 \boldsymbol{M}]\,\boldsymbol{P}^* = 0 \tag{19}$$

Natural frequencies are determined from the above equation by using an inverse iteration method.

4. Numerical examples and discussion

In order to show the influence of the shear flexibility on the dynamics of the beam and the performance of the developed finite element, the first six natural frequencies for different boundary conditions and several cross sectional shapes, whose dimensions are shown in Fig. 3, are obtained.

The considered material properties are: elastic modulus E=2.1×10^7 N/cm^2, elastic shear modulus G=E/2.6, density ρ=7.83×10 kg/cm^3.

Table 1 depicts the first six frequency values for a simply supported I-beam of length L=1200 cm and curvature radius R=2400 cm by employing 5, 10, 20, 30 and 40 finite elements. The results are compared with exact values determined by Cortínez, Piovan and Gutiérrez (1997). The convergence is evident, even the five elements solution gives an acceptable accuracy except for the first frequency. The forty-element solution may be considered as exact. From the practical

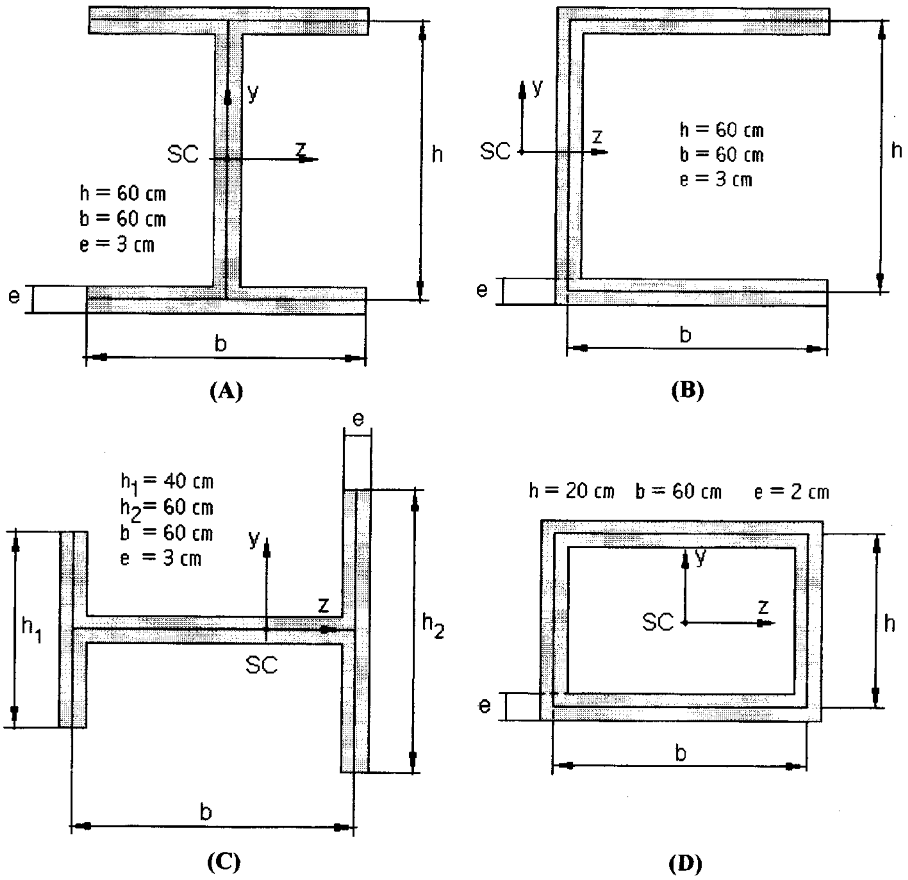


Fig. 3 Analyzed cross sectional shapes

Table 1 Convergence of natural frequencies of simply supported thin walled curved beams

Mode	Present solutions					Analitical solution (Cortínez, Piovan Gutiérrez 1997)
	Number of elements					
	5	10	20	30	40	
First	4.86	4.68	4.60	4.58	4.57	4.57
Second	30.57	30.31	29.79	29.74	29.73	29.71
Third	34.78	34.64	34.58	34.57	34.56	34.56
Fourth	65.08	64.58	64.44	64.41	64.40	64.39
Fifth	72.16	70.63	70.22	70.14	70.12	70.09
Sixth	122.76	119.26	118.41	118.25	118.20	118.13

viewpoint the use of ten elements yields very good results. However, in order to get very accurate values the calculations indicated below have been determined by taking forty elements.

The effect of shear flexibility on natural frequencies can be evaluated by means of a comparison among the results of the present model against those obtained by using the Vlasov

Table 2 Natural frequencies (Hz) of thin walled curved beams with bisymmetrical *I*-section (Case A of Fig. 3)

B.C.	*L* [cm]	Model	f1	f2	f3	f4	f5	f6
S-S	1200	Ia	2.36	22.71	62.10	63.26	82.92	116.83
		Ib	2.36	22.65	62.02	63.23	82.83	116.97
		IIa	2.36	23.18	63.39	66.14	89.32	125.03
		IIb	2.36	23.17	63.33	66.08	89.27	124.91
	600	Ia	22.66	82.89	116.70	193.07	253.05	340.88
		Ib	22.65	82.83	116.97	193.00	254.52	340.51
		IIa	23.18	89.32	125.03	250.21	287.80	512.75
		IIb	23.17	89.27	124.91	250.18	287.48	512.17
	400	Ia	63.17	129.83	252.97	340.64	508.77	576.05
		Ib	63.23	129.79	254.52	340.51	514.43	575.52
		IIa	66.14	152.08	287.80	543.71	648.95	1153.08
		IIb	66.08	152.04	287.48	543.69	648.22	1151.77
	300	Ia	116.67	193.01	418.35	496.81	791.15	818.56
		Ib	116.97	193.00	422.51	496.68	795.68	826.20
		IIa	125.03	250.21	512.74	960.33	1153.07	2047.00
		IIb	124.91	250.18	512.17	960.32	1151.77	2046.35
C-C	1200	Ia	24.26	51.64	64.00	92.96	93.55	146.31
		IIa	25.87	55.36	66.64	100.83	109.22	163.85
	600	Ia	79.30	107.27	175.06	225.33	314.02	364.47
		IIa	85.08	142.29	203.50	377.90	394.15	649.71
	400	Ia	149.52	181.25	340.56	375.00	578.04	605.61
		IIa	170.54	307.91	452.33	843.05	883.61	1459.12
	300	Ia	236.42	259.92	509.04	542.01	813.58	893.03
		IIa	294.88	544.07	801.64	1496.43	1569.03	2592.27
S-C	1200	Ia	11.76	38.04	62.91	78.72	87.89	132.10
		IIa	12.16	40.15	64.70	83.86	98.17	144.11
	600	Ia	51.60	93.52	146.75	209.83	284.21	352.81
		IIa	55.36	109.22	163.85	308.96	339.83	579.12
	400	Ia	105.95	154.35	299.12	358.32	549.07	586.38
		IIa	117.09	217.33	366.38	684.36	762.15	1301.54
	300	Ia	175.01	225.24	474.92	512.86	805.88	853.07
		IIa	203.49	377.90	649.71	1213.17	1353.25	2312.44
C-F	1200	Ia	1.44	8.53	21.56	49.56	66.55	92.72
		IIa	1.45	8.69	22.23	52.32	68.63	99.49
	600	Ia	11.10	24.79	73.29	117.68	181.30	248.52
		IIa	11.38	25.75	78.54	144.83	203.68	378.93
	400	Ia	27.28	46.16	148.52	202.28	363.42	426.37
		IIa	28.10	50.23	166.92	305.61	453.59	843.79
	300	Ia	47.50	74.62	242.30	295.36	572.11	610.36
		IIa	49.27	86.39	291.27	537.20	803.39	1497.25

I: Present model, II: Vlasov model, a: Finite element approach, b: Analytical approach (Cortínez, Piovan, Gutierrez 1997), R=1200 cm, B.C.: Boundary conditions, C: Clamped ($v_S=\theta_z=\phi=\theta=0$), S: Simply supported ($v_S=M_z=\phi=B=0$); F: Free ($M_z=Q_y=B=T_{SV}+T_W=0$)

theory, which neglects this effect.

The results determined for this study are given in Tables 2-5 for the sectional shapes A), B), C)

Table 3 Natural frequencies (Hz) of thin walled curved beams with monosymmetrical U-section (Case B of Fig. 3)

B.C.	L [cm]	Model	f1	f2	f3	f4	f5	f6
S-S	1200	Ia	1.92	13.84	35.74	65.24	75.31	100.54
		Ib	1.92	13.81	35.69	65.16	75.29	100.34
		IIa	1.92	14.11	37.53	71.21	77.71	114.79
		IIb	1.92	14.11	37.53	71.21	77.71	114.79
	600	Ia	13.82	65.18	137.53	139.84	226.87	319.70
		Ib	13.81	65.16	137.51	139.73	226.45	318.59
		IIa	14.11	71.21	153.95	168.18	304.23	470.49
		IIb	14.11	71.21	153.95	168.18	304.23	470.49
	400	Ia	35.70	139.78	229.32	272.43	413.62	556.30
		Ib	35.69	139.73	229.27	272.12	412.55	553.69
		IIa	37.53	168.18	285.29	386.87	693.17	1000.93
		IIb	37.53	168.18	285.28	386.86	693.16	1000.93
	300	Ia	65.16	226.55	338.13	413.15	602.34	789.99
		Ib	65.16	226.45	338.03	412.55	600.48	785.81
		IIa	71.21	304.23	470.49	693.17	1237.76	1744.06
		IIb	71.21	304.23	470.49	693.16	1237.75	1744.06
C-C	1200	Ia	16.28	28.27	50.68	80.42	87.09	115.47
		IIa	16.92	30.22	56.06	93.58	96.28	141.65
	600	Ia	42.00	94.24	166.41	189.99	248.58	336.81
		IIa	46.10	117.89	232.34	269.39	388.04	582.30
	400	Ia	79.59	178.69	294.35	323.65	431.03	568.55
		IIa	98.28	268.92	529.75	568.56	880.23	1317.68
	300	Ia	123.83	267.75	424.96	464.90	614.66	798.30
		IIa	174.20	481.29	946.64	989.12	1569.46	2347.32
S-C	1200	Ia	8.58	21.98	43.47	72.96	80.50	108.14
		IIa	8.84	22.96	46.89	81.80	85.84	127.91
	600	Ia	28.25	80.33	153.56	162.19	238.31	327.76
		IIa	30.22	93.58	198.16	204.40	345.00	529.51
	400	Ia	57.34	160.35	269.08	289.40	423.12	561.21
		IIa	65.84	215.95	408.31	456.22	784.01	1199.57
	300	Ia	94.21	248.18	388.54	428.45	609.28	793.05
		IIa	117.89	388.04	697.95	815.56	1398.79	2137.74
C-F	1200	Ia	2.06	6.22	14.76	28.43	51.74	82.64
		IIa	2.09	6.34	15.16	29.71	56.11	92.76
	600	Ia	11.39	22.80	43.46	100.04	177.44	200.91
		IIa	11.56	24.36	47.40	117.98	233.38	267.80
	400	Ia	19.35	63.10	84.14	196.25	325.95	341.99
		IIa	19.87	71.21	100.27	269.42	530.94	562.53
	300	Ia	30.44	114.34	132.81	301.67	475.36	492.55
		IIa	31.88	138.11	174.91	482.03	947.81	976.82

I: Present model, II: Vlasov model, a: Finite element approach, b: Analytical approach (Cortínez, Piovan, Gutiérrez 1997), R=1200 cm, B.C.: Boundary conditions, C: Clamped ($v_S=\theta_z=\phi=\theta=0$), S: Simply supported ($v_S=M_z=\phi=B=0$); F: Free ($M_z=Q_y=B=T_{SV}+T_W=0$)

and D) of the Fig. 3, respectively. In each table the first six natural frequencies are obtained for a curvature radius of 1200 cm and different lengths. The boundary conditions considered are simply

Table 4 Natural frequencies (Hz) of thin walled curved beams with monosymmetrical *I*-section (Case C of Fig. 3)

B.C.	L [cm]	Model	f1	f2	f3	f4	f5	f6
S-S	1200	Ia	2.15	20.11	28.72	37.89	56.96	61.03
		Ib	2.16	20.13	28.73	37.90	56.98	61.04
		IIa	2.15	20.28	28.79	38.21	58.06	62.37
		IIb	2.16	20.31	28.80	38.24	58.11	62.39
	600	Ia	20.10	37.87	93.44	110.10	184.75	252.32
		Ib	20.13	37.90	93.48	110.12	184.73	252.32
		IIa	20.28	38.21	96.39	114.90	198.03	276.08
		IIb	20.31	38.24	96.45	114.92	198.07	276.11
	400	Ia	56.93	61.02	184.72	252.28	369.89	534.89
		Ib	56.98	61.04	184.73	252.32	369.72	534.83
		IIa	58.06	62.37	198.03	276.08	427.41	639.96
		IIb	58.11	62.39	198.07	276.11	427.46	639.98
	300	Ia	93.43	110.09	302.83	433.22	593.62	870.84
		Ib	93.47	110.12	302.80	433.31	593.16	870.65
		IIa	96.39	114.90	340.72	502.45	748.78	1149.67
		IIb	96.44	114.92	340.77	502.47	748.82	1149.69
C-C	1200	Ia	14.01	31.00	40.48	45.73	73.29	84.05
		IIa	14.18	31.11	41.55	46.47	75.84	87.80
	600	Ia	55.10	64.87	128.48	173.87	229.83	328.27
		IIa	57.29	67.26	139.02	191.30	261.91	384.05
	400	Ia	104.68	143.87	252.60	365.75	443.94	653.61
		IIa	113.43	157.51	297.84	443.71	574.16	879.48
	300	Ia	169.09	244.90	399.84	589.71	683.89	997.73
		IIa	193.68	285.46	520.81	797.51	1011.58	1573.31
S-C	1200	Ia	7.60	28.40	31.30	41.38	66.59	70.64
		IIa	7.65	28.66	31.62	41.89	68.19	73.12
	600	Ia	40.48	45.71	110.34	141.42	207.10	290.23
		IIa	41.55	46.47	116.38	151.29	228.63	328.15
	400	Ia	80.34	97.83	218.37	309.18	407.56	595.08
		IIa	84.04	103.39	244.93	355.62	497.83	755.27
	300	Ia	128.46	173.84	352.24	513.35	640.37	943.17
		IIa	139.02	191.30	425.53	642.20	874.98	1353.50
C-F	1200	Ia	1.07	6.32	16.83	30.81	42.73	46.48
		IIa	1.08	6.35	16.94	30.95	43.51	47.15
	600	Ia	6.75	18.37	57.64	64.90	132.15	177.81
		IIa	6.81	18.46	59.15	67.03	140.48	192.52
	400	Ia	16.35	32.89	108.44	144.52	263.24	378.01
		IIa	16.57	33.21	114.86	155.92	299.53	444.92
	300	Ia	29.23	52.34	175.76	247.74	423.73	617.56
		IIa	29.81	53.31	194.18	281.85	522.65	798.83

I: Present model, II: Vlasov model, a: Finite element approach, b: Analytical approach (Cortínez, Piovan, Gutiérrez 1997), R=1200 cm, B.C.: Boundary conditions, C: Clamped ($v_S=\theta_z=\phi=\theta=0$), S: Simply supported ($v_S=M_z=\phi=B=0$), F: Free ($M_z=Q_y=B=T_{SV}+T_W=0$)

supported-simply supported, clamped-clamped, simply supported-clamped and clamped-free. For the case of simply supported ends, analytical results (Cortínez, Piovan and Gutiérrez 1997) are

Table 5 Natural frequencies (Hz) of thin walled curved beams with rectangular section (Case D of Fig. 3)

B.C.	L [cm]	Model	f1	f2	f3	f4	f5	f6
S-S	1200	Ia	4.40	19.44	43.49	75.24	90.96	113.24
		Ib	4.40	19.43	43.48	75.19	90.95	113.09
		IIa	4.42	19.85	45.61	81.68	91.14	128.06
		IIb	4.42	19.85	45.62	81.69	91.14	128.07
	600	Ia	19.44	75.20	155.87	175.87	251.24	349.61
		Ib	19.44	75.19	155.77	175.86	250.83	349.60
		IIa	19.85	81.68	176.27	184.74	329.00	350.64
		IIb	19.85	81.69	176.27	184.76	329.04	350.64
	400	Ia	43.49	155.82	262.37	301.75	459.97	525.98
		Ib	43.48	155.77	262.67	301.42	458.84	525.97
		IIa	45.61	184.74	263.05	416.57	528.19	740.95
		IIb	45.62	184.76	263.05	416.62	528.18	741.08
	300	Ia	75.20	250.93	349.60	459.48	674.21	705.11
		Ib	75.19	250.83	359.60	458.84	672.31	705.10
		IIa	81.68	329.00	350.64	709.60	740.94	1088.42
		IIb	81.69	329.04	350.64	709.58	741.08	1088.32
C-C	1200	Ia	11.01	29.89	56.84	90.28	91.78	128.91
		IIa	11.29	31.64	62.50	92.50	103.69	155.20
	600	Ia	41.94	105.00	178.60	185.95	277.05	355.31
		IIa	46.31	128.23	181.81	251.86	361.86	416.68
	400	Ia	85.53	199.23	268.62	335.66	483.10	537.78
		IIa	104.71	275.80	289.19	554.05	567.27	841.60
	300	Ia	135.58	299.05	360.50	489.96	691.55	724.35
		IIa	186.46	373.79	514.41	756.87	1007.65	1162.14
S-C	1200	Ia	7.41	24.50	50.09	82.77	91.37	121.14
		IIa	7.51	25.44	53.74	91.81	92.38	141.31
	600	Ia	29.89	90.22	171.17	177.35	264.55	352.40
		IIa	31.64	103.69	179.00	217.01	356.15	371.57
	400	Ia	63.65	178.50	265.48	319.38	471.93	531.81
		IIa	71.88	234.10	269.28	489.02	540.84	821.08
	300	Ia	104.98	276.63	355.00	475.23	683.15	714.63
		IIa	128.23	361.82	416.68	732.43	869.53	1123.84
C-F	1200	Ia	1.88	10.24	29.34	48.79	59.15	92.50
		IIa	1.88	10.39	30.55	49.40	63.46	103.61
	600	Ia	7.31	41.70	89.19	111.50	196.15	264.25
		IIa	7.40	44.52	90.82	128.92	251.27	267.50
	400	Ia	16.19	88.17	133.73	216.80	363.91	397.96
		IIa	16.59	101.11	136.75	289.30	403.87	567.24
	300	Ia	28.23	143.87	178.43	334.46	531.96	541.87
		IIa	29.45	172.63	191.56	512.93	546.45	921.39

I: Present model, II: Vlasov model, a: Finite element approach, b: Analytical approach (Cortínez, Piovan, Gutiérrez 1997), R=1200 cm, B.C.: Boundary conditions, C: Clamped ($v_S=\theta_z=\phi=\theta=0$), S: Simply supported ($v_S=M_z=\phi=B=0$), F: Free ($M_z=Q_y=B=T_{SV}+T_W=0$)

also included, which are in very good agreement with the finite element predictions.

It may be noted, from these Tables, that for slender beams (L=1200 cm) the results obtained by the present theory do not differ appreciably from those determined by the Vlasov model. However, the shear effect becomes important for shorter lengths and higher modes. As may be seen the general effect of the shear flexibility is to decrease the value of the frequency with respect to that obtained by the Vlasov model.

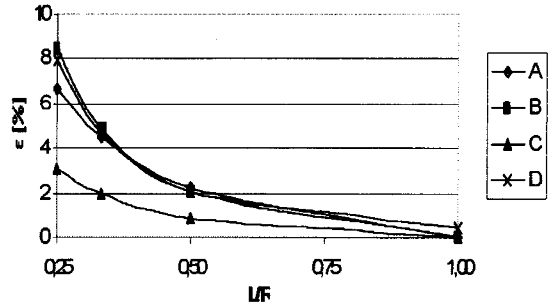
It is interesting to observe, in the case of the rectangular cross section, that for some higher frequencies the shear effect is not as important as in open sections. The explanation of this could be found in the fact that these situations correspond to torsional dominant modes, in which the warping effect is negligible in comparison with the pure torsion (Saint Venant) behavior for closed sections.

In Figs. (4a-d) the information of Tables 2-5 corresponding to the first four modes of the simply-supported case is displayed in terms of the percentage of difference of the frequencies with and without consideration of the shear effect: $100\cdot(fs-fv)/fv$, where fs and fv are the frequencies obtained with the present model and the Vlasov theory, respectively.

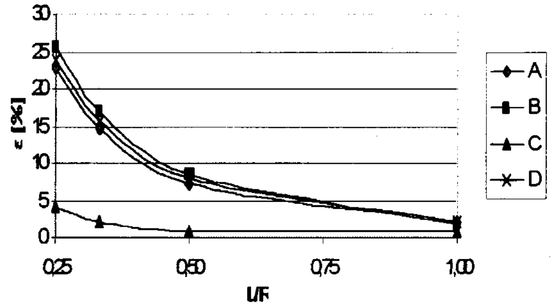
The results given in these tables show clearly that, in general, the shear effect increases with the reduction of length and the rice of the mode, excepting for some higher frequencies corresponding to the closed section.

Table 6 shows the variation of natural frequencies with the curvature radius for simply supported beams with a length of 400 cm, considering and neglecting shear effect. All of the sectional shapes shown in Fig. 3 were considered.

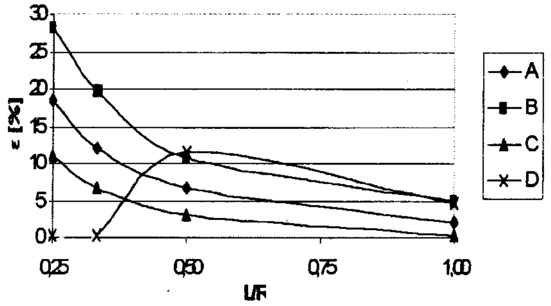
It is important to point out that the values corresponding to the greatest curvature radius (R=1$\times$ 10^{15} cm) coincide with those determined with the straight beam element developed earlier by the authors (Cortínez and Rossi 1998). As may be seen in Table 6 the increase (or decrease) of the



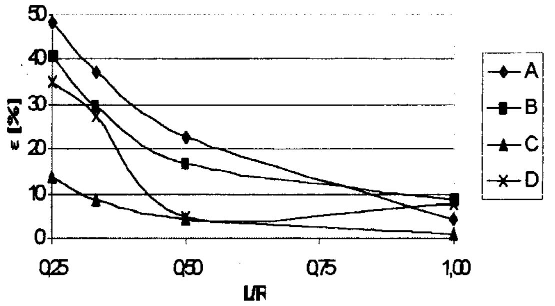
(a) First Frequency



(b) Second Frequency



(c) Third Frequency



(d) Fourth Frequency

Fig. 4 Percentage of difference of the frequencies for simply supported beams, with and without allowance for shear effect: $\varepsilon[\%]=100(fs\text{-}fv)/fv$ (A.B.C. and D are the sections of Fig. 3, R=1200 cm)

Table 6 Variation of natural frequencies with the curvature radius for simply supported thin walled curved beams

R	Case	f1		f2		f3		f4		f5		f6	
		I	II	I	II	I	II	I	II	I	II	I	II
1.00E+15	A	73.21	75.59	113.29	134.56	258.17	291.53	334.58	538.26	513.28	651.31	570.49	1154.99
[α=0°]	B	43.02	45.39	146.70	177.28	191.52	238.54	278.10	397.05	417.62	704.71	552.10	952.22
	C	47.62	48.48	73.85	75.63	172.92	185.27	270.34	296.02	357.71	412.82	553.57	663.49
	D	44.17	46.41	156.30	185.65	261.33	261.45	301.79	417.71	459.09	527.07	525.92	742.60
763.94	A	54.66	57.48	147.39	172.04	246.92	282.60	347.92	551.41	502.24	645.52	580.76	1150.26
[α=30°]		-(25.34)	-(23.96)	(30.10)	(27.85)	-(4.36)	-(3.06)	(3.99)	(2.44)	-(2.15)	-(0.89)	(1.80)	-(0.41)
	B	32.04	33.64	135.65	162.90	251.73	313.01	268.51	380.80	409.47	686.23	551.05	1029.09
		-(25.52)	-(25.89)	-(7.53)	-(8.11)	(31.44)	(31.22)	-(3.45)	-(4.09)	-(1.95)	-(2.62)	-(0.19)	(8.07)
	C	49.01	50.04	69.77	71.24	191.36	205.17	242.55	265.45	376.42	435.57	524.27	626.91
		(2.92)	(3.22)	-(5.52)	-(5.80)	(10.66)	(10.74)	-(10.28)	-(10.33)	(5.23)	(5.51)	-(5.29)	-(5.51)
	D	42.48	44.46	154.99	183.41	263.86	265.36	300.88	414.89	458.48	529.81	526.04	738.53
		-(3.83)	-(4.20)	-(0.84)	-(1.21)	(0.97)	(1.50)	-(0.30)	-(0.68)	-(0.13)	(0.52)	(0.02)	-(0.55)
381.97	A	35.48	37.30	207.12	242.38	222.31	260.18	376.29	586.37	479.47	628.95	601.25	1136.33
[α=60°]		-(51.54)	-(50.65)	(82.82)	(80.13)	-(13.89)	-(10.75)	(12.47)	(8.94)	-(6.59)	-(3.43)	(5.39)	-(1.62)
	B	23.45	24.56	124.16	148.17	257.97	363.25	315.84	391.83	400.35	665.99	543.19	1055.71
		-(45.49)	-(45.89)	-(15.37)	-(16.42)	(34.70)	(52.28)	(13.57)	-(1.31)	-(4.14)	-(5.49)	-(1.61)	(10.87)
	C	31.54	32.15	99.11	101.38	202.39	216.37	224.47	246.43	393.85	456.09	496.16	593.14
		-(33.77)	-(33.68)	(34.20)	(34.05)	(17.04)	(16.79)	-(16.97)	-(16.75)	(10.10)	(10.48)	-(10.37)	-(10.60)
	D	37.63	38.99	151.10	176.87	271.25	276.62	298.16	406.66	456.65	537.84	526.41	726.82
		-(14.81)	-(15.99)	-(3.33)	-(4.73)	(3.80)	(5.80)	-(1.20)	-(2.65)	-(0.53)	(2.04)	(0.09)	-(2.12)
254.65	A	22.44	23.54	195.86	231.44	275.15	324.07	409.87	603.60	453.02	635.61	623.95	1113.95
[α=90°]		-(69.35)	-(68.86)	(72.88)	(72.00)	(6.58)	(11.16)	(22.50)	(12.14)	-(11.74)	-(2.41)	(9.37)	-(3.55)
	B	16.41	17.16	112.31	133.25	246.54	344.59	382.09	473.08	390.28	644.16	534.45	1030.45
		-(61.85)	-(62.19)	-(23.44)	-(24.84)	(28.73)	(44.46)	(37.39)	(19.15)	-(6.55)	-(8.59)	-(3.20)	(8.22)
	C	19.66	20.02	134.11	137.38	183.25	197.06	238.95	260.91	407.21	469.92	472.07	566.68
		-(58.71)	-(58.70)	(81.60)	(81.65)	(5.97)	(6.36)	-(11.61)	-(11.86)	(13.84)	(13.83)	-(14.72)	-(14.59)
	D	30.22	30.94	144.65	166.60	282.99	294.10	293.57	393.67	453.55	550.61	527.11	708.78
		-(31.58)	-(33.33)	-(7.45)	-(10.26)	(8.29)	(12.49)	-(2.72)	-(5.76)	-(1.21)	(4.47)	(0.23)	-(4.55)

h/L=0.15 (i.e., h=60 cm, L=400 cm), I: Present model, II: Vlasov's model, The cases A, B, C and D correspond to those of Fig. 3. Percentage values $e = (f_{cur-fstrg})/f_{strg}$ [%] are presented between parentheses (f_{cur}: frequencies of curved beams, f_{strg}: frequency of the straight beam) frequencies with respect to those corresponding to the straight beam is more noticeable for the lower modes but it depends on the number of frequency and the cross section taken into account. The present model and the Vlasov theory yield a similar variation of the frequencies with the curvature for the lower modes although some discrepancies arise for higher frequencies. In fact for the case of the *U*-section, when R=763.94 cm, the present theory indicates, for the fourth frequency an increase of 13.57% with respect to the straight beam, while the Vlasov model predicts a decrease of 1.31%. This suggests the inadequacy of Vlasov theory in the range of high frequencies, where the shear effect is crucial.

5. Conclusions

A finite element for analyzing out of plane vibrations of shear deformable thin walled curved beams was developed. This element is applicable to open as well as closed sections provided the adequate cross sectional properties are selected.

Although cross sections with a single axis of symmetry were assumed, this procedure is approximately valid even for asymmetric sections when the curvature is not very large, according to Vlasov (1961) and Yoo and Feherenbach (1981). However a more deep analysis of vibrations of beams with asymmetric sections must consider the coupled motion between out of plane and in plane displacements. The convergence analysis and the comparisons with exact results show the very good performance of the present element. It may be concluded from the present numerical studies that the shear effect is noticeable for frequencies of vibration associated with higher modes or even with lower modes when the curved beam is deep. It is interesting to remark that the elements developed by Fu and Hsu (1995) and El-Amin and Brotton (1976) for statical analysis are particular cases of the present numerical procedure.

It is important to point out that the developed model is also applicable to specially laminated curved beams (where the longitudinal axis is parallel to an orthotropic axis), in which the shear flexibility could have a crucial influence because of the large value of the ratio between the equivalent longitudinal elastic modulus and the equivalent transverse elastic modulus. A numerical analysis of this last situation will be presented in the future.

Acknowledgements

The present study was sponsoréd by Secretaria de Ciencia y Tecnológía of the Universidad Tecnologica Nacional and Secretaría de Ciencia y Tecnología of the Universidad Nacional del Sur.

References

Capuani, D., Savoia M. and Laudiero, F. (1992), "A generalization of the Timoshenko beam model for coupled vibration analysis of thin walled beams", *Earthquake Engineering and Structural Dynamics,* **21**, 859-879.

Chaudhuri, S.K. and Shore, S. (1977a), "Dynamics analysis of horizontally curved I-girder bridges", *Journal of the Structural Division, ASCE,* **103**, 1589-1604.

Chaudhuri, S.K. and Shore, S. (1977b), "Thin-walled curved beam finite element", *Journal of the*

Structural Division, ASCE, **103**, 921-937.
Chidamparam P. and Leissa A.W. (1993), "Vibrations of planar curved beams, rings and arches", *Applied Mechanics Reviews* **46**(9), 467-483.
Christiano, P.P. and Culver, C.G. (1969), "Horizontally curved bridges subjected to moving load", *Journal of the Structural Division, ASCE,* **95**, 6713-6743.
Cortínez, V.H. and Rossi, R.E. (1996), "Consideración del efecto de corte en la dinámica de vigas abiertas de pared delgada", *Mecánica Computacional,* **XVI**(I), 453-462 (in spanish).
Cortínez, V.H., Piovan, M.T. and Gutiérrez, R.H. (1997), "Analytical solution for the out of plane vibrations of shear deformable thin-walled curved beams with pinned ends", *Internal Report, Mechanical Systems Analysis Group,* Universidad Tecnológica Nacional (FRBB), Bahía Blanca, Argentina.
Cortínez, V.H. and Rossi, R.E. (1998), "Dynamics of shear deformable thin walled open beams subjected to initial stresses", *Revista Internacional de Métodos Numéricos para Cálculo y Diseño en Ingeniería,* **14**(3), 293-316 (in Spanish).
Culver, C.G. (1967), "Natural frequencies of horizontally curved beams", *Journal of the Structural Division, ASCE,* **93**, 189-203.
El-Amin, F. M. and Brotton, D. M. (1976), "Horizontally curved beam finite element including warping", *Int. J. for Numerical Methods in Engineering,* **10**, 1397-1403.
Fu, C.C. and Hsu, Y.T. (1995), "The development of an improved curvilinear thin-walled Vlasov element", *Computers and Structures,* **54**, 147-159.
Gendy, A.S. and Saleeb, A.F. (1994), "Vibration of coupled extensional/flexural/torsional modes of curved beams with arbitrary thin-walled sections", *Journal of Sound and Vibration,* **174**(2), 261-274.
Heins, C.P. and Sahin, M.A. (1979), "Natural frequency of curved box girder bridges", *Journal of the Structural Division, ASCE,* **105**, 2591-2600.
Kang, K., Bert C.W. and Striz, A.G. (1996), "Vibration of horizontally curved beams with warping using DQM", *Journal of Structural Engineering, ASCE,* **122**, 657-662.
Kawakami, M., Sakiyama, T., Matsuda, H. and Morita, C. (1995), "In-plane and out-of-plane vibrations of curved beams with variable sections", *Journal of Sound and Vibration,* **187**(3), 381-401.
Krenk, S. and Gunneskov, O. (1981), "Statics of thin-walled pretwisted beams", *International Journal for Numerical Methods in Engineering,* **17**, 1407-1426.
McManus, P.F., Nasir, G.A. and Culver, C.G. (1969), "Horizontally curved girders-state of the art", *Journal of the Structural Division, ASCE,* **95**, 853-870.
Shore, S. and Chaudhuri, S. (1972), "Free vibration of horizontally curved beams", *Journal of the Structural Division, ASCE,* **98**, 793-796.
Snyder, J.M. and Wilson, J.F. (1992), "Free vibrations of continuous horizontally curved beams", *Journal of Sound and Vibration,* **157**(2), 345-355.
Tan, C.P. and Shore, S. (1968a), "Dynamic response of a horizontally curved bridge", *Journal of the Structural Division, ASCE,* **94**, 761-781.
Tan, C.P. and Shore, S. (1968b), "Response of a horizontally curved to moving load", *Journal of Structural Division, ASCE,* **94**, 2135-2152.
Task Committee on Curved Box Girders (1978), "Curved steel box-girder bridges: state of the art", *Journal of the Structural Division, ASCE,* **104**, 1719-1739.
Vlasov, V.Z. (1961), *Thin-Walled Elastic Beams,* 2nd Ed., National Science Foundation , Washington, D.C.
Yang, Y.B. and Kuo, S.R. (1986), "Static stability of curved thin-walled beams", *Journal of Engineering Mechanics, ASCE,* **112**, 821-841.
Yoo, C.H. and Feherenbach, J.P. (1981), "Natural frequencies of curved girders", *Journal of the Engineering Mechanics Division, ASCE,* **107**, 339-354.