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Eight-node field-consistent hexahedron element in dynamic problems

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Abstract. Superior performance of field consistent eight-node hexahedron element in static bending problems has already been demonstrated in literature. In this paper, its performance in free vibration is investigated. Free vibration frequencies of typical test problems have been computed using this element. The results establish its superior performance in free vibration, particularly in thin plate application and near incompressibility regimes, demonstrating that shear locking, Poisson's stiffening and volumetric locking have been eliminated.

Key words: field consistent element; eight-node hexahedron element; free vibration; shear locking; bubble functions.

1. Introduction

The eight-node hexahedron element is perhaps the most widely used element in 3D problems. The classical version of this element which employs trilinear shape funtions (e.g., see Zienkiewicz 1977) exhibits poor performance due to locking in bending situation. When a pure bending moment is applied to the element, the element does not reproduce zero shear state, a difficulty which is classically referred to as shear locking or parasitic shear. Typically, in a thin cantilever beam with a bending moment at the free end, the parasitic shear causes the element to lock to zero displacement as the thickness tends to zero irrespective of the magnitude of applied moment (e.g., see Chandra and Prathap 1989). When a cantilever is modelled using one element across the thickness, it cannot represent the variation of lateral strain through the depth (caused by Poisson's effect) corresponding to linear variation of bending stresses through the depth of the element. The element has also difficulty in representing the near-incompressible state ($\mu \rightarrow 0.5$). As $\mu \rightarrow 0.5$, the element develops spurious stiffening which adversely affects the accuracy of the element. Reduced integration of shear energy terms (e.g., see Zienkiewicz 1977), addition of bubble functions (Bathe and Wilson 1976), and assumed strain hybrid formulation (Loikkanen and Irons 1979) are a few typical methods suggested in literature to overcome these difficulties. The field consistent formulation (Chandra and Prathap 1989) provides an elegant approach to identify and alleviate these problems. This formulation does not call for reduced integration, but it rather

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rationalises why such 'tricks' work. Superior performance of field consistent eight-node hexahedron element in static bending problems has already been demonstrated in literature (Chandra and Prathap 1989). In this paper, its performance in free vibration problems is investigated. In section 2, the classical formulation of the element and typical mechanisms of locking are discussed. Formulation of field consistent elements is reviewed in section 3. In section 4, the performance of field consistent and classical elements in free vibration problems are compared for typical demonstrative problems.

2. Classical formulation

Typical classical formulation of eight-node hexahedron element employs trilinear shape functions (e.g., Zienkiewicz 1977) of the form

$$N_i(r,s,t)=\frac{1}{8}(1+rr_i)(1+ss_i)(1+tt_i),\ i=1,8 \tag{1}$$

where r, s, and t are the natural coordinates of a typical point in the element, and i refers to a typical node number. Using these shape functions, the displacement field is written as

$$\begin{Bmatrix} u(r,s,t) \\ v(r,s,t) \\ w(r,s,t) \end{Bmatrix} = \sum_{i=1}^{8} N_i(r,s,t) \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix} \tag{2}$$

where u_i, v_i and w_i refer to the nodal displacement components. The classical formulation uses a numerically exact $2\times2\times2$ Gaussian integration for computing the stiffness matrix. The performance of this element is poor in bending situation. The reason for poor performance is attributed to additional stiffening of the element. Three important stiffening mechanisms are discussed in the following.

2.1. Shear locking

For a rectangular (geometrically undistorted) element, the shape functions given by Eq. (1) lead to a trilinear polynomial displacement field as

$$u=a_1+a_2x+a_3y+a_4z+a_5xy+a_6yz+a_7zx+a_8xyz \tag{3}$$

$$v=b_1+b_2x+b_3y+b_4z+b_5xy+b_6yz+b_7zx+b_8xyz \tag{4}$$

$$w=c_1+c_2x+c_3y+c_4z+c_5xy+c_6yz+c_7zx+c_8xyz \tag{5}$$

For the case of pure bending in xy-plane, the shear strain $\gamma_{xy}=u_{,y}+v_{,x}=0$. Using Eqs. (3)-(5), it can be shown (Chandra and Prathap 1989) that this is possible only when

$$a_3+b_2=0 \tag{6}$$

$$a_6+b_7=0 \tag{7}$$

$$c_5=0 \tag{8}$$

$$b_5=0 \tag{9}$$

$$a_8=0 \tag{10}$$

$$b_8 = 0 \tag{11}$$

Eqs. (6) and (7) contain coefficients from both u and v displacement fields and hence can represent true constraint conditions. However, Eqs. (8)-(11) involve isolated coefficients and hence lead to spurious constraints responsible for locking.

2.2. Poisson's stiffening

This problem manifests itself in the case of flexure of beams, plates and shells when only one element is used across the depth. Consider, for example, pure bending of one element cantilever beam in xy-plane. For this problem $\sigma_y=\sigma_z=\varepsilon_{xy}=\varepsilon_{yz}=\varepsilon_{zx}=0$ and $\varepsilon_y=\varepsilon_z=-\mu\varepsilon_x$, and hence the strain energy can be expressed purely in terms of σ_x or ε_x. Bending in the xy-plane induces a linear variation of ε_x through the depth and hence ε_y must also vary linearly through the depth. However, it is easy to check that this requirement is not satisfied by the displacement field given by Eqs. (3)-(5). As a result $\sigma_y \neq 0$ in the finite element model. This leads to a modifed expression (Chandra and Prathap 1989) for the strain energy involving an additional stiffening parameter $1/(1-\mu^2)$ where μ is the Poisson's ratio.

Considering both Poisson's stiffening and shear locking, Prathap (1985) has derived an expression for the stiffening parameter for static flexural problems. For free vibration, it may be interpreted as

$$\frac{\bar{f}}{f} = \left(\frac{1}{1-\mu^2} + \frac{GL^2}{Et^2} \right)^{1/2} \tag{12}$$

where f and $\bar{f}$ are natural frequencies computed by theory and finite element, respectively, L and t are the length and depth of the beam, and G is the rigidity modulus of the material.

2.3. Volumetric locking

In the incompressible regime ($\mu \rightarrow 0.5$), the displacement field must be able to ensure that the volumetric strain is zero, i.e., $\varepsilon_v=u_{,x}+v_{,y}+w_{,z}=0$. The displacement field given by Eqs. (3)-(5) satisfies this condition only when

$$a_2 + b_3 + c_4 = 0 \tag{13}$$
$$b_5 + c_7 = 0 \tag{14}$$
$$a_5 + c_6 = 0 \tag{15}$$
$$a_7 + b_6 = 0 \tag{16}$$
$$a_8 = 0 \tag{17}$$
$$b_8 = 0 \tag{18}$$
$$c_8 = 0 \tag{19}$$

Here Eqs. (17)-(19) represent spurious constraints which cause locking.

The element based on classical formulation will hereinafter be referred to as FI (Field Inconsistent) element.

3. Field consistent elememt

Several methods to overcome the locking problem have been reported in literature (e.g., Zienkiewicz 1977, Bathe and Wilson 1976, Loikkanen and Irons 1979). A well known approach to overcome the problem of locking is to add bubble functions associated with nodeless variables. This removes shear locking, Poisson's stiffening and volumetric locking, in general situations. In this paper, we look at this problem from the field consistency (Prathap 1993) view point.

With the addition of bubble funtions, the displacement field (Eqs. (3) to (5)) becomes

$$u = a_1 + a_2x + a_3y + a_4z + a_5xy + a_6yz + a_7zx + a_8xyz + a_9(1-x^2) + a_{10}(1-y^2) + a_{11}(1-z^2) \quad (20)$$

$$v = b_1 + b_2x + b_3y + b_4z + b_5xy + b_6yz + b_7zx + b_8xyz + b_9(1-x^2) + b_{10}(1-y^2) + b_{11}(1-z^2) \quad (21)$$

$$w = c_1 + c_2x + c_3y + c_4z + c_5xy + c_6yz + c_7zx + c_8xyz + c_9(1-x^2) + c_{10}(1-y^2) + c_{11}(1-z^2) \quad (22)$$

For the case of pure bending in *xy*-, *yz*- and *zx*-planes, the conditions $\gamma_{xy}=0$, $\gamma_{yz}=0$ and $\gamma_{zx}=0$, respectively, need to be satisfied. Using Eqs. (20)-(22), it can be shown that these conditions lead to three spurious constraints as in Eqs. (17)-(19). Numerical experiments show that these spurious constraints do not lead to any measurable degree of locking in general 3D problems. However, in thin beam/plate application, these terms introduce additional stiffening which does not lead to locking, but affects the accuracy of the results considerably. Further, it is easy to show that with the addition of bubble functions, the Poisson's stiffening is removed automatically as strains such as ε_y can now represent variation through the depth. It can be shown that the condition for incompressibility, i.e., $\varepsilon_v=0$ again leads to the same three spurious constraints given by Eqs. (17)-(19). The element formulated using a displacement model as in Eqs. (20)-(22) will hereinafter be referred to as FIB (classical Field Inconsistent element with the addition of Bubble function).

An improvement over FIB element is to smoothen the strain expressions so that the stiffening due to the trilinear terms vanishes. This is achieved by removing the trilinear term selectively from the dilatational and shear terms in the expressions for strain. An approach to selective removal of the trilinear terms briefly described below.

The stress-strain relations for an isotropic material is written in the form,

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_x \end{Bmatrix} = 2G \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{Bmatrix} + \lambda \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{Bmatrix} \quad (23)$$

and

$$\begin{Bmatrix} \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = G \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \quad (24)$$

where $\lambda=K-2G/3=\mu E/(1-2\mu)(1+\mu)$, μ is the Poisson's ratio, and E, G and K are the Young's, shear and bulk moduli, respectively.

The second term on the right hand side of Eq. (23) corresponds to volumetric strain. Setting the coefficients $a_8=b_8=c_8=0$ in the expressions for ε_x, ε_y and ε_z appearing in the term removes the spurious constraints in the incompressibility regime. A similar operation in the expressions for γ_{xy},

γ_{yz} and γ_{zx} on the right hand side of Eq. (24) removes the spurious constraints associated with shear stain. The element formulated as above will be referred to as an FCB (Field Consistent element with Bubble functions). Removal of the trillinear term from Eqs. (20) to (22) altogether is equivalent to removing it from the strain expressions in the first term on the right hand side of Eq. (23) as well. This will introduce zero energy modes leading to spurious vibration modes which would be demonstrated numerically in section 4. The element based on such a wrong formulation is designated as an FCB* element. The reason for discussing this element is that the element in general performs well in many problems and it requires subtlety to identify its poorer performance.

4. Demonstrative problems

As a first demonstrative problem, the free vibration of a $10\times0.1\times0.1$ cantilever beam is considered. The cantilever beam has been modelled with ten elements along the length and one across the cross section. All the degrees of freedom at the fixed end are restrained. Consistent mass matrix is employed for all the computations. The Young's modulus, density and Poisson's ratio are taken as 2.1×10^{11}, 7860 and 0.3, respectively. The natural frequencies have been computed with FI, FIB, FCB and FCB* elements. The frequencies have also been computed using the 8-node CHEXA element of MSC/NASTRAN, and theory. MSC/NASTRAN uses bubble functions, and reduced integration for shear terms (MSC/NASTRAN user's Manual 1994). For computing the theoretical predictions, the effect of shear deformations and rotary inertia have been taken into account as discussed by Timoshenko *et al.* (1974).

The first few flexural, axial and torsional frequencies are listed in Table 1. The theoretical values shown in Table 1 and 3 have been computed using classical formulae available in literature (e.g., Timoshenko *et al.* 1974), which do not take into account the kinetic energy due to lateral motion induced by Poisson's ratio, and hence are meant for reference purposes only. It is seen from this Table 1 that FI element predicts too high flexural frequencies indicating the presence of stiffening effect. Table 2 shows the values of additional stiffening parameter computed and predicted theoretically using Eq. (12) for the first flexural frequency. It is seen that the predicted and computed values are in close agreement. Referring back to Table 1, it is seen that the prediction of flexural frequencies by FIB, FCB, FCB*, and MSC/NASTRAN are comparable to each other, and closer to theoretical values. Although, the FIB element has the spurious constraints, $a_8=b_8=c_8=0$, the results show that the performance of FIB element is not affected by these constraints, indicating that the stiffening effect of these constraints is negligible for this

Table 1 Natural frequencies (Hz) of cantilever beam for μ=0.3

	Theory	FI	FIB	FCB	FCB*	MSC/NASTRAN
Flexural	0.835	5.251	0.841	0.839	0.836	0.840
	5.231	33.27	5.395	5.383	5.358	5.388
	14.64	95.53	15.84	15.79	15.71	15.81
	28.66	194.7	33.49	33.36	33.12	33.41
Axial	129.2	130.5	130.3	130.3	130.3	130.4
	387.7	395.8	395.0	395.0	395.0	395.2
Torsional	80.14	80.27	80.19	80.19	80.19	80.22
	240.4	242.6	242.6	242.6	242.6	242.7

Table 2 The stiffening parameter for FI element-first flexural frequency

Thickness	$\bar{f}/f$ computed	$\bar{f}/f$ predicted
1	1.246	1.218
0.1	6.286	6.290
0.01	61.95	62.03
0.001	619.5	620.2

Table 3 Natural frequencies (Hz) of cantilever beam for μ=0.4999

	Theory	FI	FIB	FCB	FCB*	MSC/NASTRAN
Flexural	0.835	7.239	0.956	0.845	0.851	0.848
	5.231	50.07	6.086	5.424	5.360	5.440
	14.64	157.2	18.00	15.94	15.71	16.00
	28.66	325.7	38.05	33.75	33.11	33.93
Axial	129.2	383.0	133.3	133.3	133.3	133.3
	387.7	-	406.9	406.9	406.9	407.0
Torsional	74.61	74.66	74.66	74.66	74.66	74.71
	223.8	225.8	225.8	225.8	225.8	226.0

problem.

The natural frequencies computed for μ=0.4999 are shown in Table 3. Comparing Tables 1 and 3, it is seen that the flexural frequencies computed with FCB, FCB*, and MSC/NASTRAN elements are practiclly unaffected by the increase in the value of μ from 0.3 to 0.4999. However, the frequencies computed with FI and FIB elements show considerable increase. The error in the prediction of FI element is mainly due to Poisson's stiffening whereas that of FIB element is due to additional stiffening caused by the spurious constraints $a_8=b_8=c_8=0$ as $\mu \rightarrow 0.5$. The FCB and FCB* elements do not have this problem as the spurious constraints have been smoothened.

A scan through the axial and torsional frequencies listed in Table 1 shows that the frequency predictions of all the elements are comparable. This suggests that shear locking and Poisson's stiffening, and the remedies we have used to remove them do not affect the axial and torsional modes. It is seen that the torsional frequenies listed in Table 3 also show a similar trend. However, the case with the axial frequencies is different. The prediction of FI element is rather too high which indicates that there is a tendency to lock as $\mu \rightarrow 0.5$. The predictions of other elements are comparable with each other.

In Table 3, the difference between the axial frequencies predicted by theory, and those computed with FCB, FCB* and NASTRAN elements is rather large, whereas in Table 1, there is very little difference. This seems to suggest that the error in the theoretical frequencies caused by not accounting the kinetic energy due to the lateral motion is considerable only when $\mu \rightarrow 0.5$.

The second demonstrative problem considered is a square plate (of size $4\times4\times t$, with t being a variable) with clamped edges. The plate is modelled with 2×2, 4×4 and 8×8 meshes with one element through the depth. All the three degrees of freedom of the nodes on the clamped edges are restrained. Here again consistent mass matrix is used. The Young's modulus, density and Poisson's ratio are taken as 2.0×10^{11}, 7860 and 0.3, respectively. The natural frequency corresponding to the first double symmetric mode has been computed for various elements and

Table 4 First natural frequency (Hz) of thin clamped plate

t/a	0.1			0.01			0.001		
Mesh	2×2	4×4	8×8	2×2	4×4	8×8	2×2	4×4	8×8
FI	361.8	262.5	233.7	307.5	149.8	77.10	306.8	147.8	73.11
FIB	251.2	213.8	207.1	91.55	30.18	23.07	87.79	19.28	5.139
FCB	237.1	212.2	206.9	26.33	23.22	22.57	2.636	2.325	2.262
FCB*	233.6	211.5	206.7	25.83	23.11	22.55	2.585	2.315	2.259
MSC/NASTRAN	238.2	212.5	207.0	26.48	23.26	22.59	2.652	2.328	2.261
Theory[#]		223.95			22.40			2.240	

[#] shear deformation and rotory inertia not accounted
t/a is non-dimensional thickness normalized with respect to side length

Table 5 Spurious frequencies of FCB* element

FCB	8.335	8.335	51.33	51.33	80.19	130.3	141.2	141.24	242.6
	272.1	272.1	394.4	410.9	443.5	443.5			
FCB*	8.299	8.299	51.11	51.11	80.19	130.3	140.6	140.6	242.6
	270.6	270.6	**283.7**	**385.1**	**385.0**	394.4	410.9	442.1	442.1

the results are summarised in Table 4. It is seen from this Table that the frequencies computed with FI element are too high compared to theoretical values irrespective of the mesh used indicating the presence of stiffening due to spurious constraints. The frequencies of FIB element also show some stiffening although it is not so severe as FI elements. On the contrary, it may be recalled that FIB element does not exhibit stiffening in the cantilever problem. It is also seen from Table 4 that the other elements, viz. FCB, FCB* and MSC/NASTRAN's CHEXA, perform equally well and predict values close to theoretical values as t becomes small.

In both the demonstrative problems discussed so far, the performance of the FCB* element is seen to be comparable to that of FCB element. As already mentioned, the FCB* element is based on a wrong formulation and has zero energy modes. This leads to spurious modes in some problems. Consider for example a cantilever beam of $10\times1\times1$ dimensions with Young's modulus, density and Poisson's ratio equal to 2.1×10^{11}, 7860 and 0.3, respectively, The first few natural frequencies computed with FCB and FCB* element are listed in Table 5. The spurious frequencies of FCB* element are shown in bold face.

5. Conclusions

The formulation of several versions of eight-node hexahedron element has been reviewed in the light of field consistency concept. The performance of these elements in free vibration has been investigated. Free vibration frequencies for a cantilever beam and clamped square plate have been computed using these elements. The performance of the classical element (FI) is poor for both the problems. The element with bubble functions (FIB) performs well for the cantilever problem for μ =0.3, and exhibits some stiffening for μ=0.4999. However, its frequency predictions for the plate problem are too high. The field consistent element (FCB) performs uniformly well for both the problems demonstrating that shear locking, Poisson's stiffening and volumetric locking have been eliminated. The FCB* element has zero energy modes which induces spurious frequencies for some problems, and hence is unreliable.

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