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Plastic limit analysis of a clamped circular plate with unified yield criterion

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Abstract. This paper presents exact close form solutions of plastic limit loads of a clamped circular plate under uniformly distributed load with different loading radii. A unified yield criterion, which includes a family of piecewise linear yield criteria and the commonly adopted yield criteria such as the Tresca criterion and the maximum principal deviatoric stress criterion or the twin shear stress criterion that are its special cases, and the Mises criterion can be approximated by it, is employed in the analysis. The plastic limit loads, moment fields and velocity fields of the clamped circular plate are calculated based on the unified yield criterion. The influences of the yield criteria, the edge effects and the loading radius on the plastic limits of the clamped circular plate are investigated. Analytical results are calculated and compared. The exact close form solutions presented in this paper provide efficient approaches for obtaining plastic limit loads and the corresponding moments and velocities of the clamped circular plates. The previously derived solutions based on the Tresca and the Mises criteria are its special cases.

Key words: plastic limit analysis; circular plate; yield criterion; moment fields; velocity fields.

1. Introduction

Clamped circular plates form important structural elements in many branches of engineering. Efficient and accurate prediction for the load-carrying capacity of a clamped circular plate is very important to ensure that the structural design is as economical as possible. Limit analysis method has been recognized as an effective method for analysis of circular plates in the plastic limit state. Generally, the Maximum shear stress criterion or Tresca criterion, the octahedral shear stress criterion or Mises criterion and the maximum deviatoric stress criterion or twin shear stress criterion are applied to derive the plastic solutions (Hill 1950, Yu 1983, Yan and Bu 1996). Because the expression of the Tresca criterion is linear so that it is relatively easy to derive analytical plastic solutions, most of the plastic limit solutions for circular plates were obtained

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based on it (Hopkins and Prager 1953, Zaid 1958, Gamer 1983). Guven (1992) and Ghorashi (1994) extended the range of investigation of plastic limit load by using limit analysis theorems based on the Tresca criterion. Hopkins and Wang (1954) investigated the load carrying capacities of a circular plate with the Mises criterion and a parabolic criterion by numerical iterative method. They found that the plastic limit loads on the basis of the Mises criterion and that on the basis of the Tresca criterion differ by approximately 8% for a simply supported circular plate and approximately 10% for a clamped plate, respectively. These observations indicate that the plastic limit loads in terms of different yield criteria are different and the difference depends on the edge effects of the plate. Unfortunately, an analytical plastic limit solution based on the Mises criterion for a circular plate under various loading conditions is not readily derived because of its nonlinear formula. Ma *et al.* (1995) investigated the plastic limit solution of a simply supported circular plate and a rotating disk with the twin shear stress criterion and found that the plastic limit loads could differ by more than 14% with those obtained by the Tresca criterion. Although plastic solutions of circular plate based on the Tresca criterion have been intensively performed in the past years, the investigation of influence of yield criteria on the plastic behavior of circular plates with various loading conditions is still not sufficient, especially solutions based on the twin shear stress criterion is limited. This is probably because that the other criteria are not convenient to use owing to their nonlinear nature.

Yu (1991) suggested an unified yield criterion (UYC) which assumes that plastic flow is controlled by the two larger principal shear stresses. It is called unified yield criterion because the Tresca criterion and the twin shear stress criterion are its special cases and the Mises criterion can be approximated by it. Because considering the two larger principal shear stresses, the unified yield criterion actually includes all the three principal stresses into consideration when determining a material yielding property. The unified yield criterion has piecewise linear formulae and can be used to model material yielding properties in a broad spectrum of engineering application.

In this paper, close form solutions of the plastic limit loads, moment fields and velocity fields of clamped plates are derived based on the UYC. Numerical results based on the Tresca criterion, the Mises criterion and the twin shear stress criterion are calculated and compared. The differences between the results obtained according to different criteria are discussed. Although numerical analysis of a clamped circular plate is quite straightforward nowadays with powerful computers, the close form solutions derived in this paper are more easier to be applied. Besides, with author's knowledge, there is no computer code based on the UYC available yet. The theoretical solutions derived from this study can be used as benchmark for the future computer code development.

2. Unified yield criterion

Based on orthogonal octahedron of the twin shear element model (Yu 1983), a unified strength criterion was developed which specifies that a material fails when a certain function of the two larger principal shear stresses and their corresponding normal stresses reach the limit values. The mathematical expression of the unified strength criterion is (Yu 1991, 1992)

$$\tau_{13}+b\,\tau_{12}+\beta(\sigma_{13}+b\,\sigma_{12})=C \quad \text{when} \quad \tau_{12}+\beta\sigma_{12}\geq\tau_{23}+\beta\tau_{23} \tag{1a}$$

$$\tau_{13}+b\,\tau_{23}+\beta(\sigma_{13}+b\,\sigma_{23})=C \quad \text{when} \quad \tau_{12}+\beta\sigma_{12}\leq\tau_{23}+\beta\tau_{23} \tag{1b}$$

where τ_{13}, τ_{12} and τ_{23} are principal shear stresses and $\tau_{13}=(\sigma_1-\sigma_3)/2$, $\tau_{12}=(\sigma_1-\sigma_2)/2$ and $\tau_{23}=(\sigma_2-\sigma_3/2$; σ_{13}, σ_{12} and σ_{23} are normal stresses corresponding to the three principal shear stresses and $\sigma_{13}=(\sigma_1+\sigma_3)/2$, $\sigma_{12}=(\sigma_1+\sigma_2)/2$, $\sigma_{23}=(\sigma_2+\sigma_3)/2$, in which σ_1, σ_2 and σ_3 are the principal stresses and $\sigma_1 \geq \sigma_2 \geq \sigma_3$; β and C are material constants; b is a weighted coefficient that reflects the influence of the intermediate principal shear stress and corresponding normal stress on plastic behavior of the material. When the constant b varies from 0 to 1, a family of convex yield criteria that are suitable for different kinds of materials are deduced. In particular, the unified strength criterion yields to the Mohr-Coulomb criterion when b=0.

The unified yield criterion (UYC) which is specifically suitable for metal materials is degraded from the unified strength criterion, when β=0 in Eq. (1). The expression in principal stress state of the UYC is

$$\sigma_1 - \frac{1}{1+b}(b\,\sigma_2 + \sigma_3) = \sigma_0 \quad \text{when} \quad \sigma_2 \leq \frac{\sigma_1+\sigma_3}{2} \tag{2a}$$

$$\frac{1}{1+b}(\sigma_1 + b\,\sigma_2) - \sigma_3 = \sigma_0 \quad \text{when} \quad \sigma_2 \geq \frac{\sigma_1+\sigma_3}{2} \tag{2b}$$

where σ_0 is yield strength of a material. Obviously, the Tresca criterion and the maximum principal deviatoric stress criterion or the twin shear stress criterion are special cases of the UYC when b=0 and b=1, respectively. The two criteria lead to the minimum (interior) and the maximum (exterior) bounds of all the yield surfaces for stable materials. Moreover, the nonlinear mises criterion can also be approximated by the UYC by letting b=0.5. Fig. 1 and Fig. 2 show the projections in deviatoric plane of the unified strength criterion and the unified yield criterion, respectively.

It is known that the Tresca criterion predicts the pure shear strength τ_0 of a material as τ_0= 0.5σ_0. Experimental results show that the pure shear strength of most metal materials satisfy the estimation by the Mises criterion, namely τ_0=0.577σ_0, however, for some materials such as mild steel and aluminum, it is closer to the prediction of τ_0=0.667σ_0 by the twin shear stress criterion (Hill 1950, Yu 1983). The pure shear strength derived by the UYC is $\tau_0=(1+b)/(2+b)\sigma_0$ which covers all the predictions between the lower bound given by the Tresca criterion and the upper bound given by the twin shear stress criterion. In turn, the weighting coefficient b in the UYC is

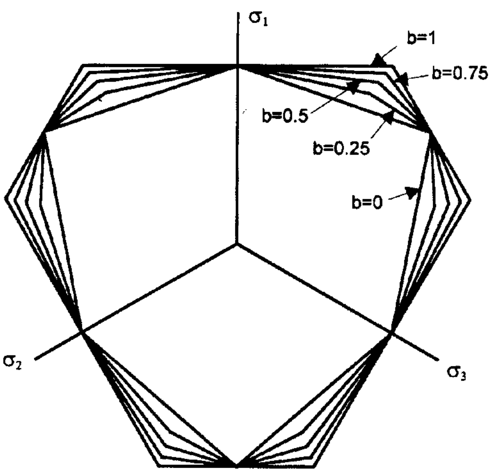


Fig. 1 Unified strength criterion

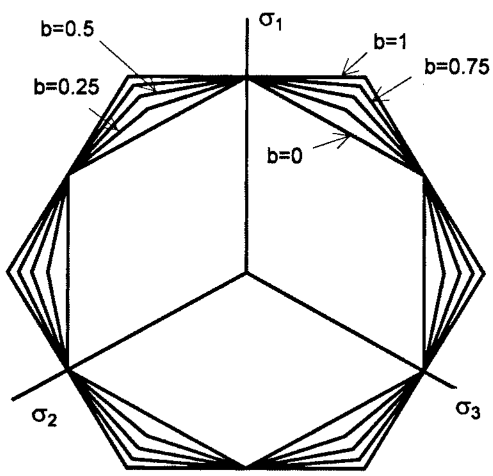


Fig. 2 Unified yield criterion β=0

equal to $(2\tau_0-\sigma_0)/(\sigma_0-\tau_0)$ corresponding to various metal materials. Thus, the application of the UYC in plastic analysis becomes very versatile because of its generality that commonly used yield criteria are its special cases, and, moreover, because it is relatively easy to be implemented due to its piecewise linear expression. It will also make a computer code more versatile by using the UYC such that users can select the yield criteria in analyses by simply choosing a few parameters.

3. Basic equations of circular plate

When a fully clamped circular plate of radius a and thickness h which is made of a rigid perfectly plastic material is subjected to a partial uniformly distributed transverse load P, the only non-zero stresses are σ_r, σ_θ and $\tau_{rz}=\tau_{zr}$ for the plate. In plastic limit state, the generalized stress components are expressed as

$$M_r=\int_{-h/2}^{h/2}\sigma_r zdz,\quad M_\theta=\int_{-h/2}^{h/2}\sigma_\theta zdz,\quad Q_{rz}=\int_{-h/2}^{h/2}\tau_{rz}zdz \quad \text{and} \quad M_0=\int_{-h/2}^{h/2}\sigma_0 zdz=\sigma_0 h^2/4 \tag{3}$$

where M_r, M_θ and M_0 are radial, tangential and ultimate (fully plastic) bending moments, respectively, and Q_{rz} is the transverse shear force which is assumed not to influence the plastic yielding. By defining dimensionless variables, $r=R/a$, $m_r=M_r/M_0$, $m_\theta=M_\theta/M_0$ and $p=Pa^2/M_0$, and because of axial symmetry, the equilibrium equations are

$$d(rm_r)/dr-m_\theta=-pr^2/2 \qquad 0\le r\le r_p \tag{4}$$

$$d(rm_r)/dr-m_\theta=-pr_p^2/2 \qquad r_p\le r\le 1 \tag{5}$$

where $r_p=R_p/a$ is the normalized loading radius of circular plate, in which R_p is the loading radius; $r_p=1$ implies the entire plate is uniformly loaded, whereas $r_p=0$ indicates a point load at the center only. When the unified yield criterion expressed by generalized stresses (shown in Fig. 3) is used, limit condition can be written as

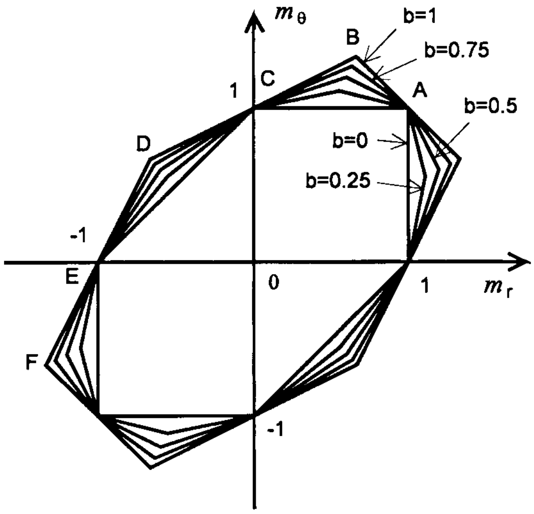


Fig. 3 Unified yield criterion in m_θ-m_r space

Table 1 Constants a_i and b_i

lines	AB (i=1)	BC (i=2)	CD (i=3)	DE (i=4)	EF (i=5)
a_i	$-b$	$b/(1+b)$	$1/(1+b)$	$1+b$	$(1+b)/b$
b_i	$1+b$	1	1	$1+b$	$(1+b)/b$

Table 2 Constant d_i

points	A (i=0)	B (i=1)	C (i=2)	D (i=3)	E (i=4)	EF (i=5)
d_i	1	$(1+b)/(2+b)$	0	$-(1+b)/(2+b)$	-1	$-(1+b)/(1+b-v_p b)$

$$m_\theta = a_i m_r + b_i \quad (i = 1\sim12) \tag{6}$$

where, a_i and b_i are constants related to the material coefficient b. Table 1 lists a_i and b_i for the five lines AB, BC, CD, DE and EF.

In elastic state, the moment fields satisfy that $m_\theta=m_r$ at the plate center (r=0), $m_\theta=vm_r$ at the clamped edge and $m_\theta \geq m_r$ on the whole plate, in which v is the elastic Poisson's ratio. In plastic limit state, the condition $m_\theta=m_r$=1 at the plate center is also satisfied. Based on the kinematically admissible requirement, one still can expect the transverse velocity $\dot{w}$ to be a decreasing function of r which implies the tangential curvature rate $\dot{k}_\theta \geq 0$. Solutions based on the Tresca criterion and the Mises criterion assume that m_θ=0 and m_θ=0.5 at the clamped edge, respectively, corresponding to the condition of $\dot{k}_\theta$=0 at the edge (Hodge 1963). In plastic state, one can assume $m_\theta=v_p m_r$ at the outer edge, where v_p is an equivalent Poisson's ratio in plastic limit state and $0 \leq v_p \leq 0.5$ due to the positive tangential curvature rate $\dot{k}_\theta$, hence the moment fields of the entire clamped plate lie on the five sides corresponding to AB, BC, CD, DE and EF shown in Fig. 3 based on the UYC. The points A, B, C, D and E in Fig. 3 correspond to dimensionless radii r_0, r_1, r_2, r_3 and r_4 on the plate, respectively, which divide the plate into five parts with $0=r_0 \leq r_1 \leq r_2 \leq r_3 \leq r_4 \leq r_5 = 1$. At the outer edge ($r$=1), the moments m_r and m_θ are located on the line EF, and they are exactly on point F when v_p is 0.5 and on point E when v_p is zero.

The continuity conditions and boundary conditions of radial moment m_r in Eqs. (4) and (5) are:

$$m_r\,(r = r_i) = d_i \quad (i = 0\sim5) \tag{7}$$

where the values of d_i (i=0~5) corresponding to the yield points A, B, C, D, E, and a point on line EF are listed in Table 2.

4. Plastic limit load and internal moment fields

For a certain loading radius r_p, r_j (j=1~4) corresponding to the yield points B, C, D, and E in Fig. 3 divides the plate into five parts. The loading radius may lie in either part, $r_{(j-1)} \leq r_p \leq r_j$, ($j$=1~5). Considering the particular cases with $r_p=r_j$, j=1~5, and defining the critical radius r_p or r_j as r_{pj} (j=1, 5), the differential Eqs. (4) and (5) of radial moment m_r are integrated directly with respect to r with the aid of the linear yield criterion expressed by Eq. (6),

$$m_{ri} = \frac{b_i}{1-a_i} - \frac{pr^2}{2(3-a_i)} + c_i r^{-1+a_i} \quad 1 \leq i \leq j \;\; or \;\; 0 \leq r \leq r_{pj} \tag{8}$$

$$m_{ri} = \frac{b_i}{1-a_i} - \frac{pr_{pj}^2}{2(1-a_i)} + c_i r^{-1+a_i} \qquad j \le i \le 5 \;\; or \;\; r_{pj} \le r \le 1 \tag{9}$$

where c_i (i=1~5) are integral constants. Eq. (8) gives the moment distribution in the loading area, while Eq. (9) is the moment distribution in the area outside the loading radius. Using the boundary conditions and continuity conditions expressed by Eq. (7), it has

$$\begin{aligned} &\frac{b_i}{1-a_i} - \frac{pr_{i-1}^2}{2(3-a_i)} + c_i r_{i-1}^{-1+a_i} = d_{i-1} \\ &\frac{b_i}{1-a_i} - \frac{pr_i^2}{2(3-a_i)} + c_i r_i^{-1+a_i} = d_i \end{aligned} \qquad 1 \le i \le j, \;\; (j = 1\text{~}5) \tag{10}$$

and

$$\left\{ \begin{aligned} &\frac{b_i}{1-a_i} - \frac{pr_{pj}^2}{2(1-a_i)} + c_i r_{i-1}^{-1+a_i} = d_{i-1} \\ &\frac{b_i}{1-a_i} - \frac{pr_{pj}^2}{2(1-a_i)} + c_i r_i^{-1+a_i} = d_i \end{aligned} \right. \qquad j+1 \le i \le 5, \;\; (j = 1\text{~}4) \tag{11}$$

Since moment at the plate center has a finite value, it has

$$c_1 = 0 \tag{12}$$

The plastic limit load is derived from Eq. (10) as

$$p = 2(3-a_1)\left(-d_1 + \frac{b_1}{1-a_1}\right) r_1^{-2} \qquad \text{or} \qquad p = \frac{2(3-a_1)}{2-a_1} r_1^{-2} \tag{13}$$

For the case of j=1, it can be derived from Eq. (11) that

$$c_i = \frac{d_i - d_{i-1}}{r_i^{-1+a_i} - r_{i-1}^{-1+a_i}}, \qquad (i = 2\text{~}5) \tag{14}$$

$$\eta_{i-1}^{-1+a_i} = 1 - (d_i - d_{i-1}) \Bigg/ \left[d_i - \frac{b_i}{1-a_i} + \frac{3-a_1}{(1-a_i)(2-a_1)} \right], \qquad (i = 2\text{~}5) \tag{15}$$

where η_i (i=1~4) are defined as:

$$\eta_{i-1} = r_{i-1}/r_i \tag{16}$$

The values of η_i for this case can be calculated from Eq. (15) directly.

For the case of $2 \le j \le 5$, it can be derived from Eq. (10) that,

$$c_i = \left[d_{i-1} - \frac{b_i}{1-a_i} + \frac{pr_{i-1}^2}{2(3-a_i)} \right] r_{i-1}^{1-a_i} \qquad (2 \le i \le j) \tag{17}$$

and η_{i-1} $(2 \le i \le j)$ satisfy

$$\frac{b_i}{1-a_i} - \frac{(3-a_1)\eta_1^{-2}\cdots\eta_{i-2}^{-2}}{(3-a_i)(2-a_1)}\eta_{i-1}^{-2} + \left[d_{i-1} - \frac{b_i}{1-a_i} + \frac{(3-a_1)\eta_1^{-2}\cdots\eta_{i-2}^{-2}}{(3-a_i)(2-a_1)}\right]\eta_{i-1}^{1-a_i} = d_i \qquad (2 \le i \le j) \tag{18}$$

The other integral constants are derived from Eq. (11) as

$$c_i = \frac{d_i - d_{i-1}}{r_i^{-1+a_i} - r_{i-1}^{-1+a_i}}, \qquad j+1 \le i \le 5 \tag{19}$$

and $\eta_{i-1}(j+1 \le i \le 5)$ satisfy

$$\eta_{i-1}^{-1+a_i} = 1 - \frac{(d_i - d_{i-1})}{\left[d_i - \frac{b_i}{1-a_i} + \frac{(3-a_1)\eta_1^{-2}\cdots\eta_{j-1}^{-2}}{(1-a_i)(2-a_1)}\right]}, \qquad j+1 \le i \le 5 \tag{20}$$

Thus, for the two cases, it has

$$r_1 = \eta_1\eta_2\eta_3\eta_4, \quad r_2 = \eta_2\eta_3\eta_4, \quad r_3 = \eta_3\eta_4, \quad r_4 = \eta_4 \quad \text{and} \quad r_5 = 1 \tag{21}$$

Substituting r_i (i=1~5) and all the integral constants into Eqs. (8), (9) and (13), moment fields and the plastic limit load for the critical cases are then determined. The critical loading radii r_{jp} (j=1~5) are also obtained from Eq. (21).

For arbitrary loading radius r_p, the moment fields can be obtained by examining which part of the plate the loading radius lies in. When r_p satisfies $r_{(j-1)p} \le r_p \le r_{jp}$ (j=1~5), the moment fields become

$$m_{ri} = \frac{b_i}{1-a_i} - \frac{pr^2}{2(3-a_i)} + c_i r^{-1+a_i} \qquad 0 \le r \le r_{(j-1)} \tag{22a}$$

$$m_{rj1} = \frac{b_j}{1-a_j} - \frac{pr^2}{2(3-a_j)} + c_{j1} r^{-1+a_j} \qquad r_{(j-1)} \le r \le r_p \tag{22b}$$

$$m_{rj2} = \frac{b_j}{1-a_j} - \frac{pr_p^2}{2(1-a_j)} + c_{j2} r^{-1+a_j} \qquad r_p \le r \le r_j \tag{22c}$$

$$m_{ri} = \frac{b_i}{1-a_i} - \frac{pr_p^2}{2(1-a_i)} + c_i r^{-1+a_i} \qquad r_j \le r \le 1 \tag{22d}$$

As compared to the previous integration results, Eq. (22) introduces one more integral constant with one more continuity condition that $m_{rj1}(r=r_p)=m_{rj2}(r=r_p)$. Therefore, the integral constants and the division radius in the above equations can be calculated with the aids of the boundary conditions and the continuity conditions in a similar manner to the particular cases discussed above.

5. Velocity fields

According to the associated flow rule, there exist

$$\dot{k}_r = \dot{\lambda}\partial F/\partial m_r, \quad \dot{k}_\theta = \dot{\lambda}\partial F/\partial m_\theta \tag{23}$$

where $\dot{\lambda}$ is a plastic flow factor and F is the plastic potential which is the same as the yield function. The relations between the curvature rate and the rate of deflection are as follows:

$$\dot{k}_r = -d^2\dot{w}/dr^2, \quad \dot{k}_\theta = -d\dot{w}/(rdr) \tag{24}$$

Substituting Eq. (24) into Eq. (23) and using the yield conditions defined by Eq. (6), velocity fields corresponding to the five parts of the plate satisfy

$$\frac{d^2\dot{w}}{dr^2} + a_i\frac{dw}{rdr} = 0 \quad (i = 1\sim5) \tag{25}$$

The velocity fields are then obtained by integrating the above equation

$$\dot{w} = c_{1i}r^{1-a_i} + c_{2i} \quad (i = 1\sim5) \tag{26}$$

where, c_{1i}, c_{2i} (i=1~5) are integral constants. The continuity and boundary conditions of velocity are (1) $\dot{w}_0$ (r=0)=$\dot{w}_0$, (2) $\dot{w}_0$ and $d\dot{w}/dr$ (r=r_i, i=1~4) are continuous and (3) $\dot{w}$ (r=1)=0. Considering these conditions, the constants c_{1i} and c_{2i} in Eq. (26) then can be determined

$$c_{11} = -\frac{\dot{w}_0}{(d_{14}+d_{24})d_{13}d_{12}d_{11} + d_{23}d_{12}d_{11} + d_{22}d_{11} + d_{21}} \tag{27a}$$

$$c_{21} = \dot{w}_0 \tag{27b}$$

and

$$\begin{bmatrix} c_{1(i+1)} \\ c_{2(i+1)} \end{bmatrix} = \begin{bmatrix} d_{1i} & 0 \\ d_{2i} & 1 \end{bmatrix} \begin{bmatrix} c_{1i} \\ c_{2i} \end{bmatrix} \quad (i = 1\sim4) \tag{28}$$

in which d_{1i} and d_{2i} (i=1~5) are constants related to the continuity conditions, and they are:

$$d_{1i} = \frac{1-a_i}{1-a_{i+1}}r_i^{a_{i+1}+a_i}, \quad d_{2i} = -\frac{a_{i+1}-a_i}{1-a_{i+1}}r_i^{1-a_i} \quad (i = 1\sim4) \tag{29}$$

Substituting these integral constants into Eq. (26), velocity field of the clamped circular plate is then obtained.

6. Analytical results and discussion

Using the above derived close form plastic limit solutions based on the UYC for clamped circular plates under uniformly distributed loads with arbitrary loading radius, analytical results of plastic limit loads, moment fields and velocity fields are calculated. The influences of loading radius, edge effect and yield criteria on the plastic limit solutions are discussed in the following. Table 3 lists the plastic limit loads p, with the load uniformly distributed over the entire plate (r_p= 1), obtained based on the three yield criteria, namely the Tresca criterion, the Mises criterion and the twin shear stress criterion. The results obtained based on the Tresca and the Mises criteria by Hopkins and Wang (1954) are also given in the table for comparison purpose. As can be noted,

Table 3 Plastic limit loads for the three common yield criterion

Yield criterion	Tresca (v_p=0)			Mises (v_p=0.5)		Hill (1950), Yu (1983)	
	Hopkins (1954)	b=0.0001	b=0.001	Hopkins (1954)	b=0.5	b=1, v_p=0	b=1, v_p=0.5
p	11.26	11.259	11.260	12.5	12.720	12.176	13.708

the UYC can indeed approximate the Mises criterion, the twin shear criterion, and the Tresca criterion result in upper bound and lower bound plastic limit solutions. The edge effect, which depends on the plastic Possion's ratio v_p, also affects the plastic limit load.

It was proved in a previous paper (Hodge 1963) that the equilibrium Eqs. (4) and (5) are invalid if the Tresca criterion is used when the m_θ~m_r trajectory is on line EF (shown in Fig. 3). They are valid to the Tresca criterion only when the point E corresponds to the clamped edge (r= 1), which leads to v_p=0, or m_θ=0 and m_r=−1 at the plate edge. On the other hand, m_θ=0.5m_r or v_p=0.5 at the plate edge is assumed by using the Mises criterion according to the plastic flow requirement. Thus, the Tresca and the Mises criteria lead to two special plastic solutions in view of the edge effect. Using the UYC, it is convenient and straightforward to extend the yield trajectory to the line EF shown in Fig. 3, the plastic Poisson's ratio can vary between 0 and 0.5, which covers the general edge effect on plastic limit solutions. Nevertheless, it should be noted that a small value of b, e.g., b=0.0001, instead of b=0 should be used when specifying the Tresca criterion to avoid the singularity problem.

Fig. 4 illustrates the moment fields of the plate in plastic limit state under different load radii for v_p=0 and v_p=0.5, respectively obtained according to the three yield criteria, that is by using b=0, b=0.5 and b=1 in the UYC. For v_p=0, it can be seen, the influences of yield criteria on the radial moment distributions are insignificant, while they are prominent on the tangential moment distributions. The same moment values at the plate center and the plate edge are obtained by the three criteria. The twin shear stress criterion (b=1) results in the larger values for both m_r and m_θ, while the Tresca criterion (b=0) gives smallest estimations. The maximum tangential moment m_θ based on the UYC of $b \neq 0$ is not equal to 1 as predicted by the Tresca criterion, indicating the largest m_θ does not occur at the plate center if other yield criteria are applied. The largest m_θ is equal to $2(1+b)/(2+b)$ and occurs at $r=r_1$. For v_p=0.5, however, the three yield criteria affect the moment distributions near the edge significantly. The twin shear stress criterion results in the largest positive and negative values of m_r and m_θ, while the Tresca criterion results in the smallest ones.

Fig. 5(a) compares the velocity fields corresponding to different loading radii for the three criteria when v_p=0.5. It can be seen, the influences of the criteria on the velocity fields are larger when the transverse load is uniformly distributed over the whole plate or concentrated at the plate center. The velocity distribution is more concentrated at the center area of the plate as the loading radius reduces. It should be noted that the velocity field is singular at the plate center for Tresca criterion because it is expressed as a linear function near the plate center. Fig. 5(b) and Fig. 5(c) illustrates the edge effects for criteria b=1 and b=0.5 respectively. The influence of v_p on velocity distribution is small when the loading radius is large, and increases with the decrease of r_p. However, the effects become small with the decrease of the value of b, there is no influence if b=0. The velocity distribution for v_p=0 is always larger than that for v_p=0.5.

Besides the moment and velocity distributions, the plastic limit load of the clamped circular

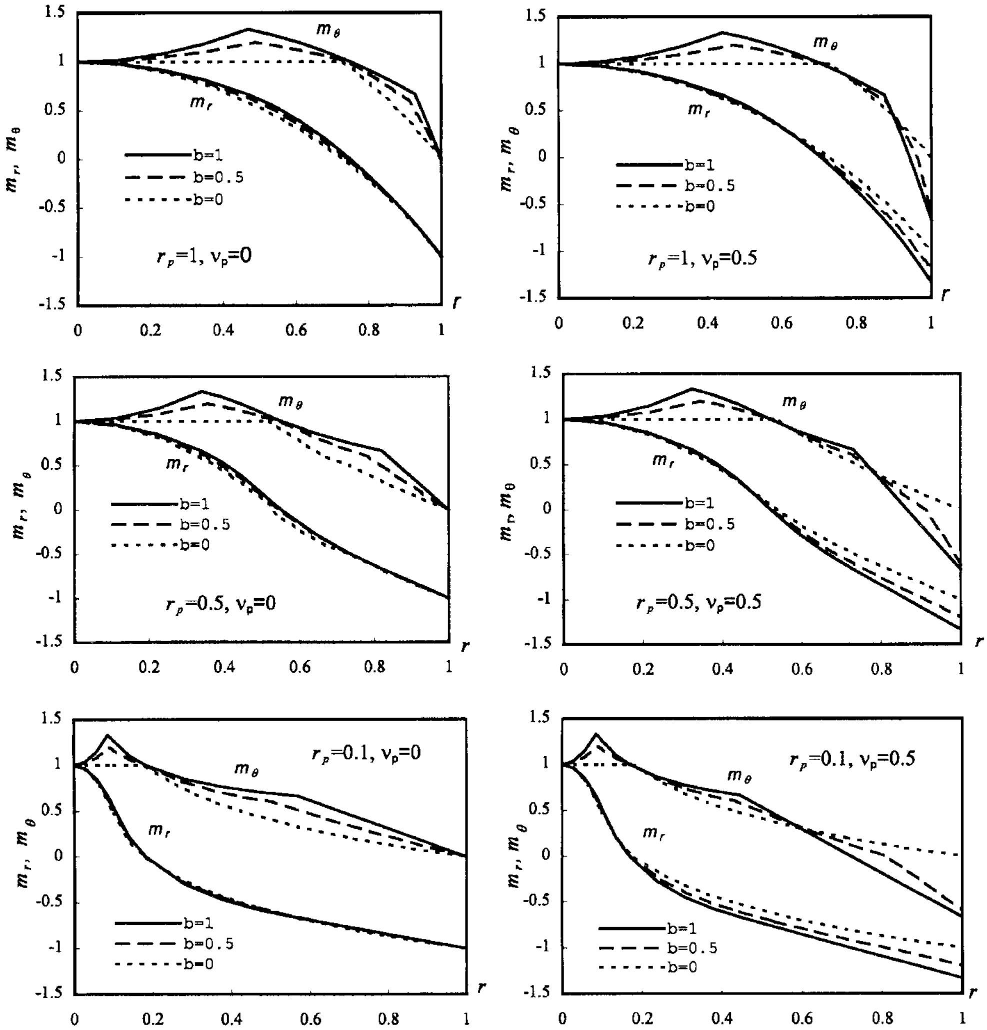
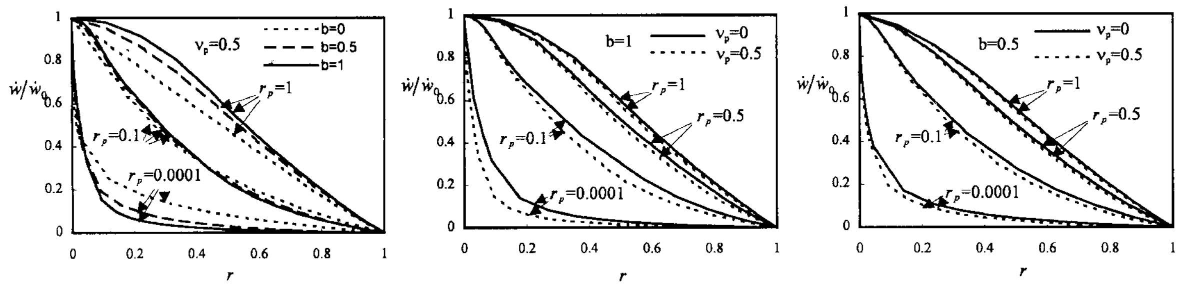


Fig. 4 Moment fields corresponding to different loading radii and edge effects



(a) ν_p = 0.5 (b) b=1 (c) b=0.5

Fig. 5 Velocity fields for clamped circular plate

plate with various loading radii is also an important factor for design. Defining the total plastic limit load P_T as $P_T = \pi r_p^2 p$, Fig. 6 illustrates the influences of yield criteria, edge effect and loading radius on P_T. As it can be seen, the total plastic load P_T increases as either b, r_p or v_p increases. The twin shear stress criterion results in the largest plastic limit loads, while the Tresca criterion gives the smallest estimations. Defining a difference ratio of plastic limit loads as

$$D_r = \frac{P_T - P_T\,(\text{Tresca})}{P_T\,(\text{Tresca})} \times 100\% \tag{30}$$

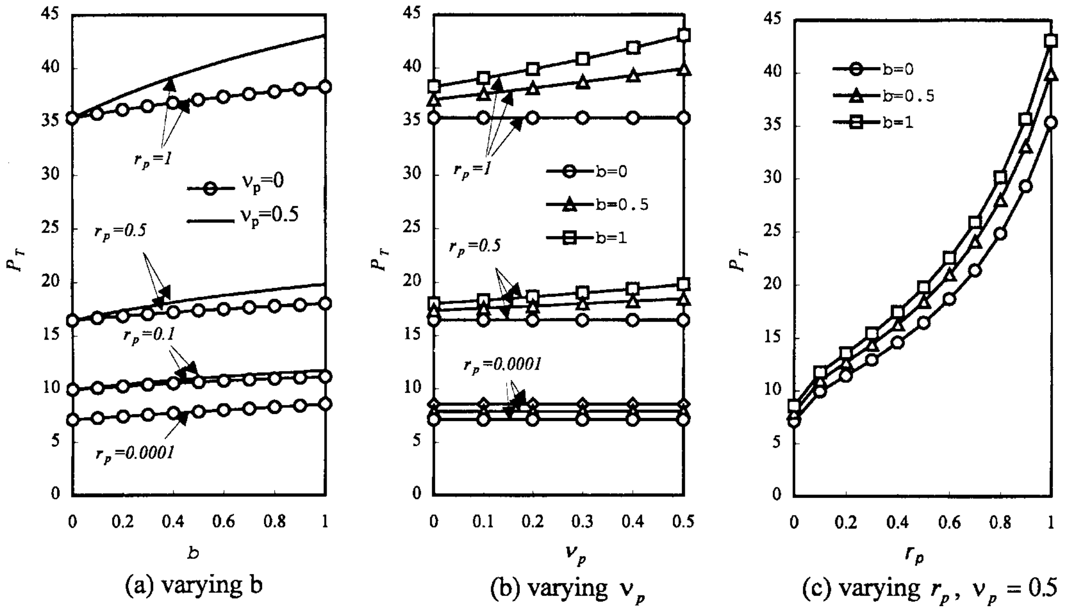


Fig. 6 Influences of yield criteria, edge effect and loading radius on the plastic limit load

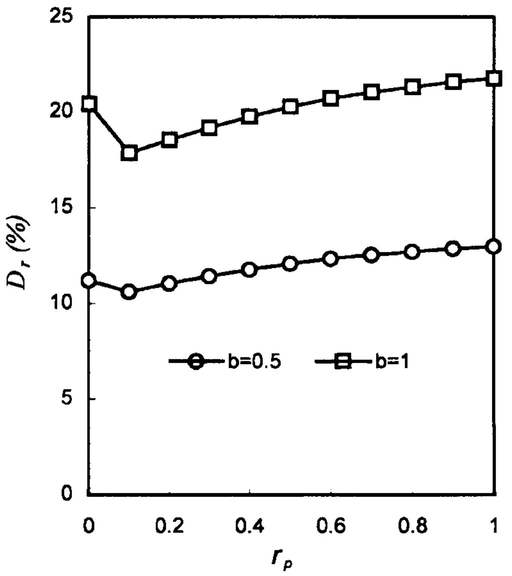


Fig. 7 Difference ratios of the plastic limit loads

where P_T (Tresca) is the plastic limit load derived based on the Tresca criterion. Fig. 7 illustrates the difference ratios corresponding to the Mises criterion and the twin shear stress criterion by specifying the weighting coefficient b as 0.5 and 1, respectively in the UYC. As can been seen, the difference ratio varies from 10.6% to 13.0% and from 17.8% to 21.7% with respect to the two criteria with different loading radii. The maximum difference ratio between the total plastic limit loads based on the Tresca criterion and the twin shear stress criterion is 21.7% when r_p=1 and v_p= 0.5. These observations imply the Tresca criterion might significantly underestimate the plastic limit load carrying capacities of a clamped circular plate. The Tresca criterion (b=0) can not properly reflect the edge effects (v_p) on the plastic limit load either.

7. Conclusions

The unified yield criterion (UYC) contains a family of piecewise and convex yield criteria by varying the parameter b from 0 to 1, and it is applicable to all the isotropic metal materials used in engineering practice. In particular, b=1 of the UYC corresponds to the twin shear stress criterion and b=0 corresponds to the Tresca criterion. It also approximates the Mises criterion by using b=0.5. For this reason, the UYC will be extremely useful in numerical analysis when it is implemented into a computer code. It allows user to specify which yield criterion to be used by simply changing the parameter b. In this paper, close form solutions of a clamped circular plate subjected to uniformly distributed load with different loading radii are derived based on the UYC. The influences of different yield criteria, edge effects and different loading radii on the plastic limit load and the corresponding moment and velocity fields are investigated. It has been demonstrated that the plastic limit load can differ by 21.7% if different yield criteria are employed in analysis.

Although the derived formulae are valid for small deformation problem only, they provide efficient estimations of the plastic limit load, the corresponding moments and velocities for a clamped circular plate with different loading radii and edge effects. Their results can be used as benchmark for general computer code developed, based on the UYC. It should be noted that when the plastic solution of a thick circular plate is analysed, the effects of shear force on the plastic behavior should also be included. These will be investigated in later studies with the aid of numerical analysis method.

References

Gamer, V. (1983), "Tresca's yield condition and the rotating disk", *J. Appl. Mech.*, **50**, 676-678.

Ghorashi, M. (1994), "Limit analysis of circular plates subjected to arbitrary rotational symmetric loading", *Int. J. Mech. Sci.*, **36**(2), 87-94.

Guven, V. (1992), "Elasto-plastic stresses in a rotating annular disk of variable thickness and variable density", *Int. J. Mech. Sci.*, **34**(2), 133-138.

Hill, R. (1950), *The Mathematical Theory of Plasticity*. Clarendon Press, Oxford.

Hodge, P.G. (1963), *Limit Analysis of Rotationally Symmetric Plates and Shells*. Prentice-Hall, Inc.

Hopkins, H.G. and Prager, W. (1953), "The load carrying capacities of circular plates", *J. Mech. Phys. Solids*, **2**, 1-13.

Hopkins, H.G. and Wang, A.J. (1954), "Load carrying capacities for circular plates of perfectly-plastic material with arbitrary yield condition", *J. Mech. Phys. Solids*, **3**, 117-129.

Ma, G.W., Yu, M.H., Iwasaki, S. and Miyamoto, Y. (1995), "Unified plastic limit solution to circular plate under portion uniform load", *J. Structural Engineering, JSCE*, **41A**, 385-392.

Ma, G.W., Yu, M.H., Iwasaki, S. and Miyamoto, Y. (1995), "Unified elastoplastic solution to rotating disc and cylinder", *J. Structural Engineering., JSCE*, **41A**, 79-85.

Yan Zongda, Bu Xiaoming (1996), "An effective characteristic method for plastic plane stress problems", *Journal of Engineering Mechanics, ASCE*, **122**(6), 502-506.

Yu Maohong (1983), "Twin shear stress yield criterion", *Int. J. Mech. Sci.*, **25**(1), 71-74.

Yu Maohong (1992), *New System of Strength Theory*, Xian Jiaotong University Press, Xian, China (In Chinese).

Yu Maohong, He Linan (1991), "A new model and theory on yield and failure of materials under complex stress state", *Mechanical Behavior of Materials-6, Pergamon Press*, **3**, 841-846.

Zaid, M. (1958), "On the carrying capacity of plates of arbitrary shape and variable fixity under a concentrated load", *Journal of Applied Mechanics, ASME*, **25**, 598-602.