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Geometrically nonlinear elastic analysis of space trusses

F. Tin-Loi† and S.H. Xia‡

School of Civil and Environmental Engineering, University of New South Wales, Sydney 2052, Australia

Abstract. A general framework for the nonlinear geometric analysis of elastic space trusses is presented. Both total Lagrangian and finite incremental formulations are derived from the three key ingredients of statics, kinematics and constitutive law. Particular features of the general methodology include the preservation of static-kinematic duality through the concept of fictitious forces and deformations, and an exact description for arbitrarily large displacements, albeit small strain, that can be specialized to any order of geometrical nonlinearity. As for the numerical algorithm, we consider specifically the finite incremental case and suggest the use of a conventional, simple and flexible arc-length based method. Numerical examples are presented to illustrate and validate the accuracy of the approach.

Key words: large displacement; nonlinear analysis; space trusses.

1. Introduction

The analysis of pin-jointed spatial structures in which geometric and material nonlinearities are present is an important problem in structural mechanics. The penetration of limit states design principles (e.g., Supple and Collins 1981) into the design of such structures has also made it almost mandatory to carry out this type of analysis.

A number of approaches, each with its own advantages of use and accuracy, have been proposed to carry out the evolutive elastoplastic analysis of space trusses in the large displacement regime. A recent state-of-the-art survey of related work (Gioncu 1995) lists some 320 references, attesting to the vigour with which research in the area has been and is being carried out. It is, however, to be noted that whether the structure remains elastic or develops, as is typically assumed, zones of plastic deformation, a key step in the analysis is the ability to accurately capture the elastic behaviour of the structure under large displacements.

The primary objective of this paper is to present a computation-oriented method which is not only suitable for arbitrarily large displacement, albeit small strain, analysis of space trusses but also one that can easily be extended to cater for elastoplasticity. This extension is possible since we adopt constitutive laws reflecting directly the behaviour of the constituent finite element members (Corradi 1978), rather than a stress-strain relation at the material level. Such element

† Professor
‡ Ph.D. Candidate

constitutive laws can either be obtained experimentally as in Maier and Zavelani-Rossi (1970) or by analysis of a theoretical strut model (e.g., Ballio *et al.* 1973, Madi 1984).

In the present work, we consider only the elastic case; inclusion of elastoplasticity will be presented in a parallel paper. The framework adopted to derive the governing system is the one popularized by Lloyd Smith and De Freitas primarily for the large displacement analysis of rigidly-connected skeletal structures (e.g., Lloyd Smith 1979, De Freitas 1979, De Freitas and Lloyd Smith 1984-85). In particular, the three fundamental conditions of equilibrium, compatibility and constitution are combined, thus avoiding the use of variational theorems. Symmetric relations are obtained through the device of 'fictitious' forces and deformations, and the forcible use of 'residuals', if necessary, to describe the constitutive laws. This form is often advantageous for both numerical and theoretical developments (De Freitas and Lloyd Smith 1984). The applicable description for space trusses, and the one we closely follow, is detailed by De Freitas *et al.* (1985) who also deal with the elastic case. However, at variance with this work, we propose the adoption of a more conventional solution algorithm based on an iterative process involving the well-known arc-length procedure; De Freitas *et al.* (1985) use a perturbation technique. We anticipate that our numerical algorithm will also form the backbone of a predictor-corrector type path-following scheme to deal with elastoplastic constitutive laws which exhibit, as an expected behaviour in space trusses, softening.

The organization of this paper is as follows. In Section 2, we detail both Lagrangian and finite incremental descriptions of statics and kinematics for a suitably discretized structure. The corresponding total and incremental forms of the elastic constitutive laws are briefly presented next. Section 4 combines the relevant symmetric, explicitly linear static-kinematic relations with the constitutive laws to produce both Lagrangian and finite incremental formulations. We deal solely with the solution of the incremental problem in Section 5 by proposing a numerical algorithm which is capable of following the equilibrium paths during structural evolution. We then present some examples to illustrate application of the method in Section 6, before concluding with some pertinent remarks.

2. Static-kinematic relations

It is attractive to develop large displacement formulations for nonlinear structural analysis based on a small displacement framework. One of the ideas used in this respect is based on original work by Denke (1960) and extensively applied by Lloyd Smith and De Freitas (e.g., Lloyd Smith 1979, De Freitas 1979, De Freitas and Lloyd Smith 1984-85). It employs the artifice of additional or fictitious forces and deformations. In addition, as shown in this section, static-kinematic duality can be preserved.

2.1. Lagrangian description

Consider the space truss as an aggregate of n finite elements. As shown in Fig. 1, let Q^m and q^m denote, respectively, the natural generalized stress (axial force) and strain resultants pertaining to a generic element m of length L^m in its undeformed configuration at some orientation specified by local axes 1-2-3, with respect to a global reference axis system. Further, let F^m and u^m represent, respectively, the vectors of unconstrained nodal forces and displacements.

The exact description of member equilibrium in its *deformed* configuration can be expressed in

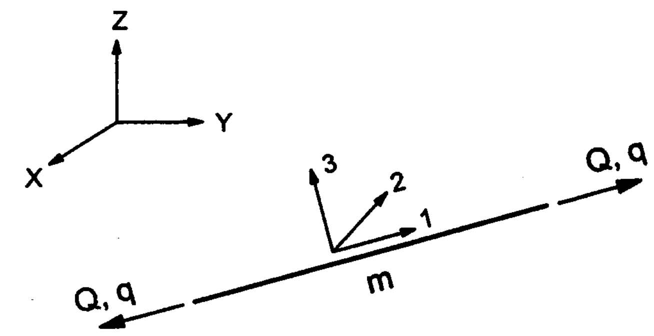

Fig. 1 Natural stresses and strains

the form

$$F^m = \begin{bmatrix} A^m & A_\pi^m \end{bmatrix} \begin{bmatrix} Q^m \\ -\pi^m \end{bmatrix} \tag{1}$$

where π^m is a vector of additional nodal forces (Fig. 2) acting also on the undeformed member. The constant matrices A^m and A_π^m are defined in terms of the direction cosines l_i^m (i=1, 2, 3) of the local axes i with respect to the Lagrangian axis system as follows:

$$A^m = \begin{bmatrix} -l_1^m \\ l_1^m \end{bmatrix} \tag{2}$$

$$A_\pi^m = \begin{bmatrix} -l_1^m & -l_2^m & -l_3^m \\ l_1^m & l_2^m & l_3^m \end{bmatrix} \tag{3}$$

In turn, the additional forces are defined by

$$\pi^m = Z^m \, Q^m \tag{4}$$

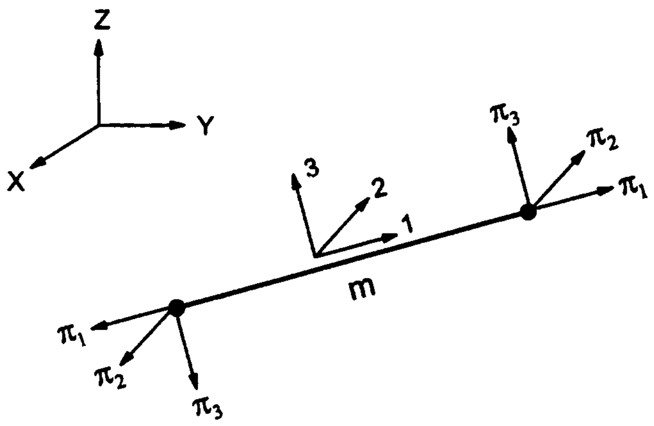

Fig. 2 Fictitious forces

with

$$Z^{mT} = \left[1 - \frac{L^m + \delta^m_{\pi 1}}{L^m_c} - \frac{\delta^m_{\pi 2}}{L^m_c} - \frac{\delta^m_{\pi 3}}{L^m_c}\right] \tag{5}$$

where L^m_c is the deformed member chord length and vector $\delta^{mT}_\pi = [\delta^m_{\pi 1}\ \delta^m_{\pi 2}\ \delta^m_{\pi 3}]$ represents auxiliary displacements (shown in Fig. 3 for the planar case for simplicity) associated with the additional forces.

At this stage it would be worthwhile commenting on the rationale behind the use of the aforementioned fictitious forces. This will also give some physical insight into their meanings. The idea behind the introduction of fictitious (pseudo, additional, or Ersatz, as they have been also called in the literature) forces can be traced back to Denke (1960). The primary motivation for their use stems from the appeal of constructing large displacement formulations of structural analysis that will make direct use of basic structural coefficients commonly used in a small displacement calculation. More specifically, we note that the equilibrium Eq. (1) represents an exact relationship for arbitrarily large displacements and yet can be written with respect to an *undeformed* configuration, provided the nodal forces F^m are 'corrected' by additional forces π^m, dependent upon the actual displacements. The role of fictitious forces is thus to report the axial force in a member from the undeformed configuration to the deformed one. Mathematically, the advantage provided by use of these forces is far greater. As will be clear in the following, symmetry of the governing relations will be preserved with the implication that extremum principles, so useful for both quantitative and qualitative characterizations of stability, existence and uniqueness of solutions, can be obtained.

Static-kinematic duality (e.g., Lloyd Smith 1979, De Freitas 1979, De Freitas and Lloyd Smith 1984, 1984-85, De Freitas *et al.* 1985) can be maintained by writing the compatibility equations in an explicitly linear format as follows:

$$\begin{bmatrix} q^m + q^m_\pi \\ \delta^m_\pi \end{bmatrix} = \begin{bmatrix} A^{mT} \\ A^{mT}_\pi \end{bmatrix} u^m \tag{6}$$

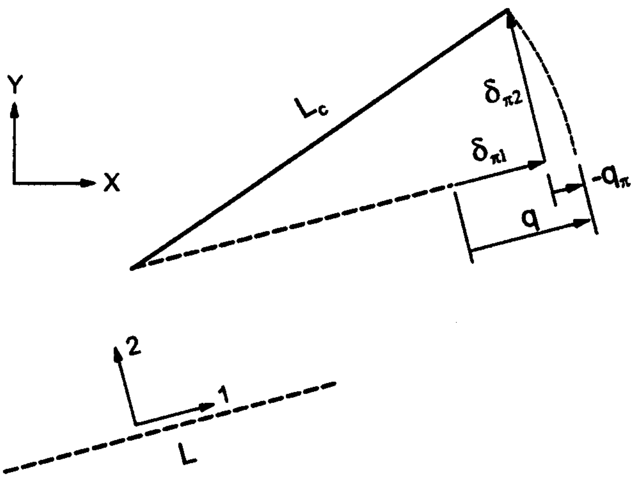

Fig. 3 Original and displaced configurations

where q_π^m is an additional fictitious deformation defined, as is obvious from Fig. 3, by

$$q_\pi^m = \delta_{\pi 1}^m - q^m \tag{7}$$

The auxiliary displacements δ_π^m are associated (dual) with the additional forces π^m. Physically, they are related to the member chord length L_c^m through an obvious compatibility condition (see Fig. 3 for the planar case).

We can now combine Eqs. (1) and (6) to form the element static-kinematic relations

$$\begin{bmatrix} \cdot & A^m & A_\pi^m \\ A^{mT} & \cdot & \cdot \\ A_\pi^{mT} & \cdot & \cdot \end{bmatrix} \begin{bmatrix} u^m \\ Q^m \\ -\pi^m \end{bmatrix} = \begin{bmatrix} F^m \\ q^m + q_\pi^m \\ \delta_\pi^m \end{bmatrix} \tag{8}$$

which clearly exhibit a duality relationship. As with Eq. (4), it will be convenient to express q_π^m in terms of δ_π^m through matrix Z^m. Simple algebraic manipulations lead to

$$q_\pi^m = Z^{mT}\, \delta_\pi^m + R_{q\pi}^m \tag{9}$$

where

$$R_{q\pi}^m = L^m \left(1 - \frac{L^m + \delta_{\pi 1}^m}{L_c^m} \right) \tag{10}$$

Hence Eq. (8) can be simplified by eliminating π^m, δ_π^m and q_π^m to give

$$\begin{bmatrix} \cdot & C^{mT} \\ C^m & \cdot \end{bmatrix} \begin{bmatrix} u^m \\ Q^m \end{bmatrix} = \begin{bmatrix} F^m \\ q^m \end{bmatrix} + \begin{bmatrix} \cdot \\ R_{q\pi}^m \end{bmatrix} \tag{11}$$

where

$$C^{mT} = A^m - A_\pi^m Z^m \tag{12}$$

The governing exact Lagrangian static-kinematic relations for the whole structure, covering all n elements then become

$$\begin{bmatrix} \cdot & C^T \\ C & \cdot \end{bmatrix} \begin{bmatrix} u \\ Q \end{bmatrix} = \begin{bmatrix} F \\ q \end{bmatrix} + \begin{bmatrix} \cdot \\ R_{q\pi} \end{bmatrix} \tag{13}$$

where u represents the vector of nodal displacements, F is the applied nodal load vector, and the indexless symbols have self-evident definitions associated with conventional finite element descriptions, e.g. $Q^T = [Q^1, \cdots, Q^n]$, $\pi^T = [\pi^{1T}, \cdots, \pi^{nT}]$, matrices A, A_π are assembled through appropriate incidence matrices and $Z = diag\ [Z^1, \cdots, Z^n]$. The term $R_{q\pi}$ is generally considered to be a vector of residuals which destroy the duality relationship given by Eq. (13); these residuals are usually treated as known quantities in the solution process.

2.2. Finite incremental description

As detailed in De Freitas (1979) and De Freitas *et al.* (1985), the finite incremental description of statics and kinematics, which is invariably more suitable for general nonlinear analysis, can be easily obtained by replacing variables in the relevant Lagrangian expressions by their increments. We adopt the notation that any such finite increment is denoted by Δ and symbols with and without hats representing known and unknown values, respectively; e.g., $x=\hat{x}+\Delta x$.

For statics, the incremental version of Eq. (1) is

$$\Delta F^m = [A^m \ A_\pi^m] \begin{bmatrix} \Delta Q^m \\ -\Delta\pi^m \end{bmatrix} \tag{14}$$

while for kinematics Eq. (6) is replaced by

$$\begin{bmatrix} \Delta q^m + \Delta q_\pi^m \\ \Delta\delta_\pi^m \end{bmatrix} = \begin{bmatrix} A^{mT} \\ A_\pi^{mT} \end{bmatrix} \Delta u^m \tag{15}$$

Further, the incremental forms of Eqs. (4) and (9) are, respectively,

$$\Delta\pi^m = \hat{Z}^m \Delta Q^m + \hat{P}^m \Delta\delta_\pi^m + \Delta R_\pi^m \tag{16}$$

$$\Delta q_\pi^m = \hat{Z}^m \Delta\delta_\pi^m + \Delta R_{q\pi}^m \tag{17}$$

where, with $\hat{Z}^{mT} = [\hat{Z}_1^{mT} \ \hat{Z}_2^{mT} \ \hat{Z}_3^{mT}]$,

$$\hat{P}^m = \frac{\hat{Q}^m}{L_c^m} \begin{bmatrix} \hat{Z}_1^m \hat{Z}_1^m - 2\hat{Z}_1^m & (\hat{Z}_1^m - 1)\hat{Z}_2^m & (\hat{Z}_1^m - 1)\hat{Z}_3^m \\ (\hat{Z}_1^m - 1)\hat{Z}_2^m & \hat{Z}_2^m \hat{Z}_2^m - 1 & \hat{Z}_2^m \hat{Z}_3^m \\ (\hat{Z}_1^m - 1)\hat{Z}_3^m & \hat{Z}_2^m \hat{Z}_3^m & \hat{Z}_3^m \hat{Z}_3^m - 1 \end{bmatrix} \tag{18}$$

$$\Delta R_{q\pi}^m = \frac{1}{2\hat{L}_c^m} (\Delta q^m \Delta q^m - \Delta\delta_\pi^{mT} \Delta\delta_\pi^m) \tag{19}$$

$$\Delta R_\pi^m = \frac{\Delta Q^m}{\hat{Q}^m} \hat{P}^m \Delta\delta_\pi^m + \frac{\Delta R_{q\pi}^m}{\hat{L}_c^m} (\hat{Q}^m + \Delta Q^m) \begin{bmatrix} \hat{Z}_1^m - 1 \\ \hat{Z}_2^m \\ \hat{Z}_3^m \end{bmatrix} \tag{20}$$

The key incremental description of statics can be obtained by substituting Eq. (16) into Eq. (14) and using Eq. (15) to eliminate $\Delta\delta_\pi^m$. Similarly, for kinematics, we substitute Eq. (17) into the first part of Eq. (15) and then use the second part of Eq. (15) to eliminate $\Delta\delta_\pi^m$. The resulting static-kinematic relations then read

$$\begin{bmatrix} \hat{K}_G^m & \hat{C}^{mT} \\ \hat{C}^m & \cdot \end{bmatrix} \begin{bmatrix} \Delta u^m \\ \Delta Q^m \end{bmatrix} = \begin{bmatrix} \Delta F^m \\ \Delta q^m \end{bmatrix} + \begin{bmatrix} A_\pi^m \Delta R_\pi^m \\ \Delta R_{q\pi}^m \end{bmatrix} \tag{21}$$

where

$$\hat{K}_G^m = -A_\pi^m \hat{P}^m A_\pi^{mT} \tag{22}$$

$$\hat{C}^{mT} = A^m - A_\pi^m \hat{Z}^m \tag{23}$$

At the structure level, the statics-kinematic relations become

$$\begin{bmatrix} \hat{K}_G & \hat{C}^T \\ \hat{C} & \cdot \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta Q \end{bmatrix} = \begin{bmatrix} \Delta F \\ \Delta q \end{bmatrix} + \begin{bmatrix} A_\pi \Delta R_\pi \\ \Delta R_{q\pi} \end{bmatrix} \tag{24}$$

by a familiar reinterpretation of the indexless form of Eq. (21), as was explained for the Lagrangian case.

3. Constitutive relations

The third and last ingredient of the formulation are the constitutive relations associating the stress and strain resultants. These laws thus link the static and kinematic variables and are generally obtained from an analysis embodying both geometric and material properties. As for the static and kinematic relations, they can be described in both Lagrangian and finite incremental formats, which we now briefly present. We assume throughout that the behaviour is purely elastic.

3.1. Lagrangian description

In its most general form, the relationship associating the generalized stress to the generalized strain for a typical member m is simply given as

$$Q^m = S^m q^m \tag{25}$$

where S^m is a nonlinear member stiffness which is a function of Q^m and which can also include imperfection (end and out-of-straightness) effects.

As an example, we state in the following a simple expression especially developed for thin-walled circular tubes by Madi and Lloyd Smith (1984). The member stiffness for a tube with an inner radius R^m, cross-sectional area A^m, Young's modulus E, initial midspan deformation e^m, and Euler buckling load Q_e^m (assumed negative) is given by

$$S^m = \begin{cases} s^m & \text{if } Q^m \geq 0 \\ s^m \left[1 + v^2 (1 + 0.5r) r \dfrac{Q_e^m}{Q^m} \right]^{-1} & \text{if } Q^m < 0 \end{cases} \tag{26}$$

where $s^m = EA^m/L^m$, $v = e^m/R^m$, $r = Q^m/(Q_e^m - Q^m)$. De Freitas and Lloyd Smith (1983), among other researchers, have also derived accurate and useful elastic constitutive laws for planar beam-columns which can be specialized to truss members.

Finally, Eq. (25) can be interpreted at the structural level to give

$$Q = Sq \tag{27}$$

where $S = diag[S^1, \cdots, S^n]$, etc.

3.2. Finite incremental description

The Lagrangian constitutive laws can easily be expressed in finite incremental form to give a general expression, at the element level, of the form

$$\Delta Q^m = \hat{S}_\Delta^m \Delta q^m + \Delta R_s^m \tag{28}$$

where $\hat{S}_\Delta^m$ represents the constant element incremental stiffness and ΔR_s^m collects all nonlinear residual terms. We refer the interested reader to De Freitas and Lloyd Smith (1983) and Tin-Loi and Misa (1996) for typical examples of incremental stiffnesses.

For the whole structure, Eq. (28) becomes

$$\Delta Q = \hat{S}_\Delta \Delta q + \Delta R_s \tag{29}$$

where, as usual $\hat{S}_\Delta = diag[\hat{S}_\Delta^1, \cdots, \hat{S}_\Delta^n]$.

4. Formulations

Total Lagrangian and finite incremental formulations can now be easily obtained through appropriate combinations of the relevant statics, kinematics and constitutive relations. In particular, we eliminate the stress and strain variables to generate the governing systems in displacement variables.

4.1. Lagrangian

Simple manipulations of Eqs. (13) and (27) lead to

$$Ku = F + R \tag{30}$$

where

$$K = C^T S\, C \tag{31}$$

$$R = C^T S\, R_{q\pi} \tag{32}$$

It is evident that the governing system represented by Eq. (30) has basically the same familiar form as a linear stiffness equation, except for the nonlinearity of K and the presence of residual R. In fact, as shown for instance by Tin-Loi and Vimonsatit (1996), such exact formulations as given by Eqs. (30)~(32) can be systematically and consistently approximated to any analysis order by carrying out the necessary series expansions followed by appropriate truncations.

For the elastic (reversible) case, the Lagrangian formulation evidently leads to the same solution as a finite incremental approach. However, the latter is usually preferred as it is not only typically easier to solve but is also directly applicable, albeit in an approximate way, to inelastic irreversible constitutive laws.

Finally, an interesting and challenging problem would be to try and capture *all* solutions to Eq. (30), or show that none exists, for a given load level F, as has been attempted recently in the quasibrittle fracture context (Bolzon *et al.* 1994).

4.2. Finite incremental

In this case, we combine and simplify Eqs. (24) and (29) to generate the governing finite incremental formulation represented by

$$\hat{K}_{\Delta}\Delta u = \Delta F + \Delta R \tag{33}$$

where

$$\hat{K}_{\Delta} = \hat{K}_G + \hat{C}^T \hat{S}_{\Delta} \hat{C} \tag{34}$$

$$\Delta R = A_{\pi} \Delta R_{\pi} + \hat{C}^T \hat{S}_{\Delta} \Delta R_{q\pi} - \hat{C}^T \Delta R_s \tag{35}$$

The form of Eq. (33) is worthy of note. In particular, symmetry of $\hat{K}_{\Delta}$ has been preserved and all nonlinear terms have been collected in residual ΔR. It is also easy to see that such a form also suggests direct application of a simple iterative solution algorithm, as described next. Finally, as for the Lagrangian formulation, any order approximation can be obtained from Eq. (34) through suitable series expansions.

5. Solution algorithm for incremental problem

In this section, we outline our preferred solution algorithm for solving the incremental problem, a typical step of which is represented by Eqs. (33)~(35).

Two popular schemes in current use are the perturbation method (e.g. De Freitas *et al.* 1985, Hangai and Kawamata 1973) and the Newton-Raphson iterative method or a variant thereof. In a perturbation approach, Eq. (33) is replaced by an equivalent infinite sequence of recursive linear equation systems by expanding intervening variables in a power series about some perturbation parameter (load, displacement or work rate) and equating terms of the same order as the parameter. The infinite sequence of equations is recursive since the nonlinear residuals of any order are functions of variables of lower order. This technique can generate highly accurate results since it is capable of providing monotonically improving approximations to the solution. A Newton-Raphson method, on the other hand, may be less robust and accurate, but is a simple and natural solution technique. Its robustness and hence ability to traverse critical points and trace unstable equilibrium paths can be greatly enhanced if modified by the popular arc-length procedure. Such considerations and the fact that we required a solution scheme that could easily be extended to accommodate irreversible material behaviour prompted us to adopt an arc-length based algorithm.

In the first instance, assume a constant specified load step ΔF. The nonlinear Eq. (33) can be solved iteratively if rewritten in the computational form

$$\hat{K}_{\Delta}\Delta u_i = \Delta F + \Delta R_{i-1} \tag{36}$$

where subscript i denotes the iteration number, and $\Delta R_0=0$. The basic algorithm is then as follows:

Step 1 (Initialization)

- $i=1$, $\Delta R_0=0$.
- Assemble $\hat{K}_{\Delta}$.

Step 2 (General iteration)

- Solve Eq. (36) for Δu_i.
- Calculate Δq_i, ΔQ_i, $\cdots$, ΔR_i.
- If $\| \Delta R_i - \Delta R_{i-1} \| \leq \varepsilon \| \Delta F \|$ (e.g., $\varepsilon=10^{-4}$) then stop, else go to start of Step 2.

The following remarks are worthy of note at this stage.

(a) In the above basic scheme, the nonlinear residual ΔR is explicitly calculated. In fact, it is not necessary to do so since the difference in successive residuals obviously represents an out-of-balance load. Thus, Eq. (36) can be written as

$$\hat{K}_{\Delta}\Delta u_i - \hat{K}_{\Delta}\Delta u_{i-1} = \hat{F} + \Delta F - C_{i-1}^T Q_{i-1} \tag{37}$$

where Δu_0=0 and $\hat{F}=C_0^T Q_0$.

(b) An alternative expression for ΔR_{i-1} is clearly available from Eqs. (36) and (37).

(c) Our implementation involves the use of a spherical arc-length method (Forde and Stiemer 1987) on the basic system

$$\hat{K}_{\Delta}\Delta u_i = \Delta F_{i-1} + \Delta R_{i-1} \tag{38}$$

to calculate improved values of Δu_i and ΔF_i, with ΔF_0 being the specified load step at the start of the iteration.

(d) As mentioned earlier, the algorithm is well suited for the so-called 'predictor' phase of a numerical approach for elastoplastic constitutive laws.

6. Numerical examples

Three examples are given in this section to illustrate application of the computational procedure. Example 1 is a simple two-bar truss analysed for two cases: linear elastic constitutive law and nonlinear elastic law according to Eq. (26). Example 2 is a twelve-bar space truss used by several researchers (e.g., Yang and Leu 1991, Krenk and Hededal 1995) to illustrate the complex behaviour that can be exhibited under large displacements. Finally, example 3 involves a shallow truss dome and is usually considered to be a benchmark problem for evaluation of geometrically nonlinear solution algorithms.

6.1. Example 1

This example concerns the simple two-bar truss loaded as shown in the inset of Fig. 4. Bar lengths L are 1 m each with each bar inclined at 15° to the horizontal. The bars have tubular cross-sections with 0.02 m inner radius R and 0.004 m wall thickness t. It was analysed for two cases: (a) linear elastic with $E=2\times10^8$ kPa, and (b) nonlinear elastic in compression according to Eq. (26) with $e=0.00003\ L^2/R$.

The results of the analyses in the form of load P versus vertical deflection v are shown in Fig. 4. Crosses are used for the linear case and solid circles for the nonlinear case; complete agreement with the easily obtainable closed form results, shown as solid lines, can be observed.

Convergence was obtained within six iterations near critical points and within four iterations elsewhere.

6.2. Example 2

The symmetrically loaded twelve-bar truss shown in Fig. 5 is believed (Krenk and Hededal 1995) to be a difficult test for nonlinear finite element analysis solution algorithms. Key structure dimensions are: h=35.355 cm, a=50 cm and b=60 cm. We assumed linear elasticity with A=10 cm^2 and E=0.1 kN/cm^2.

We managed to capture the complete deformation history of this structure. This is illustrated in

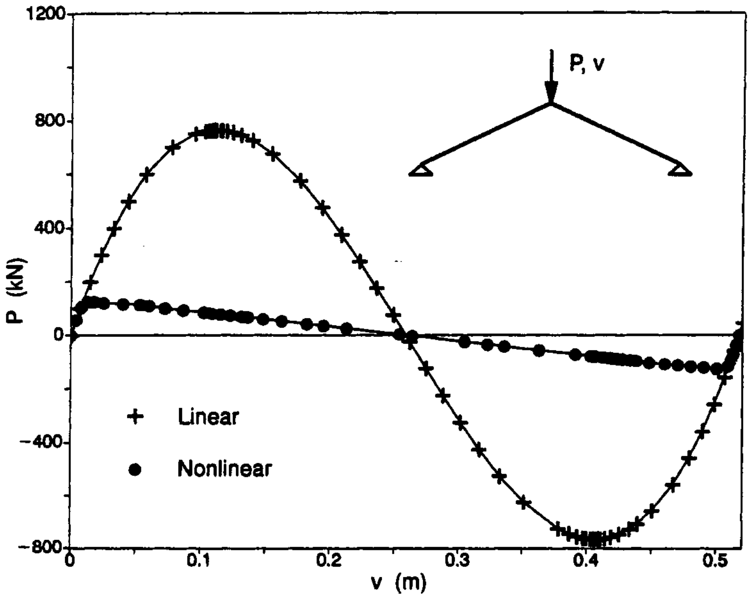

Fig. 4 Example 1: two-bar truss and load-deflection resulsts

Figs. 6~8 representing, load P versus, respectively, u, v and w components of deflection. Solid circles represent actual computed points; dashed lines have been added to clarify the evolution of the complex equilibrium paths. The analyses by Yang and Leu (1991), shown as solid lines, are in excellent agreement with our results but terminated prematurely. Recent work by Krenk and Hededal (1995) also traversed the series of snap-through and traced the complete equilibrium paths. Their results are fairly close to ours, the difference being possibly due to their use of a different strain measure; see Yang and Leu (1991) for explicit expressions of these different strain definitions.

Again no numerical difficulties were encountered with convergence being attained within eight iterations in the vicinity of critical points and five iterations elsewhere.

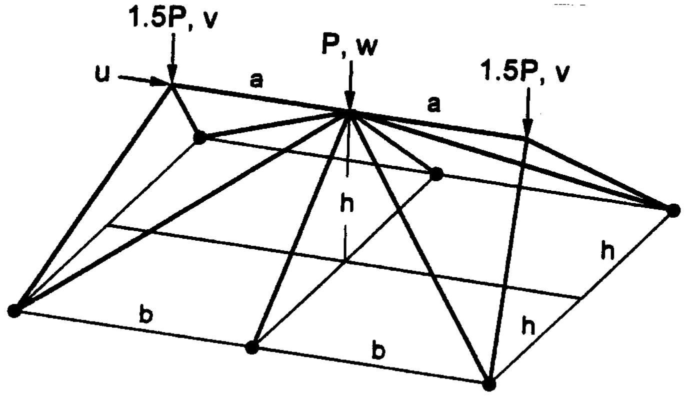

Fig. 5 Example 1: twelve-bar truss

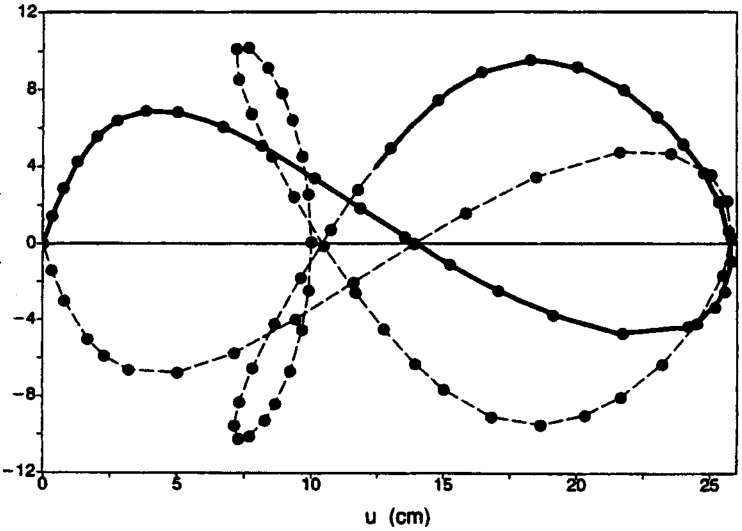

Fig. 6 Example 2: load versus u deflection component results

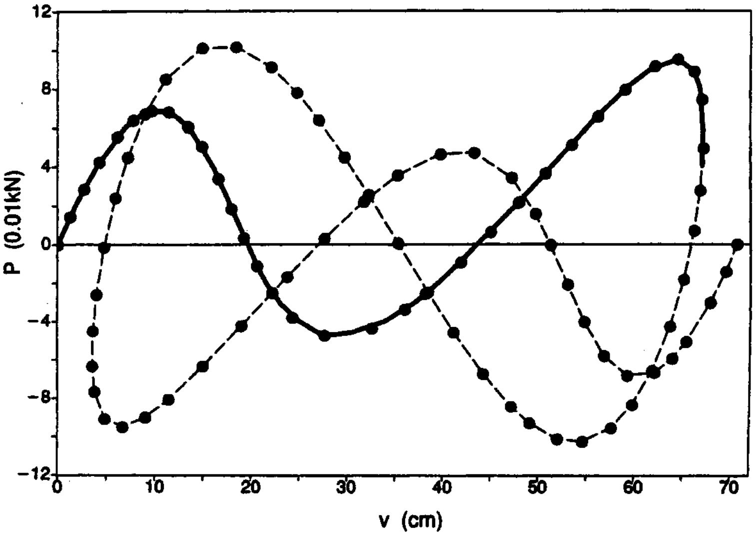

Fig. 7 Example 2: load versus v deflection component results

6.3. Example 3

This third example is concerned with the shallow truss dome shown in Fig. 9. Under the assumption of a linear elastic material with A=1 cm^2, we analysed it for two load cases: (a) a point load P applied vertically downwards at node 1 together with point loads of $2P$ applied also vertically downwards at each of nodes 2~7, and (b) the same pattern of loading as for the previous case but with an asymmetric imperfection introduced into the structure consisting of reducing by 0.2 cm the vertical heights above the ground of nodes 2, 4 and 6.

For these load cases, Figs. 10~12 show our results in the form of load versus vertical or radial deflections at the indicated nodes; solid circles refer to the original structure and crosses to the

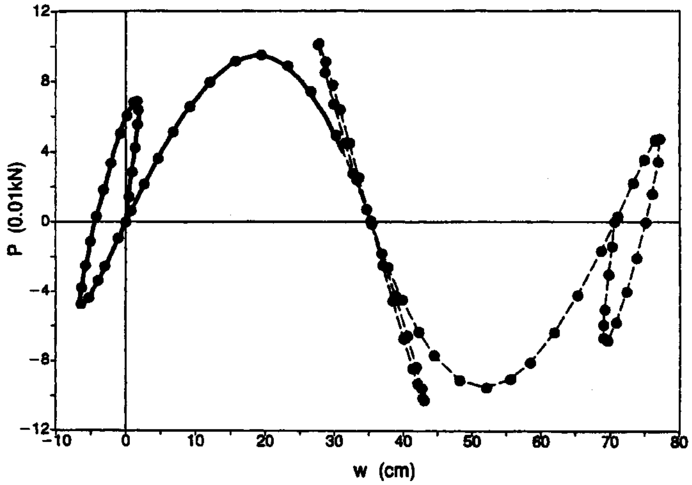

Fig. 8 Example 2: load versus w deflection component results

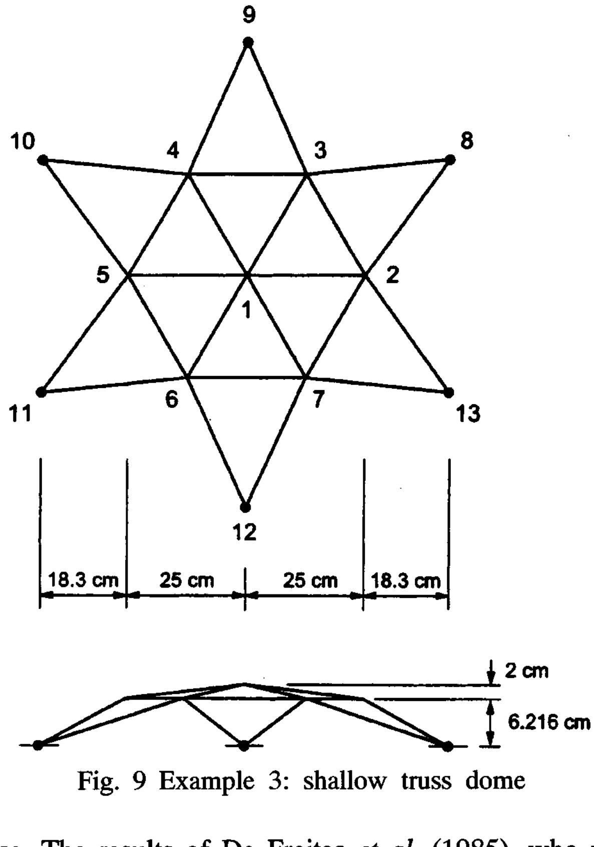

Fig. 9 Example 3: shallow truss dome

imperfection induced case. The results of De Freitas *et al.* (1985), who used a work perturbation approach, are shown as solid lines for comparison. Very good agreement was obtained for both the original structure and for the topographically altered structure.

A maximum of twelve iterations were required for convergence near critical points; elsewhere, convergence was achieved within six iterations.

7. Concluding remarks

We have presented a systematic and unified framework for the formulation and numerical solution of the large displacement elastic analysis of space trusses under quasistatic loading. Using the three fundamental conditions of statics, kinematics and constitutive law, both total Lagrangian and finite incremental formulations are easily developed in parallel. The device of fictitious forces and deformations leads to the preservation of static-kinematic duality and enables the governing equations to be written using the undeformed configuration. Since we also intend to consider elastoplastic materials which can exhibit a softening behaviour at the member level, only the more appropriate finite incremental formulation has been considered for numerical solution. A simple

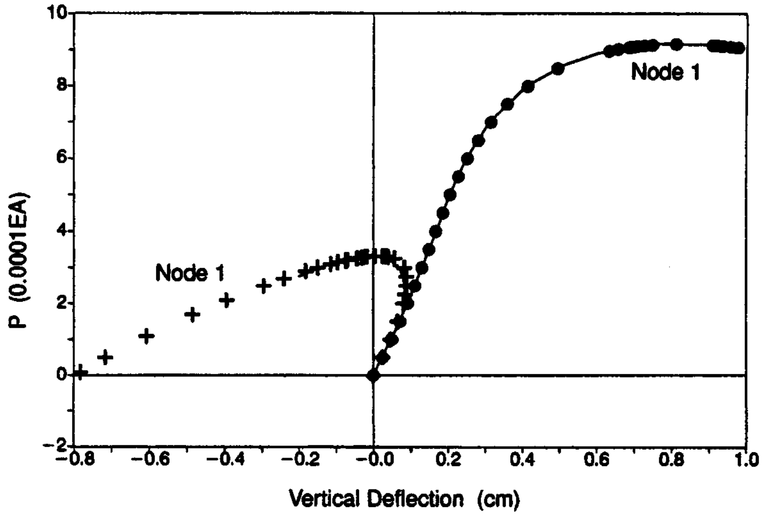

Fig. 10 Example 3: load versus vertical deflection at node 1

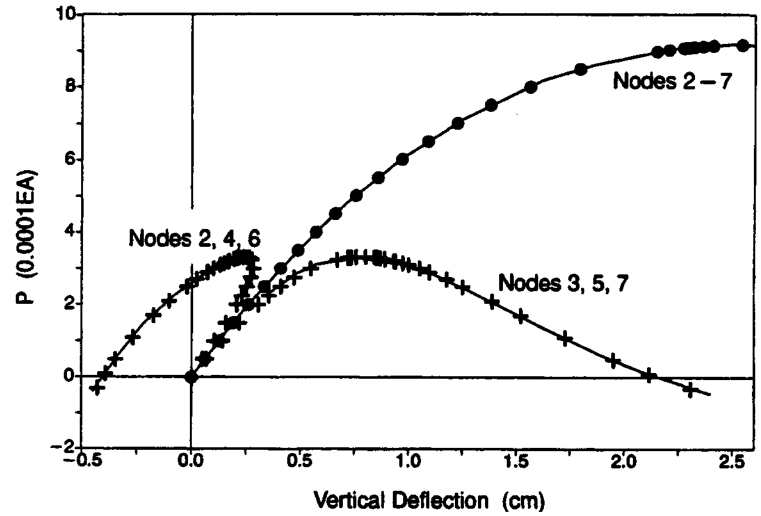

Fig. 11 Example 3: load versus vertical deflection at nodes 2~7

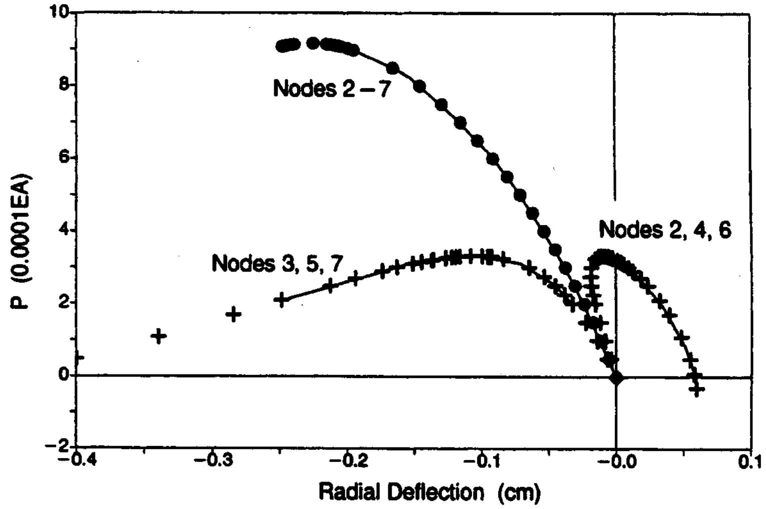

Fig. 12 Example 3: load versus radial deflection at nodes 2~7

arc-length based procedure, also believed to be suitable for inelastic behaviour, has been adopted as the solver. We have solved numerous examples, including some notoriously difficult ones, but have had no difficulty in tracing efficiently and completely all equilibrium paths. Three representative examples are presented in this paper. Current work is aimed at solving large-scale structures, extending the framework to inelastic softening behaviour and at obtaining in such a case theoretical, albeit implementable, convergence results.

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