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Fundamental theory of curved structures from a non-tensorial point of view

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Abstract. The present paper shows a new non-tensorial approach to derive basic equations for various structural analyses. It can be used directly in numerical computation procedures. The aim of the paper is, however, to show that the approach serves as an excellent tool for analytical purposes also, working as a link between analytical and numerical techniques. The paper gives a method to derive, at first, expressions for strains in general beam and shell analyses, and secondly, the governing equilibrium equations. The approach is based on the utilization of local fixed Cartesian coordinate systems. Applying these, all the definitions required are the simple basic ones, well-known from the analyses in common global coordinates. In addition, the familiar principle of virtual work has been adopted. The method will be, apparently, most powerful in teaching the theories of curved beam and shell structures for students not familiar with tensor analysis. The final results obtained have no novelty value in themselves, but the procedure developed opens through its systematic and graphic progress a new standpoint to theoretical considerations.

Key words: fundamental structural mechanics; curved structures; local coordinate systems; equilibrium equations; principle of virtual work.

1. Introduction

Nowadays, when the amount of time allotted to be used for teaching traditional theoretical topics at universities tends to diminish, it is important to improve the efficiency of the use of time. Realizing this fact encourages teachers to look for new ideas which could make learning easier and more interesting, lowering at the same time the threshold which has too often prevented fresh interest towards mechanical problems. Analytical calculation methods have traditionally taken the biggest part of the capacity reserved for teaching in engineering mechanics. But, the importance of numerical procedures, and particularly that of the finite element method has become more and more central in solving problems in this field. The time used for teaching both these themes has now to be shared optimally, and if possible, to find out new methods which could be applied both in analytical and numerical techniques instead of their traditional totally separated roles. It seems that under these circumstances the principle of virtual work plays an extremely important role. It is a direct starting point for deriving the governing finite element equations but it can equally well be used for obtaining the corresponding analytical equilibrium equations. The principle describes any structural problem as a scalar equation, in which scalar quantities - intensities of internal and

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external virtual work done - are integrated over the volume and surface of the structure. It is essential that the evaluation of these invariant scalars can be performed in any coordinate system. The system does not need to be conformable with the geometry of the structure. This property can be made use of in structural analyses, so that only familiar rectangular Cartesian coordinates are employed. Strains and stresses can be expressed at various points using different coordinate systems, in which the directions of coordinate axes can vary from point to point.

Curved structures, say beams and shells, are usually considered using coordinate systems coinciding with certain characteristic lines of the structure. In analytical procedures it is thus natural to express the displacement field using the unit basis vectors of these systems. As the directions of these vectors are usually not constant, quite complicated expressions for strains are inevitably obtained. In the literature there are mainly two alternative ways which are used in deriving the basic equations. The first one, utilizing various figures of the differential geometry for incremental elements, e.g., Flügge (1966) and Oden (1967), may be useful in connection with rather simple geometries. In particular, it has advantages when the figure needed is prepared skillfully. Much depends on the quality of the figure. When more complicated structures are considered it leaves, however, the reader a little unsure about the exactness of final results. The other method, applied frequently, can be called tensor formalism method, e.g., Flügge (1972) and Malvern (1969). There is no doubt about the correctness of this procedure, but it demands a lot of previous knowledge of tensor analysis. Students, especially, usually do not have this background, and consequently this method can not be utilized in teaching.

The present paper introduces a non-tensorial procedure for deriving expressions for strains and equilibrium equations for structures, supporting both numerical and analytical purposes. Same type of ideas have before been thought up by Morley (1984), (1987a, b). In the numerical computation technique, particularly in thin shell analyses, these ideas have been utilized starting with Irons and Ahmad (1980). The procedure has the advantages of both above-mentioned approaches: demonstrative presentation of simple geometrical figures and mathematical exactness of the tensor formalism method. In addition, the procedure requires knowledge only of very fundamental vector calculus not too demanding for students. The presentation is based on the utilization of local fixed Cartesian coordinates, in which the expressions needed can be formulated as simply as in any common global Cartesian coordinate system. The role of local coordinates serves as a tool to execute all these mathematical operations required. The transformation to relevant curvilinear coordinates takes place at the final stage of the derivation. The method has proved to be of some educational value. It, for instance, serves the possibility to derive first the basic equations of general Timoshenko's beam and Reissner-Mindlin's plate theories and then the equations of the traditional Euler-Bernoulli's and Kirchhoff-Love's theories simply as special cases. Until now, this has been a bit problematic.

As framework, attention is firstly paid to the principal technique and to the mathematical tools required. Consideration continues by applying the theory to derive expressions for strains in the general beam analysis. To avoid too complicated expressions in this context, for shells the same procedure is applied here only in the case of rotational symmetry. Finally, corresponding equilibrium equations are derived by using, in addition, the principle of virtual work.

2. Coordinate systems

Three different coordinate systems are used in the following presentation. The whole structure under consideration is studied in a global Cartesian coordinate system x, y, z with unit vectors $\vec{i}$, $\vec{j}$,

$\vec{k}$. These basis vectors, likewise the ones all through this presentation, form a right-handed system. Consideration is limited here to one-dimensional beam and two-dimensional surface structures. Therefore, the geometry of the structure is defined by determining a reference line for a beam and reference surface for a shell. For a beam this line may be say the beam axis and for a shell the middle surface. The geometry is described, as usually, using curvilinear line and surface coordinates. The position vectors on the reference line or surface are

$$\vec{r}_o = \vec{r}_o(\alpha) = x(\alpha)\vec{i} + y(\alpha)\vec{j} + z(\alpha)\vec{k} \tag{1a}$$

$$\vec{r}_o = \vec{r}_o(\alpha, \beta) = x(\alpha, \beta)\vec{i} + y(\alpha, \beta)\vec{j} + z(\alpha, \beta)\vec{k} \tag{1b}$$

in which the line coordinate is denoted by α and the surface coordinates by α and β. The general position vectors are obtained by adding to these the unit normal vectors:

$$\vec{r}(\alpha, n) = \vec{r}_o(\alpha) + n\,\vec{e}_n(\alpha) \tag{2a}$$

$$\vec{r}(\alpha, \beta, n) = \vec{r}_o(\alpha, \beta) + n\,\vec{e}_n(\alpha, \beta) \tag{2b}$$

which are pointwise perpendicular to the reference line or surface. The normal coordinate n is rectilinear measuring the distance from the reference line or surface. The unit vectors in the directions of coordinate lines are denoted by $\vec{e}_\alpha, \vec{e}_\beta, \vec{e}_n$. In analytical presentations the consideration is limited here to the cases in which coordinate lines α and β are orthogonal. In numerical manipulations, instead, the orthogonality assumption is not necessary.

An additional local Cartesian coordinate system X, Y, Z, employed as a temporary tool, is erected in general at the point of consideration, i.e., at the point where, for example, expressions for strains are pursued, or, at an integration point in a numerical procedure. Here, in accordance with the one- or two-dimensional nature of problem, the origin of the local system is put on the reference line or surface. Coordinate lines X and Y are taken to be tangents to the reference surface on which they can be oriented otherwise arbitrarily. In beam analyses X-coordinate is tangent to the reference axis. Coordinate Z coincides with normal n at each point. The Lamé coefficients or scale factors $H_\alpha = |\partial\vec{r}_o/\partial\alpha|$ and $H_\beta = |\partial\vec{r}_o/\partial\beta|$ likewise the radii of curvatures R_α, R_β and torsion $R_{\alpha\beta} = R_{\beta\alpha}$ are defined, as is usual, only on the reference surface or line so that they do

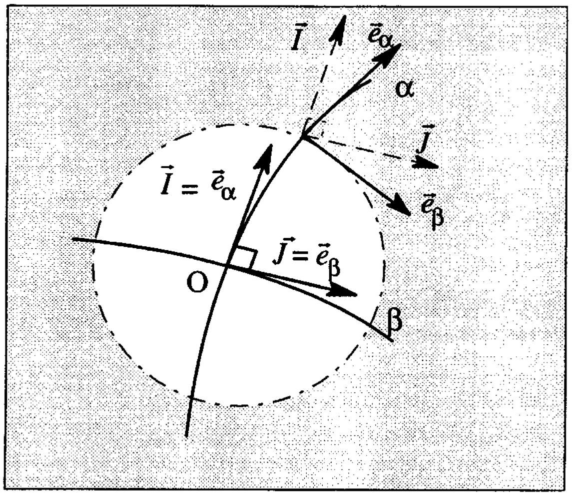

Fig. 1 Visualization of a local fixed coordinate system

not depend on n. Fig. 1, focusing on the immediate neighbourhood of point O, shows the relation between the general curvilinear and the fixed local coordinate systems, and explains the directions of the relevant unit vectors. Unit vectors in the fixed directions are denoted by $\vec{I}$, $\vec{J}$, $\vec{K}$. There is no need for any continuity requirement for these coordinates which are defined at each point of consideration, separately. Neither do the coordinates have to conform any way with the geometry of the structure. In the continuation, when orthogonal systems are considered only and the aim is to find the expressions for strains in curvilinear coordinates, it is appropriate to choose the unit vectors of the local Cartesian coordinate system to coincide with the ones of a curvilinear one at each point.

Since the geometry of the structure is given in curvilinear line and surface coordinates and the expressions of strains are searched for in local orthogonal ones the interdependence between derivatives evaluated in both these systems is needed. For this purpose the well-known chain rule

$$\begin{aligned}\frac{\partial}{\partial\alpha} &= \frac{\partial X}{\partial\alpha}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial\alpha}\frac{\partial}{\partial Y} + \frac{\partial Z}{\partial\alpha}\frac{\partial}{\partial Z}\\ \frac{\partial}{\partial\beta} &= \frac{\partial X}{\partial\beta}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial\beta}\frac{\partial}{\partial Y} + \frac{\partial Z}{\partial\beta}\frac{\partial}{\partial Z}\\ \frac{\partial}{\partial n} &= \frac{\partial X}{\partial n}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial n}\frac{\partial}{\partial Y} + \frac{\partial Z}{\partial n}\frac{\partial}{\partial Z}\end{aligned} \tag{3}$$

will be utilized. This is in matrix form

$$\begin{Bmatrix}\dfrac{\partial}{\partial\alpha}\\ \dfrac{\partial}{\partial\beta}\\ \dfrac{\partial}{\partial n}\end{Bmatrix} = \begin{bmatrix}\dfrac{\partial X}{\partial\alpha} & \dfrac{\partial Y}{\partial\alpha} & \dfrac{\partial Z}{\partial\alpha}\\ \dfrac{\partial X}{\partial\beta} & \dfrac{\partial Y}{\partial\beta} & \dfrac{\partial Z}{\partial\beta}\\ \dfrac{\partial X}{\partial n} & \dfrac{\partial Y}{\partial n} & \dfrac{\partial Z}{\partial n}\end{bmatrix} \begin{Bmatrix}\dfrac{\partial}{\partial X}\\ \dfrac{\partial}{\partial Y}\\ \dfrac{\partial}{\partial Z}\end{Bmatrix} \tag{4}$$

The terms of the coefficient matrix or Jacobian are easily evaluated by differentiating the expressions

$$\begin{aligned}X &= (\vec{r}-\vec{r}_O)\cdot\vec{I}\\ Y &= (\vec{r}-\vec{r}_O)\cdot\vec{J}\\ Z &= (\vec{r}-\vec{r}_O)\cdot\vec{K}\end{aligned} \tag{5}$$

of which the two first ones only, for simplicity, are shown in Fig. 2. The figure describes an orientation of the local coordinate system on a two-dimensional surface at point O. It is noteworthy that the position vector $\vec{r}_O$ and the fixed basis vectors are constant with respect to differentiation. Thus, the differentiation gives

$$\begin{bmatrix}\dfrac{\partial X}{\partial\alpha} & \dfrac{\partial Y}{\partial\alpha} & \dfrac{\partial Z}{\partial\alpha}\\ \dfrac{\partial X}{\partial\beta} & \dfrac{\partial Y}{\partial\beta} & \dfrac{\partial Z}{\partial\beta}\\ \dfrac{\partial X}{\partial n} & \dfrac{\partial Y}{\partial n} & \dfrac{\partial Z}{\partial n}\end{bmatrix} = \begin{bmatrix}\dfrac{\partial\vec{r}}{\partial\alpha}\cdot\vec{I} & \dfrac{\partial\vec{r}}{\partial\alpha}\cdot\vec{J} & \dfrac{\partial\vec{r}}{\partial\alpha}\cdot\vec{K}\\ \dfrac{\partial\vec{r}}{\partial\beta}\cdot\vec{I} & \dfrac{\partial\vec{r}}{\partial\beta}\cdot\vec{J} & \dfrac{\partial\vec{r}}{\partial\beta}\cdot\vec{K}\\ \dfrac{\partial\vec{r}}{\partial n}\cdot\vec{I} & \dfrac{\partial\vec{r}}{\partial n}\cdot\vec{J} & \dfrac{\partial\vec{r}}{\partial n}\cdot\vec{K}\end{bmatrix} \tag{6}$$

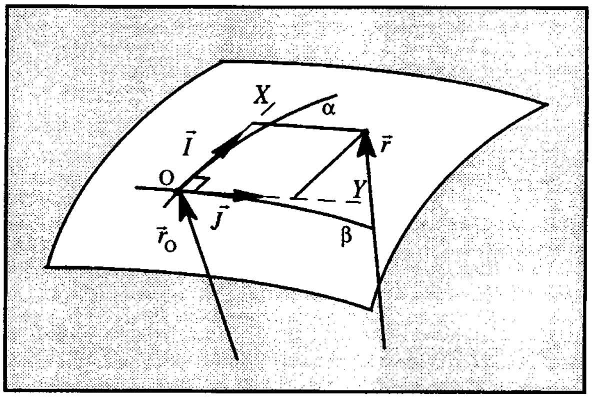

Fig. 2 Evaluation of the local coordinates

for the Jacobian. The final form introduced by Irons and Ahmad (1980) and applied in this paper is

$$\begin{Bmatrix} \dfrac{\partial}{\partial X} \\ \dfrac{\partial}{\partial Y} \\ \dfrac{\partial}{\partial Z} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial \vec{r}}{\partial \alpha}\cdot\vec{I} & \dfrac{\partial \vec{r}}{\partial \alpha}\cdot\vec{J} & \dfrac{\partial \vec{r}}{\partial \alpha}\cdot\vec{K} \\ \dfrac{\partial \vec{r}}{\partial \beta}\cdot\vec{I} & \dfrac{\partial \vec{r}}{\partial \beta}\cdot\vec{J} & \dfrac{\partial \vec{r}}{\partial \beta}\cdot\vec{K} \\ \dfrac{\partial \vec{r}}{\partial n}\cdot\vec{I} & \dfrac{\partial \vec{r}}{\partial n}\cdot\vec{J} & \dfrac{\partial \vec{r}}{\partial n}\cdot\vec{K} \end{bmatrix}^{-1} \begin{Bmatrix} \dfrac{\partial}{\partial \alpha} \\ \dfrac{\partial}{\partial \beta} \\ \dfrac{\partial}{\partial n} \end{Bmatrix} \tag{7}$$

Applying basic vector calculus, the derivatives of the unit vectors $\vec{e}_\alpha$, $\vec{e}_\beta$, $\vec{e}_n$ with respect to surface coordinates

$$\frac{\partial}{\partial \alpha}\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} 0 & -\dfrac{1}{H_\beta}\dfrac{\partial H_\alpha}{\partial \beta} & -\dfrac{H_\alpha}{R_\alpha} \\ \dfrac{1}{H_\beta}\dfrac{\partial H_\alpha}{\partial \beta} & 0 & \dfrac{H_\alpha}{R_{\alpha\beta}} \\ \dfrac{H_\alpha}{R_\alpha} & -\dfrac{H_\alpha}{R_{\alpha\beta}} & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n \end{Bmatrix}$$

$$\frac{\partial}{\partial \beta}\begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n \end{Bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{H_\alpha}\dfrac{\partial H_\beta}{\partial \alpha} & \dfrac{H_\beta}{R_{\alpha\beta}} \\ -\dfrac{1}{H_\alpha}\dfrac{\partial H_\beta}{\partial \alpha} & 0 & -\dfrac{H_\beta}{R_\beta} \\ -\dfrac{H_\beta}{R_{\alpha\beta}} & \dfrac{H_\beta}{R_\beta} & 0 \end{bmatrix} \begin{Bmatrix} \vec{e}_\alpha \\ \vec{e}_\beta \\ \vec{e}_n \end{Bmatrix} \tag{8}$$

which are needed in the continuation are obtained. The derivation is given in most text-books of shell theory.

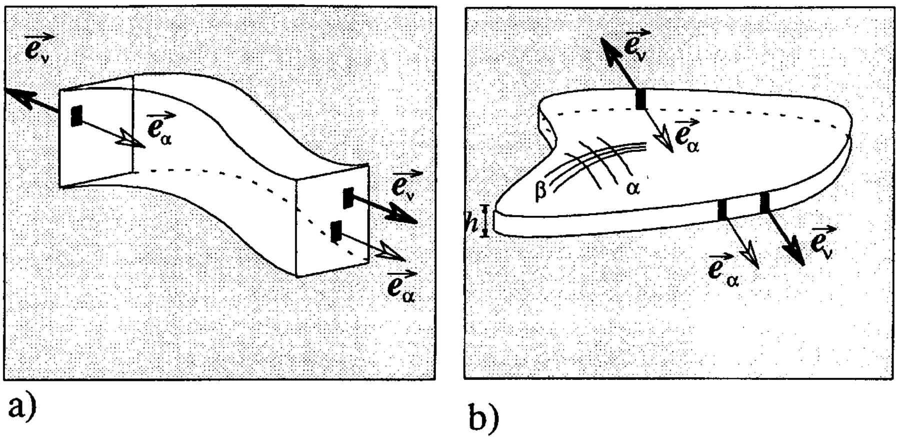

Fig. 3 Boundaries in one- and two-dimensional problems

In addition, the general rule for integration by parts, called also Green's or divergence theorem, will be needed. In the beam analysis it is simply

$$\int_\alpha g\,\frac{dh}{d\alpha}\,d\alpha=[gh\,n_\alpha]_\alpha-\int_\alpha\frac{dg}{d\alpha}\,h\,d\alpha \tag{9}$$

Here, $g=g(\alpha)$ and $h=h(\alpha)$ and the term in brackets includes substitutions from each end of the beam, with ν the normal to the end surfaces and $n_\alpha=\vec{e}_\nu\cdot\vec{e}_\alpha$, Fig. 3a.

In two dimensions the corresponding formula takes a more complicated form, e.g., Wempner (1981):

$$\begin{aligned}\int_{\alpha,\beta} g\frac{\partial h}{\partial\alpha}\,d\alpha d\beta&=\int_s\frac{gh}{H_\beta}\,n_\alpha ds-\int_{\alpha,\beta}\frac{\partial g}{\partial\alpha}\,h\,d\alpha d\beta\\ \int_{\alpha,\beta} g\frac{\partial h}{\partial\beta}\,d\alpha d\beta&=\int_s\frac{gh}{H_\alpha}\,n_\beta ds-\int_{\alpha,\beta}\frac{\partial g}{\partial\beta}\,h\,d\alpha d\beta\end{aligned} \tag{10}$$

with $g=g(\alpha,\beta)$ and $h=h(\alpha,\beta)$, and ν the normal to the boundary surface. The notations $n_\alpha=\vec{e}_\nu\cdot\vec{e}_\alpha$ and $n_\beta=\vec{e}_\nu\cdot\vec{e}_\beta$ for the direction cosines are used, Fig. 3b.

3. Displacement and strain fields

The displacement vector $\vec{u}$ of each point in a structure can be expressed using the basis vectors of any of the three above-mentioned coordinate systems. In global coordinates

$$\vec{u}=u\vec{i}+v\vec{j}+w\vec{k} \tag{11}$$

and in local Cartesian and curvilinear ones, correspondingly

$$\vec{u}=U\vec{I}+V\vec{J}+W\vec{K} \tag{12a}$$

$$\vec{u}=u_\alpha\vec{e}_\alpha+u_\beta\vec{e}_\beta+u_n\vec{e}_n \tag{12b}$$

Here, the displacement components as coefficients are

$$u = \vec{u} \cdot \vec{i}, \quad v = \vec{u} \cdot \vec{j}, \quad w = \vec{u} \cdot \vec{k} \tag{13a}$$

$$U = \vec{u} \cdot \vec{I}, \quad V = \vec{u} \cdot \vec{J}, \quad W = \vec{u} \cdot \vec{K} \tag{13b}$$

$$u_\alpha = \vec{u} \cdot \vec{e}_\alpha, \quad u_\beta = \vec{u} \cdot \vec{e}_\beta, \quad u_n = \vec{u} \cdot \vec{e}_n \tag{13c}$$

The strains defined in local Cartesian coordinates can now be obtained easily by applying the familiar expressions introduced usually in common global orthogonal coordinates. These components, expressed here in the local coordinates, are

$$\begin{aligned}
\varepsilon_X &= \frac{\partial U}{\partial X} = \frac{\partial \vec{u}}{\partial X} \cdot \vec{I}, \quad \gamma_{YZ} = \frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y} = \frac{\partial \vec{u}}{\partial Z} \cdot \vec{J} + \frac{\partial \vec{u}}{\partial Y} \cdot \vec{K} \\
\varepsilon_Y &= \frac{\partial V}{\partial Y} = \frac{\partial \vec{u}}{\partial Y} \cdot \vec{J}, \quad \gamma_{ZX} = \frac{\partial W}{\partial X} + \frac{\partial u}{\partial Z} = \frac{\partial \vec{u}}{\partial X} \cdot \vec{K} + \frac{\partial \vec{u}}{\partial Z} \cdot \vec{I} \\
\varepsilon_Z &= \frac{\partial W}{\partial Z} = \frac{\partial \vec{u}}{\partial Z} \cdot \vec{K}, \quad \gamma_{XY} = \frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X} = \frac{\partial \vec{u}}{\partial Y} \cdot \vec{I} + \frac{\partial \vec{u}}{\partial X} \cdot \vec{J}
\end{aligned} \tag{14}$$

The latter part of each formula is obtained by substituting the coefficient from Eqs. (13b) into the former part and taking into account that the unit vectors of the local coordinate system are constants. The scalar product expressions are computationally very useful. In numerical analyses for example, strains can immediately be evaluated using these after the displacement vector or its interpolant is known. These type of expressions were also given in Green and Zerna (1954) and applied later for numerical purposes by Irons, Irons and Ahmad (1980).

4. Principle of virtual work

The equilibrium equations and boundary conditions for stresses and stress resultants are derived most often also utilizing various differential geometrical figures. The equations required are deduced by considering free body diagrams of infinitesimal body elements. This approach is often graphic and offers an excellent tool in simple cases and should be used in these, but as soon as the structure under consideration is a little more complicated the method will be, however, rather unwieldy. As is well known, the principle of virtual work offers an alternative and systematic way to derive the relevant equations. Here it is used emphasizing the simultaneous application of local Cartesian coordinate systems.

The principle of virtual work is

$$\delta W^i + \delta W^e = 0 \tag{15}$$

or in a body in equilibrium the sum of virtual works done by (i)nternal and (e)xternal forces vanishes with respect to each virtual displacement state. The general expressions for the virtual work done are

$$\delta W^i = -\int_V \overleftrightarrow{\sigma} : \delta \overleftrightarrow{\varepsilon} \, dV \tag{16a}$$

$$\delta W^e = \int_V \vec{f} \cdot \delta \vec{u} \, dV + \int_{S_t} \vec{t} \cdot \delta \vec{u} \, dS \tag{16b}$$

where $\overleftrightarrow{\sigma}$ and $\overleftrightarrow{\varepsilon}$ are the stress and strain tensors, and $\vec{f}$ and $\vec{t}$ the body force and traction vectors, respectively. V is the volume of the body and S_t the surface loaded by given tractions. On the remaining body surface S_u the geometrical boundary conditions are given. In a three-dimensional body these expressions can be written in a familiar component form using global coordinates x, y, z as

$$\delta W^i = -\int_V [\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx}]\, dV \tag{17a}$$

$$\delta W^e = \int_V [f_x \delta u_x + f_y \delta u_y + f_z \delta u_z]\, dV + \int_{S_t} [t_x \delta u_x + t_y \delta u_y + t_z \delta u_z]\, dS \tag{17b}$$

In the linear analysis considered here the expressions for the variations $\delta\varepsilon_x$, $\delta\varepsilon_y \cdots$ are obtained directly by the replacement $\vec{u} \to \delta\vec{u}$ in Eq. (14) or by components $u \to \delta u$, $v \to \delta v$ and $w \to \delta w$ in the corresponding definitions Eq. (11).

The expressions for the internal and external work can also be presented by applying the definition of an integral, i.e., expressing the volume integrals as limits of Riemannian sums. This means that the domain in the consideration is splitted into a infinite number of subdomains (ΔV_j, $\Delta S_{t_j} \to 0$) in which the expressions are written. Hence

$$\delta W^i = -\lim_{n\to\infty} \sum_{j=1}^{n} [\sigma_x \delta\varepsilon_x + \sigma_y \delta\varepsilon_y + \sigma_z \delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{yz}\delta\gamma_{yz} + \tau_{zx}\delta\gamma_{zx}]\, \Delta V_j \tag{18a}$$

$$\delta W^e = \lim_{n\to\infty} \sum_{j=1}^{n} [f_x \delta u_x + f_y \delta u_y + f_z \delta u_z]\, \Delta V_j + \lim_{n\to\infty} \sum_{j=1}^{n} [t_x \delta u_x + t_y \delta u_y + t_z \delta u_z]\, \Delta S_{t_j} \tag{18b}$$

But, since the terms under integration signs and in the sums: the intensities of the virtual work due to internal forces $-\overleftrightarrow{\sigma} : \delta\overleftrightarrow{\varepsilon}$, due to volume forces $\vec{f} \cdot \delta\vec{u}$ and due to surface tractions $\vec{t} \cdot \delta\vec{u}$ are scalars, invariants which are independent of the coordinate system applied, relations (18) can also be expressed at each subdomain j applying the local Cartesian coordinate system X_j, Y_j, Z_j connected exactly to each point in question, i.e.,

$$\begin{aligned} \delta W^i = -\lim_{n\to\infty} \sum_{j=1}^{n} [&\sigma_{X_j} \delta\varepsilon_{X_j} + \sigma_{Y_j} \delta\varepsilon_{Y_j} + \sigma_{Z_j} \delta\varepsilon_{Z_j} \\ &+ \tau_{X_jY_j}\delta\gamma_{X_jY_j} + \tau_{Y_jZ_j}\delta\gamma_{Y_jZ_j} + \tau_{Z_jX_j}\delta\gamma_{Z_jX_j}]\, \Delta V_j \end{aligned} \tag{19a}$$

$$\begin{aligned} \delta W^e = &\lim_{n\to\infty} \sum_{j=1}^{n} [f_{X_j} \delta u_{X_j} + f_{Y_j} \delta u_{Y_j} + f_{Z_j} \delta u_{Z_j}]\, \Delta V_j \\ &+ \lim_{n\to\infty} \sum_{j=1}^{n} [t_{X_j} \delta u_{X_j} + t_{Y_j} \delta u_{Y_j} + t_{Z_j} \delta u_{Z_j}]\, \Delta S_{t_j} \end{aligned} \tag{19b}$$

In these the components $f_{X_j} = \vec{f} \cdot \vec{e}_{X_j}$, $f_{Y_j} = \vec{f} \cdot \vec{e}_{Y_j}$ $\cdots$ $t_{Z_j} = \vec{t} \cdot \vec{e}_{Z_j}$ are introduced. The unconventional expressions Eq. (19) cannot be represented here with integral notation over the whole domain as an infinite number of coordinate systems are defined. These forms are, however, important for instance in taking advantages due to one- or two-dimensionality. This means that the simplified assumptions about certain vanishing strain and stress components can be easily taken into account by directing the local coordinate systems appropriately at each point. In numerical calculations the expressions can, however, be evaluated 'numerically' over the whole domain or over the finite number of subdomains by modifying the expressions with the correcting or weighting factors as

$$\delta W^i = -\sum_{j=1}^{N} w_j\, [\sigma_{X_j} \delta\varepsilon_{X_j} + \sigma_{Y_j} \delta\varepsilon_{Y_j} + \sigma_{Z_j} \delta\varepsilon_{Z_j}$$

$$+ \tau_{X_jY_j}\delta\gamma_{X_jY_j} + \tau_{Y_jZ_j}\delta\gamma_{Y_jZ_j} + \tau_{Z_jX_j}\delta\gamma_{Z_jX_j}] \, \Delta V_j \tag{20a}$$

$$\begin{aligned} \delta W^e &= \sum_{j=1}^{N} w_j \, [f_{X_j}\delta u_{X_j} + f_{Y_j}\delta u_{Y_j} + f_{Z_j}\delta u_{Z_j}] \, \Delta V_j \\ &+ \sum_{j=1}^{N} w_j \, [t_{X_j}\delta u_{X_j} + t_{Y_j}\delta u_{Y_j} + t_{Z_j}\delta u_{Z_j}] \, \Delta S_{t_j} \end{aligned} \tag{20b}$$

Here, N is the number of subdomains or integration points. In addition, the expressions are written having fixed abscissae values and weighting factors w_j defined in the integration scheme applied.

The procedure adopted is closely related to various techniques of numerical integration, in which the infinite Riemannian sums are replaced by finite ones. The present consideration, actually, demonstrates the fact that integration applied is independent of the coordinate transformation. In other respects, it follows the same idea applied in the derivation of the expressions for strains: all the definitions and mathematical manipulations needed are performed using the local Cartesian coordinates.

In analytical derivations the expressions, Eq. (19) can be brought back to integral forms by introducing finally a suitable curvilinear coordinate system conforming with the geometry of the structure. This procedure will be applied in the following when deriving the basic equilibrium equations.

5. Curved beam analysis

As an introduction to shell problems, a curved beam in the global x, z plane shown in Fig. 4 is considered. The curvilinear coordinate α coincides with the beam axis and normal n measures the distance from the axis in the plane of the beam. The corresponding unit vectors are $\vec{e}_\alpha$ and $\vec{e}_n$. The expressions for strains, for example, at an arbitrary point P are to be found. A local Cartesian XZ-coordinate system is spanned, following the above-mentioned practice, at point O where the normal through P intersects the beam axis. The rule of Eq. (7) for differentiation is simplified in this case to the form

$$\begin{Bmatrix} \dfrac{\partial}{\partial X} \\ \dfrac{\partial}{\partial Z} \end{Bmatrix} = \begin{bmatrix} \dfrac{\partial \vec{r}}{\partial \alpha} \cdot \vec{I} & \dfrac{\partial \vec{r}}{\partial \alpha} \cdot \vec{K} \\ \dfrac{\partial \vec{r}}{\partial n} \cdot \vec{I} & \dfrac{\partial \vec{r}}{\partial n} \cdot \vec{K} \end{bmatrix}^{-1} \begin{Bmatrix} \dfrac{\partial}{\partial \alpha} \\ \dfrac{\partial}{\partial n} \end{Bmatrix} \tag{21}$$

From Eq. (2a),

$$\begin{aligned} \frac{\partial \vec{r}}{\partial \alpha} &= \frac{d\vec{r}_o}{d\alpha} + n\frac{d\vec{e}_n}{d\alpha} = H_\alpha \vec{e}_\alpha + n\frac{d\vec{e}_n}{d\alpha} = H_\alpha\left(1 + \frac{n}{R_\alpha}\right)\vec{e}_\alpha \\ \frac{\partial \vec{r}}{\partial n} &= \vec{e}_n \end{aligned} \tag{22}$$

In these the expressions (8) for the derivatives of unit vectors are utilized. Making use of Eqs. (22) in the differentiation rule, Eq. (21) and taking into account the vector relations $\vec{e}_\alpha = \vec{I}$ and $\vec{e}_n = \vec{K}$ at point P yields the rule

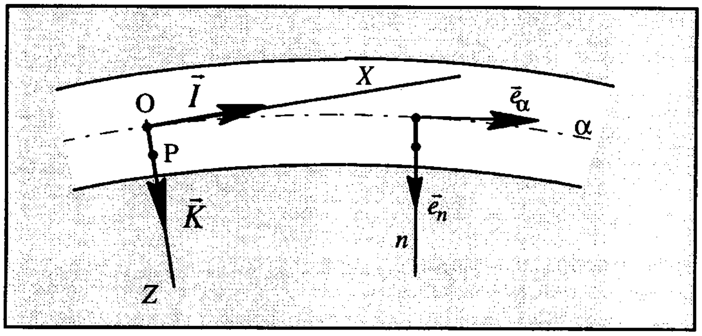

Fig. 4 General curved beam with relevant coordinates

$$\begin{Bmatrix} \dfrac{\partial}{\partial X} \\ \dfrac{\partial}{\partial Z} \end{Bmatrix} = \begin{bmatrix} H_\alpha \left(1 + \dfrac{n}{R_\alpha}\right) & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{Bmatrix} \dfrac{\partial}{\partial \alpha} \\ \dfrac{\partial}{\partial n} \end{Bmatrix} \tag{23}$$

or

$$\frac{\partial}{\partial X} = H_\alpha^{-1} \left(1 + \frac{n}{R_\alpha}\right)^{-1} \frac{\partial}{\partial \alpha}$$
$$\frac{\partial}{\partial Z} = \frac{\partial}{\partial n} \tag{24}$$

in which the Jacobian matrix and its inverse take a simple diagonal form.

The displacement of the beam is defined applying Timoshenko's beam theory as (Fig. 4)

$$\vec{u}\,(\alpha, n) = [u\,(\alpha) - n\,\theta(\alpha)]\,\vec{e}_\alpha(\alpha) + w\,(\alpha)\,\vec{e}_n\,(\alpha) \tag{25}$$

The functions $u=u_\alpha(\alpha)$ and $w=u_n(\alpha)$ represent the displacement components of the corresponding point on the beam axis, and $\theta(\alpha)$ the rotation of a material fiber originally perpendicular to the axis. In fact $\theta(\alpha)$, which is positive in the clockwise direction, is the rotation component about the axis perpendicular to the beam plane, i.e., $\theta_\beta(\alpha)$, but here the subscript β is dropped out because no confusion can arise. Timoshenko's beam theory takes approximately into account shear deformations in the beam, by allowing a normal to the beam axis to deviate from its perpendicular position under deformation.

The only nonzero strain components ε_α and $\gamma_{n\alpha}$ are from Eqs. (14) and (25) at P with $\vec{I} = \vec{e}_\alpha$ and $\vec{K} = \vec{e}_n$.

$$\varepsilon_X = \varepsilon_\alpha = \frac{\partial \vec{u}}{\partial X} \cdot \vec{I} = H_\alpha^{-1} \left(1 + \frac{n}{R_\alpha}\right)^{-1} \frac{\partial \vec{u}}{\partial \alpha} \cdot \vec{e}_\alpha$$
$$\gamma_{ZX} = \gamma_{n\alpha} = \frac{\partial \vec{u}}{\partial X} \cdot \vec{K} + \frac{\partial \vec{u}}{\partial Z} \cdot \vec{I} = H_\alpha^{-1} \left(1 + \frac{n}{R_\alpha}\right)^{-1} \frac{\partial \vec{u}}{\partial \alpha} \cdot \vec{e}_n + \frac{\partial \vec{u}}{\partial n} \cdot \vec{e}_\alpha \tag{26}$$

The derivatives needed are

$$\frac{\partial \vec{u}}{\partial \alpha} = \left(\frac{du}{d\alpha} - n\,\frac{d\theta}{d\alpha}\right) \vec{e}_\alpha + (u - n\,\theta)\,\frac{d\,\vec{e}_\alpha}{d\alpha} + \frac{dw}{d\alpha}\,\vec{e}_n + w\,\frac{d\,\vec{e}_n}{d\alpha}$$

$$\frac{\partial \vec{u}}{\partial n} = -\theta \vec{e}_\alpha \tag{27}$$

Taking into account formulas, Eq. (8) yields

$$\frac{\partial \vec{u}}{\partial \alpha} = \left(\frac{du}{d\alpha} - n\frac{d\theta}{d\alpha} + \frac{w}{R_\alpha}H_\alpha\right)\vec{e}_\alpha + \left(\frac{dw}{d\alpha} - \frac{u - n\theta}{R_\alpha}H_\alpha\right)\vec{e}_n$$

$$\frac{\partial \vec{u}}{\partial n} = -\theta \vec{e}_\alpha \tag{28}$$

Thus, the strain components obtain the final formulas

$$\varepsilon_X = \varepsilon_\alpha = \left(1 + \frac{n}{R_\alpha}\right)^{-1}\left(\frac{du}{H_\alpha d\alpha} - n\frac{d\theta}{H_\alpha d\alpha} + \frac{w}{R_\alpha}\right)$$

$$\gamma_{ZX} = \gamma_{n\alpha} = \left(1 + \frac{n}{R_\alpha}\right)^{-1}\left(\frac{dw}{H_\alpha d\alpha} - \frac{u - n\theta}{R_\alpha}\right) - \theta \tag{29}$$

It is often of interest to find out the terms dependent at most linearly on the normal coordinate n. Eqs. (29) can be presented as a series according to the increasing powers of n as follows

$$\varepsilon_\alpha = \frac{du}{H_\alpha d\alpha} + \frac{w}{R_\alpha} - \frac{n}{R_\alpha}\left(\frac{du}{H_\alpha d\alpha} + \frac{w}{R_\alpha} + \frac{R_\alpha d\theta}{H_\alpha d\alpha}\right) + O\,(n^2)$$

$$\gamma_{n\alpha} = \left(1 - \frac{n}{R_\alpha} + O(n^2)\right)\left(\frac{dw}{H_\alpha d\alpha} - \frac{u}{R_\alpha} - \theta\right) \tag{30}$$

In the Euler-Bernoulli beam theory the shear deformation $\gamma_{n\alpha}$ is assumed to vanish. This condition applied in the latter one of Eqs. (29) or (30) constraints the rotation:

$$\theta = \frac{dw}{H_\alpha d\alpha} - \frac{u}{R_\alpha} \tag{31}$$

Substituting this into the expression of axiai strain yields an exact result

$$\varepsilon_\alpha = \frac{du}{H_\alpha d\alpha} + \left(1 + \frac{n}{R_\alpha}\right)^{-1}\left[\frac{w}{R_\alpha} - n\left(\frac{u}{R_\alpha^2}\frac{dR_\alpha}{H_\alpha d\alpha} + \frac{d}{H_\alpha d\alpha}\left(\frac{dw}{H_\alpha d\alpha}\right)\right)\right] \tag{32}$$

and a corresponding series expression

$$\varepsilon_\alpha = \frac{du}{H_\alpha d\alpha} + \frac{w}{R_\alpha} - n\left[\frac{w}{R_\alpha^2} + \frac{u}{R_\alpha^2}\frac{dR_\alpha}{H_\alpha d\alpha} + \frac{d}{H_\alpha d\alpha}\left(\frac{dw}{H_\alpha d\alpha}\right)\right] + O(n^2) \tag{33}$$

Results obtained can easily be simplified for some special cases. For example, applying them for a straight beam, in which $R_\alpha = \infty$ and changing still the symbols to the familiar ones as $\alpha \to x$ with $H_\alpha = 1$ and $n \to y$ results in an equation

$$\varepsilon_x = \frac{du}{dx} - y\frac{d^2 w}{dx^2} \tag{34}$$

which is one of the most well-known formulas of engineering beam theory.

6. Equilibrium equations

The principle of virtual work is recalled for general beam analysis. Two nonzero strain components of Eq. (26) exist at point P (Fig. 3). The formulas for the internal and external work are expressed applying the kinematic assumption of Eq. (25) for the displacement and the local Cartesian coordinate system spanned again at point O on the beam axis. The expression for the internal work in Eq. (19a) takes the form

$$\delta W^i = -\lim_{n\to\infty}\sum_{j=1}^{n}(\sigma_{X_j}\delta\varepsilon_{X_j} + \tau_{Z_jX_j}\delta\gamma_{Z_jX_j})\Delta V_j \tag{35}$$

Using the definitions (26) gives

$$\delta W^i = -\lim_{n\to\infty}\sum_{j=1}^{n}\left[\sigma_{X_j}\left(\frac{\partial\delta\vec{u}}{\partial X_j}\cdot\vec{I}_j\right) + \tau_{Z_jX_j}\left(\frac{\partial\delta\vec{u}}{\partial X_j}\cdot\vec{K}_j + \frac{\partial\delta\vec{u}}{\partial Z_j}\cdot\vec{I}_j\right)\right]\Delta V_j \tag{36}$$

At point P on the normal through the origin of the local Cartesian coordinate system, Fig. 4, where the local coordinate system coincides with the curvilinear one, the inverse of the Jacobian matrix takes the form (23). The elementar volume in Eq. (36) is

$$\lim_{\Delta V_j\to 0}\Delta V_j = \lim_{\Delta X_j\to 0}\Delta X_j\, dA = H_\alpha\left(1 + \frac{n}{R_\alpha}\right)d\alpha dA \tag{37}$$

The notation A for the cross-sectional area is used. Evaluating the derivatives required at point P following the formulas (23) to (29) produces as substituted into Eq. (36) together with the vector equalities $\vec{I} = \vec{e}_\alpha$ and $\vec{K} = \vec{e}_n$ the expression

$$\delta W^i = -\int_\alpha\left\{\int_A\left[\sigma_\alpha\left(1+\frac{n}{R_\alpha}\right)^{-1}\left(\frac{d\delta u}{H_\alpha d\alpha} - n\frac{d\delta\theta}{H_\alpha d\alpha} + \frac{\delta w}{R_\alpha}\right)\right.\right.$$

$$\left.\left. + \tau_{n\alpha}\left(1+\frac{n}{R_\alpha}\right)^{-1}\left(\frac{d\delta w}{H_\alpha d\alpha} - \frac{\delta u - n\delta\theta}{R_\alpha}\right) - \tau_{n\alpha}\delta\theta\right] H_\alpha\left(1+\frac{n}{R_\alpha}\right)dA\right\}d\alpha \tag{38}$$

Defining the normal force, shear force and bending moment resultants as

$$N_\alpha = \int_A \sigma_\alpha dA, \quad Q_\alpha = \int_A \tau_{n\alpha} dA, \quad M_\alpha = \int_A \sigma_\alpha n\, dA \tag{39}$$

and incorporating them into expression (38) gives

$$\delta W^i = -\int_\alpha \left[N_\alpha \left(\frac{d\,\delta u}{H_\alpha d\alpha} + \frac{\delta w}{R_\alpha} \right) - M_\alpha \frac{d\,\delta\theta}{H_\alpha d\alpha} + Q_\alpha \left(\frac{d\,\delta w}{H_\alpha d\alpha} - \frac{\delta u}{R_\alpha} - \delta\theta \right) \right] H_\alpha d_\alpha \tag{40}$$

The expression for the work (20b) done by external forces takes in the beam analysis the form

$$\delta W^e = \lim_{n \to \infty} \sum_{j=1}^{n} [f_{X_j} \delta U_j + f_{Z_j} \delta W_j] \Delta V_j + \lim_{n \to \infty} \sum_{j=1}^{n} [t_{X_j} \delta U_j + t_{Z_j} \delta W_j] \Delta S_{t_j} \tag{41}$$

which can be formulated at point P correspondingly using the variation of the displacement vector (25) as

$$\delta W^e = \int_\alpha \int_A [f_\alpha (\delta u - n\,\delta\theta) + f_n\,\delta w] H_\alpha \left(1 + \frac{n}{R_\alpha} \right) dA d\alpha$$

$$+ \int_{S_t} [t_\alpha (\delta u - n\,\delta\theta) + t_n\,\delta w]\, dS \tag{42}$$

Defining the external load resultants on the beam axis and corresponding ones at the end surfaces as

$$\bar{f}_\alpha = \int_A f_\alpha \left(1 + \frac{n}{R_\alpha} \right) dA \qquad \bar{t}_\alpha = \int_{A_t} t_\alpha dA$$

$$\bar{f}_n = \int_A f_n \left(1 + \frac{n}{R_\alpha} \right) dA \qquad \bar{t}_n = \int_{A_t} t_n\, dA$$

$$\bar{f}_\theta = \int_A f_\alpha n \left(1 + \frac{n}{R_\alpha} \right) dA \qquad \bar{t}_\theta = \int_{A_t} t_\alpha n dA \tag{43}$$

gives for the work due to external loads the expression

$$\delta W^e = \int_\alpha [\bar{f}_\alpha \delta u + \bar{f}_n\, \delta w - \bar{f}_\theta \delta\theta]\, H_\alpha d\alpha + [\bar{t}_\alpha \delta u + \bar{t}_n\, \delta w - \bar{t}_\theta \delta\theta]_\alpha \tag{44}$$

Possible tractions on the mantle surface of the beam can be included, if necessary, with obvious expressions. Now, the expressions of the internal (41) and external (44) work done are inserted into the principle of virtual work (15) which gives

$$-\int_\alpha \left[N_\alpha \left(\frac{d\,\delta u}{H_\alpha d\alpha} + \frac{\delta w}{R_\alpha} \right) - M_\alpha \frac{d\,\delta\theta}{H_\alpha d\alpha} + Q_\alpha \left(\frac{d\,\delta w}{H_\alpha d\alpha} - \frac{\delta u}{R_\alpha} - \delta\theta \right) \right.$$

$$\left. - \bar{f}_\alpha \delta u - \bar{f}_n\, \delta w + \bar{f}_\theta \delta\theta \right] H_\alpha d\alpha + [\bar{t}_\alpha \delta u + \bar{t}_n\, \delta w - \bar{t}_\theta \delta\theta]_\alpha = 0 \tag{45}$$

Integrating once by parts using the rule (9) yields an equation

$$\int_\alpha \left\{ \left[\frac{dN_\alpha}{H_\alpha d\alpha} + \frac{Q_\alpha}{R_\alpha} + \bar{f}_\alpha \right] \delta u + \left[\frac{dQ_\alpha}{H_\alpha d\,\alpha} - \frac{N_\alpha}{R_\alpha} + \bar{f}_n \right] \delta w \right.$$

$$- \left[\frac{dM_\alpha}{H_\alpha d\alpha} - Q_\alpha + \bar{f}_\theta \right] \delta\theta \Bigg\} H_\alpha d\alpha$$

$$+ [(-N_\alpha + \bar{t}_\alpha)\,\delta u + (-Q_\alpha + \bar{t}_n)\,\delta w + (M_\alpha - \bar{t}_\theta)\,\delta\theta]_\alpha = 0 \tag{46}$$

which gives the equilibrium equations for a general beam analysis

$$\begin{aligned} &\frac{dN_\alpha}{H_\alpha d\alpha} + \frac{Q_\alpha}{R_\alpha} + \bar{f}_\alpha = 0 \\ &\frac{dQ_\alpha}{H_\alpha d\alpha} - \frac{N_\alpha}{R_\alpha} + \bar{f}_n = 0 \\ &\frac{dM_\alpha}{H_\alpha d\alpha} - Q_\alpha + \bar{f}_\theta = 0 \end{aligned} \tag{47}$$

and corresponding boundary conditions

$$\left.\begin{aligned} -N_\alpha + \bar{t}_\alpha &= 0 \\ -Q_\alpha + \bar{t}_n &= 0 \\ M_\alpha - \bar{t}_\theta &= 0 \end{aligned}\right\} \text{ at } \alpha = \alpha_t \qquad \left.\begin{aligned} u - \bar{u} &= 0 \\ w - \bar{w} &= 0 \\ \theta - \bar{\theta} &= 0 \end{aligned}\right\} \text{ at } \alpha = \alpha_u \tag{48}$$

at each end of the beam. The mechanical boundary conditions are applied on α_t with boundary tractions. The geometrical ones are a consequence of the requirements for kinematically admissible virtual displacements due to the conditions $\delta u = \delta w = \delta\theta = 0$ on α_u, with given displacements denoted in this context with an overbar. Also, mixed conditions can be applied. When the Euler-Bernoulli assumption with constraint (31) for the rotation is assumed, the equilibrium equations obtained are directly those two which can be derived from Eqs. (47) by eliminating the shear force Q_α. Now, the boundary conditions take the form

$$\left.\begin{aligned} -N_\alpha - \frac{M_\alpha}{R_\alpha} + \bar{t}_\alpha + \frac{\bar{t}_\theta}{R_\alpha} &= 0 \\ -\frac{dM_\alpha}{H_\alpha d\alpha} + \bar{t}_n &= 0 \\ M_\alpha - \bar{t}_\theta &= 0 \end{aligned}\right\} \text{on } \alpha_t \qquad \left.\begin{aligned} u - \bar{u} &= 0 \\ w - \bar{w} &= 0 \\ \frac{dw}{H_\alpha d\alpha} - \frac{d\bar{w}}{H_\alpha d\alpha} &= 0 \end{aligned}\right\} \text{on } \alpha_u \tag{49}$$

The expression of the shear force is obtained from the last one of Eqs. (47). The results can be examined, for example, with the well-known formulas in the case of a straight beam.

7. General shell geometry

Intention is now shifted to shell structures (Fig. 5). The coordinate system applied is α, β, n with unit vectors $\vec{e}_\alpha$, $\vec{e}_\beta$, $\vec{e}_n$. Here, α and β are the surface coordinates and n the coordinate in

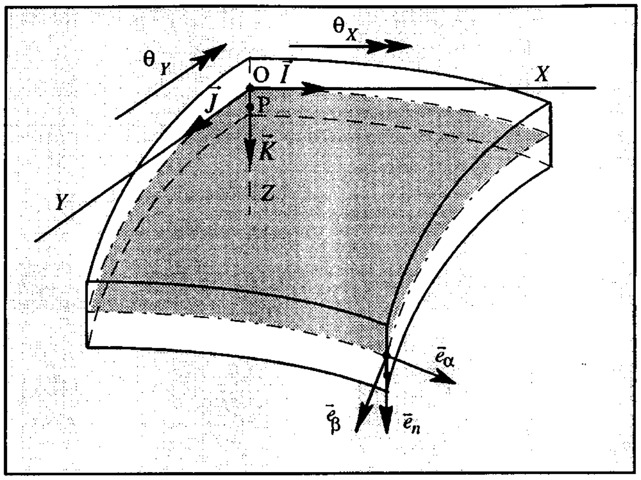

Fig. 5 General shell with relevant coordinates

the normal direction. The coordinate lines for the surface coordinates are assumed to be mutually orthogonal on the reference surface of a shell. The derivatives of the position vector (2b) are

$$\begin{aligned}
\frac{\partial \vec{r}}{\partial \alpha} &= \frac{\partial \vec{r}_o}{\partial \alpha} + n\,\frac{\partial \vec{e}_n}{\partial \alpha} = H_\alpha\left(1+\frac{n}{R_\alpha}\right)\vec{e}_\alpha - H_\alpha\frac{n}{R_{\alpha\beta}}\vec{e}_\beta \\
\frac{\partial \vec{r}}{\partial \beta} &= \frac{\partial \vec{r}_o}{\partial \beta} + n\frac{\partial \vec{e}_n}{\partial \beta} = -H_\beta\frac{n}{R_{\alpha\beta}}\vec{e}_\alpha + H_\beta\left(1+\frac{n}{R_\beta}\right)\vec{e}_\beta \\
\frac{\partial \vec{r}}{\partial n} &= \vec{e}_n
\end{aligned} \tag{50}$$

in which formulas (8) are exploited, are substituted into the Jacobian matrix in Eq. (6). Utilizing the fact that the unit vectors of local rectangular and curvilinear coordinate systems are specified to coincide at point O, i.e., $\vec{e}_\alpha=\vec{I}$, $\vec{e}_\beta=\vec{J}$ and $\vec{e}_n=\vec{K}$, results in a nondiagonal Jacobian matrix

$$\begin{bmatrix} \frac{\partial \vec{r}}{\partial \alpha}\cdot\vec{I} & \frac{\partial \vec{r}}{\partial \alpha}\cdot\vec{J} & \frac{\partial \vec{r}}{\partial \alpha}\cdot\vec{K} \\ \frac{\partial \vec{r}}{\partial \beta}\cdot\vec{I} & \frac{\partial \vec{r}}{\partial \beta}\cdot\vec{J} & \frac{\partial \vec{r}}{\partial \beta}\cdot\vec{K} \\ \frac{\partial \vec{r}}{\partial n}\cdot\vec{I} & \frac{\partial \vec{r}}{\partial n}\cdot\vec{J} & \frac{\partial \vec{r}}{\partial n}\cdot\vec{K} \end{bmatrix} = \begin{bmatrix} H_\alpha\left(1+\frac{n}{R_\alpha}\right) & -H_\alpha\frac{n}{R_{\alpha\beta}} & 0 \\ -H_\beta\frac{n}{R_{\alpha\beta}} & H_\beta\left(1+\frac{n}{R_\beta}\right) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{51}$$

It is obvious that the reason for the nondiagonal matrix is due to the fact that coordinates α and β are not necessarily orthogonal on the surface through point P despite of the orthogonality on the reference surface. Likewise, the strain components in various coordinates at each point outside the reference surface are no more equal, i.e., $\varepsilon_\alpha \neq \varepsilon_X$ etc.

To avoid unnecessary complications in the continuation, α and β are assumed to be principal coordinates and the corresponding directions principal directions which is the usual practice in most textbooks on shell theory. Hence the torsion or geodesic torsion of the structure vanishes i.e., $1/R_{\alpha\beta}=0$ which makes that the Jacobian matrix in Eq. (51) diagonal and its inversion is trivial.

According to the Reissner-Mindlin theory the displacement of a generic point of a shell is (Fig. 5)

$$\vec{u}(\alpha,\beta,n)=[u_\alpha(\alpha,\beta)-n\,\theta_\beta(\alpha,\beta)]\vec{e}_\alpha(\alpha,\beta)+[u_\beta(\alpha,\beta)-n\,\theta_\alpha(\alpha,\beta)]\vec{e}_\beta(\alpha,\beta)$$
$$+w(\alpha,\beta)\,\vec{e}_n(\alpha,\beta) \tag{52}$$

This is an obvious counterpart of the Timoshenko's beam theory expression (25).

8. Shells of revolution

Only shells of revolution are considered in the following. The method presented is applicable to general shell analyses as well, but to avoid too complicated expressions in this context the consideration is limited to a simpler case. In the continuation, α is the meridional and β the circumferential coordinate. Due to the axisymmetric geometry and loading, the displacement component in the circumferential direction vanishes yielding $u_\beta=\theta_\alpha\equiv 0$. In addition, all the other displacement functions are constant with respect to the coordinate β. The axisymmetric kinematics is

$$\vec{u}(\alpha,n)=[u(\alpha)-n\,\theta(\alpha)]\vec{e}_\alpha(\alpha,\beta)+w(\alpha)\vec{e}_n(\alpha,\beta) \tag{53}$$

Function $\theta(\alpha)=\theta_\beta(\alpha)$ is the rotation component of a material fibre, originally perpendicular to the surface, about the coordinate line parallel to β-axis. Its positive direction is chosen as shown in Fig. 6. Strain components are again to be determined at point P on a normal through O. Due to the kinematics in Eq. (53), the strains $\varepsilon_n=\gamma_{\alpha\beta}=\gamma_{\beta n}\equiv 0$.

Substituting now the derivatives of the displacement vector into the expressions of strain components, Eq. (14), taking into account the differentiation rule Eq. (7) with Jacobian matrix Eq. (51) the strains at point P will be

$$\varepsilon_\alpha=\left(1+\frac{n}{R_\alpha}\right)^{-1}\left(\frac{du}{H_\alpha d\alpha}-n\frac{d\theta}{H_\alpha d\alpha}+\frac{w}{R_\alpha}\right)$$
$$\varepsilon_\beta=\left(1+\frac{n}{R_\beta}\right)^{-1}\left((u-n\theta)\frac{1}{H_\beta}\frac{dH_\beta}{H_\alpha d\alpha}+\frac{w}{R_\beta}\right)$$
$$\gamma_{n\alpha}=\left(1+\frac{n}{R_\alpha}\right)^{-1}\left(\frac{dw}{H_\alpha d\alpha}-\frac{u-n\theta}{R_\alpha}\right)-\theta \tag{54}$$

It may be noted that the expressions for ε_α and $\gamma_{n\alpha}$ are exactly of the same form as for the Timoshenko beam; formulas, Eq. (29). These can be presented - if wanted - again according to the increasing powers of coordinate n. By introducing the abbreviations, associated with a traditional geometrical interpretation of the deformation state, the strains are

$$\varepsilon_\alpha=\varepsilon_\alpha^o+n\,\kappa_\alpha+O(n^2)$$
$$\varepsilon_\beta=\varepsilon_\beta^o+n\,\kappa_\beta+O(n^2)$$
$$\gamma_{n\alpha}=\gamma_{n\alpha}^o+n\,\kappa_{n\alpha}+O(n^2) \tag{55}$$

The terms with a superscript 'circle' refer to the strains on the reference surface

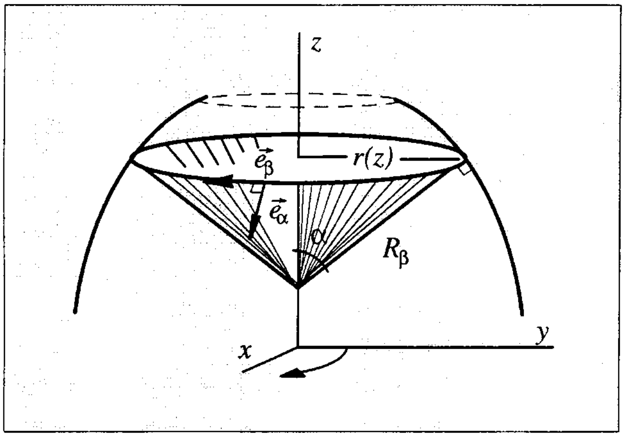

Fig. 6 Shell of revolution

$$\varepsilon_\alpha^o = \frac{du}{H_\alpha d\alpha} + \frac{w}{R_\alpha}$$
$$\varepsilon_\beta^o = \frac{u}{H_\beta}\frac{dH_\beta}{H_\alpha d\alpha} + \frac{w}{R_\beta}$$
$$\gamma_{n\alpha}^o = \frac{dw}{H_\alpha d\alpha} - \frac{u}{R_\alpha} - \theta \tag{56}$$

The coefficients κ correspond traditionally to the changes of curvature. But, according to the Reissner-Mindlin assumptions their exact geometrical meaning does no more, due to shear deformations, represent pure changes of curvatures. The coefficients are

$$\kappa_\alpha = -\frac{1}{R_\alpha}\left(\frac{du}{H_\alpha d\alpha} + \frac{w}{R_\alpha} + \frac{R_\alpha d\theta}{H_\alpha d\alpha}\right)$$
$$\kappa_\beta = -\frac{1}{R_\beta}\left(\frac{u + R_\beta\theta}{H_\beta}\frac{dH_\beta}{H_\alpha d\alpha} + \frac{w}{R_\beta}\right)$$
$$\kappa_{n\alpha} = -\frac{\gamma_{n\alpha}^o}{R_\alpha} \tag{57}$$

Kirchhoff-Love's theory for thin plates assumes the normals to the reference surface to remain normals also after deformation. It yields vanishing shear deformations $\gamma_{n\alpha}$=0. The rotation is constrained due to this requirement. Here, setting the last one of Eqs. (54) or (56) equal to zero gives

$$\theta = \frac{dw}{H_\alpha d\alpha} - \frac{u}{R_\alpha} \tag{58}$$

Incorporating this into strains (54) results in the traditional expressions of shell theory, Washizu (1975), including two nonzero strain components:

$$\varepsilon_\alpha = \frac{du}{H_\alpha d\alpha} + \left(1 + \frac{n}{R_\alpha}\right)^{-1}\left[\frac{w}{R_\alpha} - n\left(\frac{u}{R_\alpha^2}\frac{dR_\alpha}{H_\alpha d\alpha} + \frac{d}{H_\alpha d\alpha}\left(\frac{dw}{H_\alpha d\alpha}\right)\right)\right]$$

$$\varepsilon_\beta = \left(1 + \frac{n}{R_\beta}\right)^{-1}\left[\left(1 + \frac{n}{R_\alpha}\right)\frac{u}{H_\beta}\frac{dH_\beta}{H_\alpha d\alpha} + \frac{w}{R_\beta} - \frac{n}{H_\beta}\frac{d\,w}{H_\alpha d\alpha}\frac{dH_\beta}{H_\alpha d\alpha}\right] \tag{59}$$

Also these can be arranged according to the increasing powers of n, corresponding to Eqs. (55). Thus the deformations of the reference surface are naturally still those given by the first three of expressions (56), but the coefficients of n are instead of Eqs. (57) of the form

$$\kappa_\alpha = -\frac{u}{R_\alpha^2}\frac{dR_\alpha}{H_\alpha d\alpha} - \frac{w}{R_\alpha^2} - \frac{d}{H_\alpha d\alpha}\left(\frac{dw}{H_\alpha d\alpha}\right)$$

$$\kappa_\beta = \left(\frac{1}{R_\alpha} - \frac{1}{R_\beta}\right)\frac{u}{H_\beta}\frac{dH_\beta}{H_\alpha d\alpha} - \frac{w}{R_\beta^2} - \frac{1}{H_\beta}\frac{dw}{H_\alpha d\alpha}\frac{dH_\beta}{H_\alpha d\alpha} \tag{60}$$

It is easy to verify the correctness of equations derived, by comparing them with the corresponding ones presented in many fundamental textbooks, for instance with Washizu (1975). Instead, some books, Novozhilov (1964) for example, include certain additonal approximations during the derivation resulting in some slight deviations in the corresponding equations.

9. Equilibrium equations

Recalling the principle of virtual work for deriving the equilibrium equations for a shell follows precisely the procedure applied above in the beam problem. The local Cartesian coordinate system is spanned again at point O, Fig. 5. The coordinate transformation defines the Jacobian matrix (51) and determinant for differentiation and integration operations at point P at which the discrete integration is replaced by the continuous one applying curvilinear coordinates. The elementar volume is

$$\lim_{\Delta V_j \to 0} \Delta V_j = \lim_{\Delta X_j, \Delta Y_j \to 0} \Delta X_j \Delta Y_j dZ = H_\alpha H_\beta \left(1 + \frac{n}{R_\alpha}\right)\left(1 + \frac{n}{R_\beta}\right) d\alpha\, d\beta\, dn \tag{61}$$

This together with the variations of strain components (54) and vector equalities $\vec{I} = \vec{e}_\alpha$, $\vec{J} = \vec{e}_\beta$ and $\vec{K} = \vec{e}_n$ at point P incorporated in the expression for the internal virtual work (20a) yield

$$\delta W^i = -\int_{\alpha,\beta}\left[N_\alpha\left(\frac{d\,\delta u}{H_\alpha d\alpha} + \frac{\delta w}{R_\alpha}\right) - M_\alpha \frac{d\,\delta\theta}{H_\alpha d\alpha} + N_\beta\left(\frac{\delta u}{H_\beta}\frac{dH_\beta}{H_\alpha d\alpha} + \frac{\delta w}{R_\beta}\right)\right.$$

$$\left. - M_\beta \frac{\delta\theta}{H_\beta}\frac{dH_\beta}{H_\alpha d\alpha} + Q_\alpha\left(\frac{d\,\delta w}{H_\alpha d\alpha} - \frac{\delta u}{R_\alpha} - \delta\theta\right)\right] H_\alpha H_\beta\, d\alpha\, d\beta \tag{62}$$

This corresponds to the expression (40) of the beam analysis and includes the notations

$$N_\alpha=\int_n \sigma_\alpha\left(1+\frac{n}{R_\beta}\right)dn \qquad M_\alpha=\int_n \sigma_\alpha n\left(1+\frac{n}{R_\beta}\right)dn$$
$$N_\beta=\int_n \sigma_\beta\left(1+\frac{n}{R_\alpha}\right)dn \qquad M_\beta=\int_n \sigma_\beta n\left(1+\frac{n}{R_\alpha}\right)dn$$
$$Q_\alpha=\int_n \tau_{n\alpha}\left(1+\frac{n}{R_\beta}\right)dn \tag{63}$$

for the stress resultants. The formula for the work due to external loads, the counterpart of Eq. (44) of the beam analysis, is given in the form

$$\begin{aligned}\delta W^e &= \int_{\alpha,\beta}[\bar f_\alpha\,\delta u+\bar f_n\delta w-\bar f_\theta\delta\theta]H_\alpha H_\beta\, d\alpha\, d\beta\\ &\quad+\int_{s_t}[\bar t_\alpha\delta u+\bar t_n\,\delta w-\bar t_\theta\delta\theta]\,d\beta\end{aligned} \tag{64}$$

in which the boundary line coincides with the circumferential coordinate line β. In addition, the notations for the volume forces and surface tractions on the reference surface of a shell, as follows

$$\bar f_\alpha=\int_n f_\alpha\left(1+\frac{n}{R_\alpha}\right)\left(1+\frac{n}{R_\beta}\right)dn \qquad \bar t_\alpha=\int_n t_\alpha dn$$
$$\bar f_n=\int_n f_n\left(1+\frac{n}{R_\alpha}\right)\left(1+\frac{n}{R_\beta}\right)dn \qquad \bar t_n=\int_n t_n\, dn$$
$$\bar f_\theta=\int_n f_\alpha n\left(1+\frac{n}{R_\alpha}\right)\left(1+\frac{n}{R_\beta}\right)dn \qquad \bar t_\theta=\int_n t_\alpha n dn \tag{65}$$

are introduced. Possible tractions on the upper and lower surface of the shell can be included, if necessary, with obvious expressions. Substituting the expressions (62) and (64) due to internal and external work into the principle of virtual work (15) and applying the two-dimensional Green's theorem (10) results directly in an equation

$$\begin{aligned}&\int_{\alpha,\beta}\left\{\left[\frac{1}{H_\alpha H_\beta}\left(\frac{d\,(H_\beta N_\alpha)}{d\alpha}-N_\beta\frac{dH_\beta}{d\alpha}\right)+\frac{Q_\alpha}{R_\alpha}+\bar f_\alpha\right]\delta u\right.\\
&\quad-\left[\frac{1}{H_\alpha H_\beta}\left(\frac{d\,(H_\beta M_\alpha)}{d\alpha}-M_\beta\frac{dH_\beta}{d\alpha}\right)-Q_\alpha+\bar f_\theta\right]\delta\theta\\
&\quad\left.+\left[\frac{1}{H_\alpha H_\beta}\frac{d\,(H_\beta Q_\alpha)}{d\alpha}-\frac{N_\alpha}{R_\alpha}-\frac{N_\beta}{R_\beta}+\bar f_n\right]\delta w\right\}H_\alpha H_\beta\, d\alpha\, d\beta\\
&\quad+\int_\beta[(-N_\alpha+\bar t_\alpha)\,\delta u+(M_\alpha-\bar t_\theta)\,\delta\theta+(-Q_\alpha+\bar t_n)\,\delta w]\,d\beta=0\end{aligned} \tag{66}$$

This yields the well-known general equilibrium equations for a shell of revolution

$$\begin{aligned}
&\frac{dN_\alpha}{H_\alpha d\alpha}+\frac{N_\alpha-N_\beta}{H_\beta}\frac{dH_\beta}{H_\alpha d\alpha}+\frac{Q_\alpha}{R_\alpha}+\bar{f}_\alpha=0\\
&\frac{dM_\alpha}{H_\alpha d\alpha}+\frac{M_\alpha-M_\beta}{H_\beta}\frac{dH_\beta}{H_\alpha d\alpha}-Q_\alpha+\bar{f}_\theta=0\\
&\frac{dQ_\alpha}{H_\alpha d\alpha}+\frac{Q_\alpha}{H_\beta}\frac{d\,H_\beta}{H_\alpha d\alpha}-\frac{N_\alpha}{R_\alpha}-\frac{N_\beta}{R_\beta}+\bar{f}_n=0
\end{aligned}\tag{67}$$

From these the shear force Q_α can be eliminated resulting in only two independent equilibrium equations, finally. The general boundary conditions of a shell

$$\left.\begin{aligned}-N_\alpha+\bar{t}_\alpha&=0\\ M_\alpha-\bar{t}_\theta&=0\\ -Q_\alpha+\bar{t}_n&=0\end{aligned}\right\}\text{ on } s_t \qquad \left.\begin{aligned}u-\bar{u}&=0\\ \theta-\bar{\theta}&=0\\ w-\bar{w}&=0\end{aligned}\right\}\text{on } s_u \tag{68}$$

are obtained from Eq. (66). Geometrical boundary conditions, with given displacement components provided with an overbar, are again due to the requirements of a kinematically admissible virtual displacements. It is still to be noticed when the Kirchhoff-Love theory with the constrained rotation component (58) is assumed, that the boundary conditions will take a little different forms, like

$$\left.\begin{aligned}-N_\alpha-\frac{M_\alpha}{R_\alpha}+\bar{t}_\alpha+\frac{\bar{t}_\theta}{R_\alpha}&=0\\ M_\alpha-\bar{t}_\theta&=0\\ -\frac{dM_\alpha}{H_\alpha d\alpha}-\frac{M_\alpha-M_\beta}{H_\beta}\frac{dH_\beta}{H_\alpha d\alpha}+\bar{t}_n&=0\end{aligned}\right\}\text{ on } s_t \qquad \left.\begin{aligned}u-\bar{u}&=0\\ \frac{dw}{d\alpha}-\frac{d\bar{w}}{d\alpha}&=0\\ w-\bar{w}&=0\end{aligned}\right\}\text{ on } s_u \tag{69}$$

The results obtained can again be verified by comparing them with the ones in any text-book about the shell theory, for example with Washizu (1975). All the results derived are well-known, but the way in which they were deduced here is different, for educational purposes more feasible compared with traditional procedures.

10. Conclusions

The present paper shows an approach which introduces a new non-tensorial way to derive basic equations for any problem in curvilinear geometries. The paper proves local Cartesian coordinate systems to work well, in addition to numerical computation procedures, in analytical considerations of curved beam and shell structures. The coordinates which do not have to conform with the geometry of the structure serve as a convenient tool for deriving the equations needed. The geometry may be very complicated, but expressions formulated using local orthogonal coordinates are formally as simple as the familiar expressions in global coordinates. The method is very systematic depending only on rather basic mathematics which makes it usable, particularly, for educational purposes. Equally well, the procedure introduced can be utilized in generating the basic equations for geometrically non-linear analyses.

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Notations

A, A_t, A_u	cross-sectional areas
x, y, z	global Cartesian coordinates
$\vec{f}\,(f_x f_y f_z)$	volume force vector with references to relevant coordinates
α, β, n	curvilinear coordinates
$\gamma_{xy}\,\gamma_{yz}\,\gamma_{zx}$	shear strains with references to relevant coordinates
$H_\alpha H_\beta$	Lamé coefficients
$\vec{I}\,\vec{J}\,\vec{K}$	local unit vectors
$\varepsilon_x\,\varepsilon_y\,\varepsilon_z$	axial strains with references to relevant coordinates
$\vec{i}\,\vec{j}\,\vec{k}$	global unit vectors
$M_\alpha M_\beta$	bending moment resultants
$\overleftrightarrow{\varepsilon}$	strain tensor
$N_\alpha N_\beta$	membrane force resultants
$\theta_\alpha\,\theta_\beta$	rotation components
n_α, n_β	direction cosines
$\kappa_\alpha\,\kappa_\beta\,\kappa_{\alpha\beta}$	curvatures with references
$Q_\alpha Q_\beta$	shear force resultants
$\kappa_{\beta n}\,\kappa_{n\alpha}$	to relevant coordinates
$R_\alpha R_\beta R_{\alpha\beta}$	radii of curvatures and torsion
ν	normal out of the boundary
$\vec{r}, \vec{r}_o$	position vectors
$\sigma_x\sigma_y\sigma_z$	axial stresses with references to relevant coordinates
S, S_t, S_u	surfaces
s, s_t, s_u	coordinate along the boundary
$\overleftrightarrow{\sigma}$	stress tensor
$\vec{t}\,(t_x\,t_y\,t_z)$	surface traction vector with references to relevant coordinates
$\tau_{xy}\tau_{yz}\tau_{zx}$	shear stresses with references

$\vec{u}$	displacement vector
U, V, W	local displacement components
δ	variation
u, v, w	global displacement components
∂	partial derivative
V	volume
$(\cdot)\cdot(\cdot)$	scalar product of vectors
δW^i, δW^e	virtual works due to (i)nternal and (e)xternal forces
$(\cdot):(\cdot)$	scalar product of tensors
$\vert\cdot\vert$	determinant
X, Y, Z	local Cartesian coordinates
$O(\cdot)$	order estimate