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Dynamic analysis of trusses including the effect of local modes

Eldad Levy†

Faculty of Agricultural Engineering, Technion, Israel Institute of Technology, Haifa 32000, Israel

Moshe Eisenberger‡

Faculty of Civil Engineering, Technion, Israel Institute of Technology, Haifa 32000, Israel

Abstract. The dynamic analysis of trusses using the finite element method tends to overlook the effect of local member dynamic behavior on the overall response of the complete structure. This is due to the fact that the lateral inertias of the members are omitted from the global inertia terms in the structure mass matrix. In this paper a condensed dynamic stiffness matrix is formulated and used to calculate the exact dynamic properties of trusses without the need to increase the model size. In the examples the limitations of current solutions are presented together with the exact results obtained from the proposed method.

Key words: dynamic stiffness matrix; local mode shape; large structure; condensed dynamic stiffness matrix; finite elements.

1. Introduction

The dynamic analysis of large structures has become in recent years a practical problem. Large space stations and huge structures on the ground (like Energy Towers) require the dynamic analysis of systems that are modeled by hundred thousands of degrees of freedom. This kind of analysis is an impasse even for the most advanced computers on the market today. Thus, it becomes a real obstacle to technological advance in a number of fields.

The vibrational analysis of large scale trusses is currently done by the finite elements method. The members of the structure are modeled with elements, one elements per member, and this results in a stiffness matrix of size ($2N \times 2N$) for plane trusses, or ($3N \times 3N$) for space trusses, where N is the number of joints in the truss. For large scale structures this results in a very large eigenvalue problem, which demands enormous computational efforts and very long execution time, even with the most modern computers. This finite element model, however, is approximate, and the results may still have significant errors due to several shortcomings of the finite element method that will be pointed out.

The dynamic stiffness method differs from the finite elements method in several respects. The

‡ Research Assistant
† Associate Professor

first and most important is that it is exact and the results are not dependent on the selection of the element type and discretization parameters for the structure. The second difference is that there is no separate modeling of the mass inertia (as the mass matrix in finite element method) and its effect is included in the stiffness matrix that is frequency dependent. The third difference is that all the inertias of the members, along its axis, and transversely are taken into account and it will be shown that they have a significant effect on the vibration frequencies of the structure. The fourth difference is in the fact that the size reduction that is done in the finite element solution introduces additional inaccuracies, and in the dynamic stiffness approach the results from the condensed model are still exact.

2. Limitations of the finite element method in dynamic analysis

The Finite Element Method (FEM) is based on assuming the local behavior within the element (in our case member) to derive the stiffness and mass matrices of the structure which are needed for dynamic analysis. The quality of the results of the analysis are therefore highly dependent on the quality of the assumed element behavior. For frame structures, as it is in our case, the shape functions are exact and the stiffness matrix is thus the exact static stiffness matrix. It is exact since the shapes function which are used are the exact solution of the differential equation that represents the element response, but this is so only when the inertia terms are omitted. The mass matrix, which is derived based on the same static shape function is thus not exact. It is well known that due to this limitation, the vibration frequencies that are found using FEM are only approximation to the exact results.

Another problem that arises in the dynamic analysis using FEM is that for the higher frequencies of vibration, the accuracy is deteriorating, and at some level these become totally unreliable. For large scale structures, the resulting matrices, stiffness and mass, are very big, and thus very time consuming for the computation of the eigenvalues. Size reduction techniques that were employed in the past have introduced additional approximations, and reduced even further the accuracy of the resulting natural frequencies.

A very important and ignored problem of FEM is the fact that the local inner element or local modes of vibrations cannot be found. The shape functions for the axial degrees of freedom of the element are linear, and the local axial modes are sinusoidal. Thus, the element is not capable of representing the local axial vibration even at minimal accuracy, for the first axial mode, and even worse for higher axial modes. As for the lateral local modes the third degree polynomial that is used in FEM to approximate lateral displacements is very limited in modeling the sinusoidal behavior of such motions, for all the modes.

3. Derivation of the dynamic stiffness method (DSM)

The dynamic stiffness matrix is frequency dependent (Paz and Dung 1975, Fricker 1975, Leung 1993). It is derived, as shown below, by solving exactly the equations of motion of the member. The single dynamic stiffness matrix combines the stiffness and inertia effects in one. The natural frequencies of vibrations are the values of the frequency ω that cause the dynamic stiffness matrix to become singular. These values can be found by several search methods and by the Wittrik-Williams algorithm (Wittrick and Williams 1971).

The differential equation that represents the lateral vibrations ***y*(*x*, *t*)** of a beam element with flexural rigidity ***EI***, mass per unit length ***m***, and length ***L***, loaded by time dependent distributed lateral load ***p*(*x*, *t*)** is:

$$E \cdot I \cdot \frac{\partial^4}{\partial x^4} y(x,t) + m \cdot \frac{\partial^2}{\partial t^2} y(x,t) = p(x,t) \tag{1}$$

The stiffness matrix is derived for an unloaded member i.e., $p(x, t)=0$. The general solution of this equation is:

$$Y(\xi) = A \cdot \cos(\alpha \cdot \xi) + B \cdot \sin(\alpha \cdot \xi) + C \cdot \cosh(\alpha \cdot \xi) + D \cdot \sinh(\alpha \cdot \xi) \tag{2}$$

where:

$$\alpha^4 = \omega^2 \cdot \frac{m}{E \cdot I} = \omega^2 \cdot \frac{\rho \cdot a}{E \cdot I} \tag{3}$$

and where: ρ-material density, $\boldsymbol{a}$-beam cross section, and

$$\xi = \frac{x}{L} \qquad 0 \leq \xi \leq 1 \tag{4}$$

Applying the unit end conditions for the 4 degrees of freedom (two end translations and two end rotations) one at a time, one can write down the force-displacement relationship as:

$$\tilde{F} = \tilde{K}(\omega) \cdot \tilde{D} \tag{5}$$

with $\tilde{F}$ the end forces, $\tilde{D}$ the end displacements and rotations, and $\tilde{K}(\omega)$ the frequency dependent element dynamic stiffness matrix. Written in full we have:

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} = B \cdot \begin{bmatrix} -a^3 \cdot Z_1 & -\alpha^2 \cdot Z_2 & \alpha^3 \cdot Z_4 & \alpha^2 \cdot Z_5 \\ -\alpha^2 \cdot Z_2 & \alpha \cdot Z_3 & -\alpha^2 \cdot Z_5 & \alpha \cdot Z_6 \\ \alpha^3 \cdot Z_4 & -\alpha^2 \cdot Z_5 & -\alpha^3 \cdot Z_1 & \alpha^2 \cdot Z_2 \\ \alpha^2 \cdot Z_5 & \alpha \cdot Z_6 & \alpha^2 \cdot Z_2 & \alpha \cdot Z_3 \end{bmatrix} \cdot \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} \tag{6}$$

$$B = \frac{E \cdot I_z}{\cos(\alpha \cdot L) \cdot \cosh(\alpha \cdot L) - 1} \tag{6a}$$

$$Z_1 = \cos(\alpha \cdot L) \cdot \sinh(\alpha \cdot L) + \sin(\alpha \cdot L) \cdot \cosh(\alpha \cdot L) \tag{6b}$$

$$Z_2 = \sin(\alpha \cdot L) \cdot \sinh(\alpha \cdot L) \tag{6c}$$

$$Z_3 = \cos(\alpha \cdot L) \cdot \sinh(\alpha \cdot L) - \sin(\alpha \cdot L) \cdot \cosh(\alpha \cdot L) \tag{6d}$$

$$Z_4 = \sin(\alpha \cdot L) + \sinh(\alpha \cdot L) \tag{6e}$$

$$Z_5 = \cos(\alpha \cdot L) - \cosh(\alpha \cdot L) \tag{6f}$$

$$Z_6 = \sin(\alpha \cdot L) - \sinh(\alpha \cdot L) \tag{6g}$$

$\boldsymbol{v}$, $\boldsymbol{\theta}$, $\boldsymbol{V}$ and $\boldsymbol{M}$ are the displacement, rotation, shear force, and bending moment at the two ends, 1 and 2, respectively.

For axial vibrations the differential equation is:

$$-E \cdot a \cdot \frac{\partial^2 u(x,t)}{\partial x^2} + m \cdot \frac{\partial^2 u(x,t)}{\partial t^2} = q(x,t) \tag{7}$$

with ***Ea*** the axial stiffness, ***q***(***x***, ***t***) the time dependent distributed axial load, and ***u***(***x***, ***t***) is the axial displacement. Following the same procedure as above the axial dynamic stiffness matrix is found as:

$$\begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} = E \cdot a \cdot \mu \cdot \begin{bmatrix} \cot(\mu \cdot L) & -\csc(\mu \cdot L) \\ -\csc(\mu \cdot L) & \cot(\mu \cdot L) \end{bmatrix} \cdot \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} \tag{8}$$

where:

$$\mu = \omega \cdot \sqrt{\frac{m}{a \cdot E}} = \omega \cdot \sqrt{\frac{\rho}{E}} \tag{9}$$

and $\tilde{u}$ and $\tilde{F}$ are the axial displacement and axial force at the ends, 1 and 2 respectively.

For three dimensional members the same terms are combined into the appropriate locations in the space frame member stiffness (with terms for torsional stiffness similar to those for axial stiffness).

4. Local modes within members in FEM analysis

The problem of obtaining the local modes in the FEM can be overcome at the high price of introducing additional nodes along the members. The number of additional nodes per each member will determine the ability of the model to predict the order of local modes, where sinusoidal lateral and longitudinal behavior should be approximated. This will cause the increase of the numerical eigenvalue problem to be solved by a factor depending on the number of additional nodes, and even by an order of magnitude.

Another strategy is to use the component mode method suggested by Weaver and Loh (1985). Here, a number of exact local modes are added to the member stiffness matrix formulation, and resulting in a larger element matrix. Again, the size of the eigenvalue problem is increased significantly. The advantage of this approach is that the local modes are calculated exactly, rather than approximately as in the FEM solution, but only for the number of modes that is added in the member formulation.

5. Local modes in dynamic stiffness method

The formulation of the DSM includes in it inherently all the local modes for a member. For truss type structures, the inertias related to rotations of the end-joints which are required for local transverse modes, are retained by obtaining the required stiffness matrix for the end translations only, by condensation of the rotational degrees of freedom. In the DSM, this condensation is exact (as proved in, Levy 1997).

For the truss member we partition the dynamic stiffness matrix to be:

$$\tilde{D} = \begin{bmatrix} \tilde{D}_{11} & \tilde{D}_{12} \\ \tilde{D}_{21} & \tilde{D}_{22} \end{bmatrix} \tag{10}$$

where subscript 1 relates to the retained degrees of freedom (the end translations), and subscript 2 relates to the condensed (the end rotations), and the condensed matrix $\tilde{D}^*$ is

$$\tilde{D}^* = \tilde{D}_{11} - \tilde{D}_{12} \cdot \tilde{D}_{22}^{-1} \cdot \tilde{D}_{21} \tag{11}$$

or in detail for a plane truss member

$$\tilde{D}^* = \begin{bmatrix} \dfrac{E_x \cdot A \cdot \mu \cdot \cos(\mu \cdot L)}{\sin(\mu \cdot L)} & 0 & \dfrac{E_x \cdot A \cdot \mu}{\sin(\mu \cdot L)} & 0 \\ 0 & \dfrac{E_x \cdot I_z \cdot \alpha^3 \cdot Z_3}{2 \cdot Z_2} & 0 & \dfrac{E_x \cdot I_z \cdot \alpha^3 \cdot Z_6}{2 \cdot Z_2} \\ \dfrac{E_x \cdot A \cdot \mu}{\sin(\mu \cdot L)} & 0 & \dfrac{E_x \cdot A \cdot \mu \cdot \cos(\mu \cdot L)}{\sin(\mu \cdot L)} & 0 \\ 0 & \dfrac{E_x \cdot I_z \cdot \alpha^3 \cdot Z_6}{2 \cdot Z_2} & 0 & \dfrac{E_x \cdot I_z \cdot \alpha^3 \cdot Z_3}{2 \cdot Z_2} \end{bmatrix} \tag{12}$$

This matrix is used in the example in the next section to calculate all the vibrational modes of a two-bar plane truss. A 3D version of this matrix is used to calculate two space truss example.

6. Examples

The first example is of a very simple two bar truss and it is used to compare finite element,

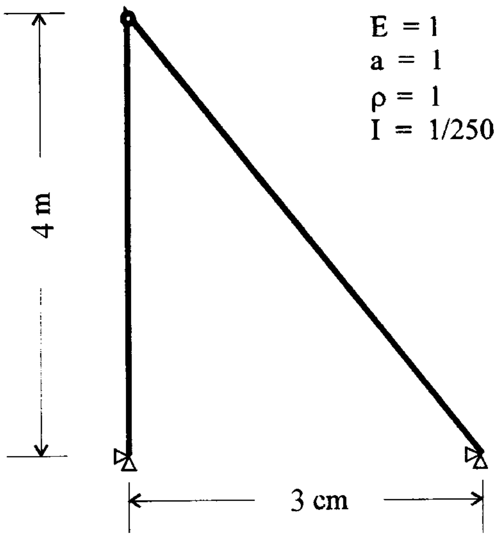

Fig. 1 Example of two member truss

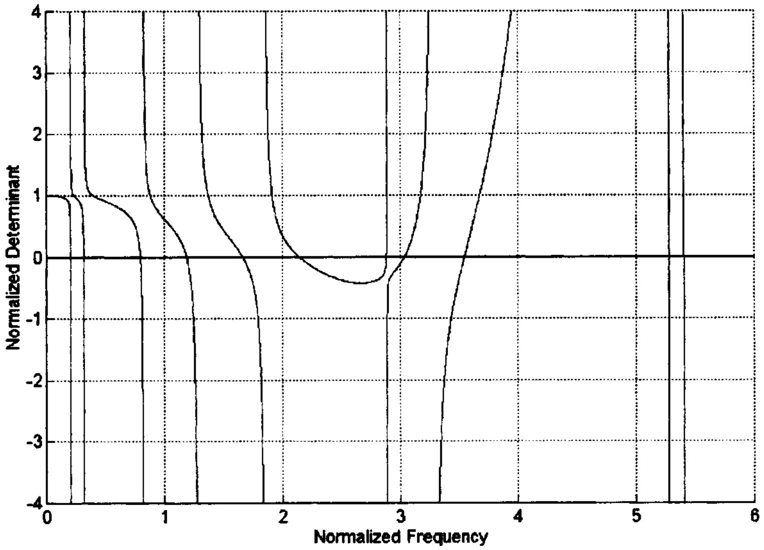

Fig. 2 Normalized determinant vs. normalized frequency plot for the plane truss

Table 1 Normalized frequencies for standard FEM

Mode number	F.E.	F.E.
	Consistent mass matrix	Lumped mass matrix
1	1.00000	0.81650
2	3.02330	2.46851

Table 2 Results from refined FEM, CMM and DSM

Mode number	Condensed dynamic stiffness method	Component mode method		F.E. beam element method (Coupled at connection)	
	exact	5 Function par member	20 Function par member	1 Element par member	10 Elements par member
1	0.20396	0.20397	0.20397	0.22580	0.20455
2	0.31658	0.31659	0.31659	0.34975	0.31621
3	0.79562	0.79657	0.79567	0.97634	0.79351
4	1.18867	1.18941	1.18933	1.42832	1.18448
5	1.66325	1.66787	1.66727	2.30692	1.65516
6	2.14448	2.17098	2.16838	3.19301	2.13019
7	2.88168	3.03368	3.03162		2.88118
8	3.03502	3.16326	3.16281		3.00243
9	3.54627	3.62789	3.61621		3.50462
10	5.13264	5.14617	5.14612		5.04856
11	5.28080	5.57522	5.53014		5.20794
12	6.66417	8.26739	7.53583		6.62913
Member matrix size	4	9	24	6	33
DOF model size	2	12	42	6	60

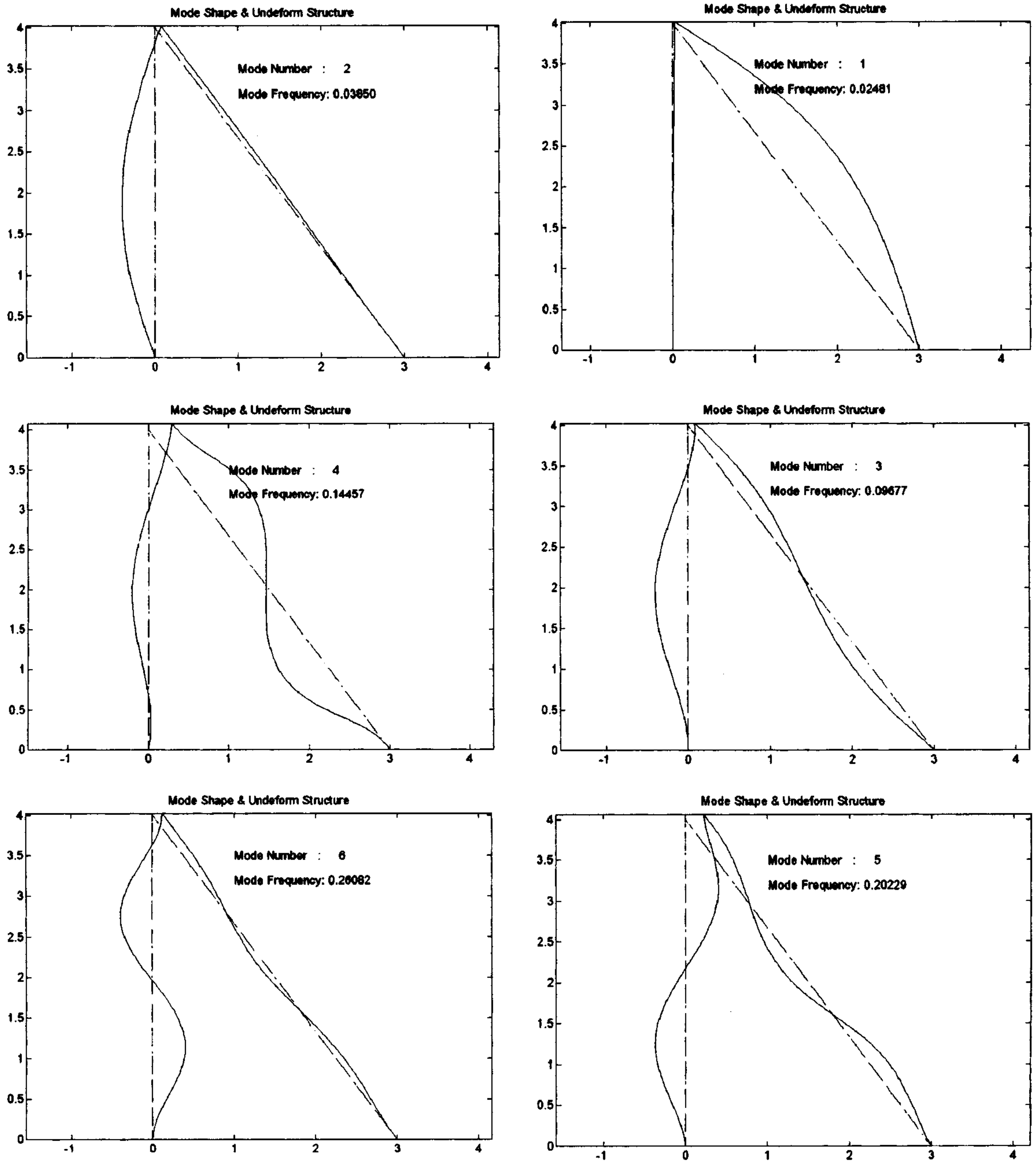

Fig. 3 The first 6 modes of the example truss

component mode and dynamic stiffness results. The example truss shown in Fig. 1 is taken from Weaver and Loh (1985). The numerical data is taken as unity for all material and section properties, except for the moment of inertia which is taken as 1/250. This was done so deliberately in the original analysis by Weaver and Loh (1985), in order to make the influence of the transverse local modes more important. The first analysis that was done is a regular two bar two degrees of freedom model solved by FEM. Two methods for mass modeling were used - the

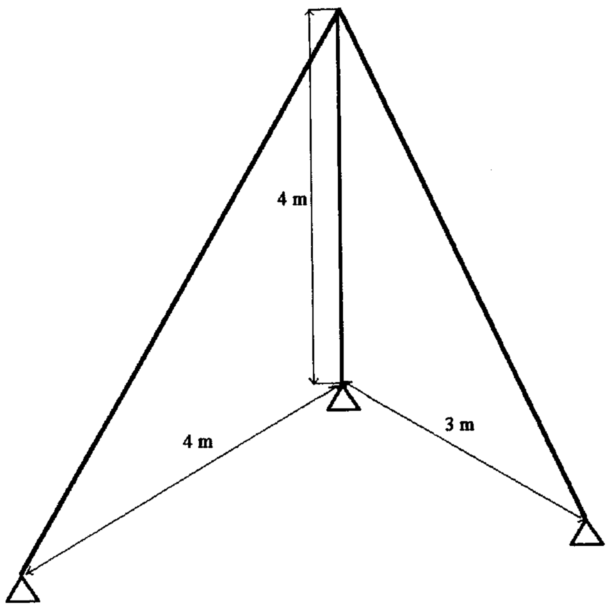

Fig. 4 Example of 3 member space truss

lumped mass model and the consistent mass model. The results for the first two modes (this model is not capable of predicting more frequencies), are given in Table 1, normalized with respect to the first frequency from the consistent mass solution (ω_1=0.12162 [rad/s]). As the consistent mass model yields upper bound on the correct eigenvalues, and the lumped mass model yields lower bounds, we can observe that the difference between these values is in the order of 20%. This difference is too large to bracket the frequencies adequately. In order to study the effect of local modes, the same problem was also analyzed by the component mode method. Two modeling schemes were used: 5 sinusoidal shape functions added to each member (as in Weaver and Loh 1985), and 20 sinusoidal shape functions added. These resulted in (12×12) and (42×42) eigenvalue problems, respectively. Also a finite element solution was obtained using the ANSYS Code. There, in order to obtain local modes, each member was modeled by 1 and 10 beam elements with the connections simulated by coupling the displacements degrees of freedom at the ends. The sizes of these models were (6×6) and (60×60), respectively.

The proposed method was employed with 2 elements i.e., (2×2) dynamic stiffness matrix. For this case, the determinant of the dynamic stiffness matrix was found for values of ω. The natural frequencies are the values of ω, for which the determinant is equal to zero. The plot of the normalized determinant (with respect to the value at ω=0, the static determinant) is shown in Fig. 2. The exact values are found by bracketing the eigenvalues by two values with opposite determinant sign, and then by bisections to the desired accuracy.

The results form the three methods normalized with respect to the first frequency in the FE solution in Table 1 (ω_1=0.12162 [rad/s]) , are given in Table 2. The first 6 modes (from the proposed method) are plotted in Fig. 3. From this figure it is very clear that local transverse modes dominate the behavior of the truss. From the results in Table 2, we see that the FE model, using 20 elements per member, but with beam elements that include the lateral mode shapes, and coupling the end displacements at the connection of the members, gives very different results

Table 3 Normalized frequencies for standard FEM of the 3 member space truss

Mode number	F.E.	F.E.
	Consistent mass matrix	Lumped mass matrix
1	1.00000	0.81650
2	1.46789	1.19853
3	3.64114	2.97298

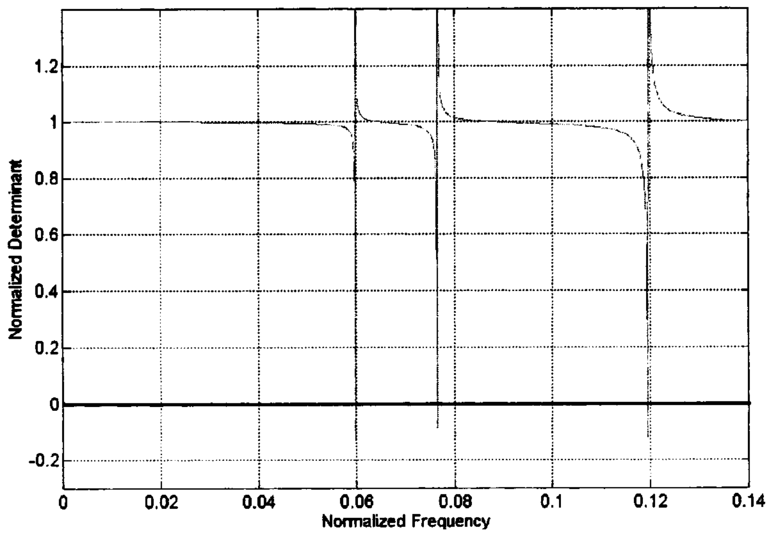

Fig. 5 Normalized determinant vs. normalized frequency plot for the space truss

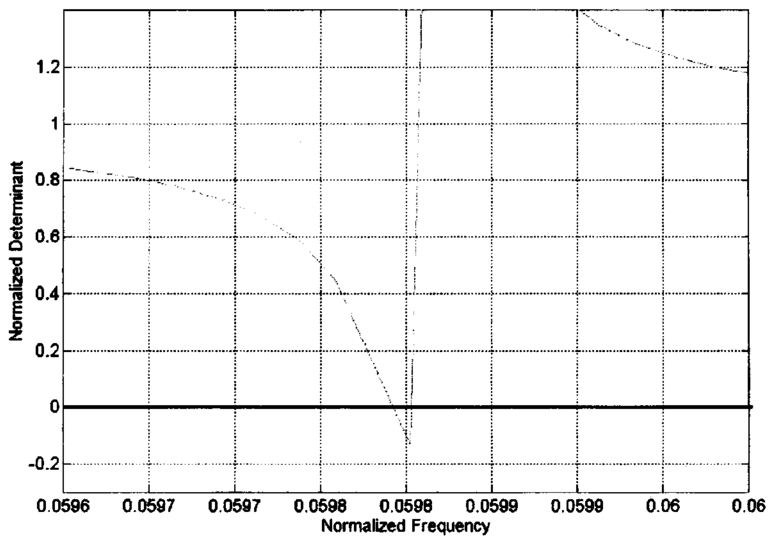

Fig. 6 Normalized determinant vs. normalized frequency plot for the space truss, detailed behavior in the close neighborhood of the first pair

Table 4 FEM, CMM and DSM results for the 3 member space truss example

Mode number	Condensed dynamic stiffness method	Component mode method		F.E. beam element method (Coupled at connection)	
	exact	5 Function par member	20 Function par member	1 Element par member	10 Elements par member
1	0.05979	0.05979	0.05979	0.06635	0.05979
2	0.05980	0.05980	0.05980	0.06637	0.05980
3	0.07652	0.07652	0.07652	0.08491	0.07651
4	0.07652	0.07654	0.07654	0.08494	0.07654
5	0.11949	0.11949	0.11949	0.13258	0.11948
6	0.11950	0.11956	0.11956	0.13267	0.11955
⋮	⋮	⋮	⋮	⋮	⋮
17	0.94896	0.94900	0.94896		0.94983
18	0.95445	0.95445	0.95444		0.95533
19	1.06386	1.06397	1.06388		1.06361
20	1.07041	1.07045	1.07043		1.07013
21	1.21174	1.21193	1.21177		1.21263
Member matrix size	6	16	46	12	66
DOF model size	3	33	123	27	189

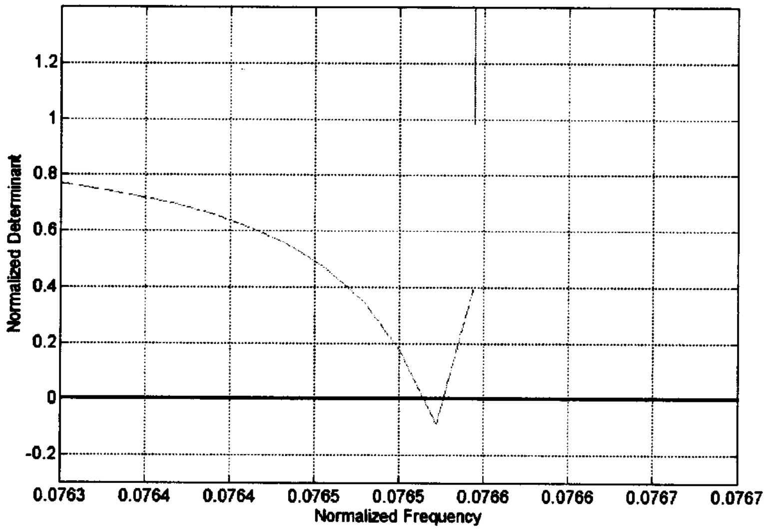

Fig. 7 Normalized determinant vs. normalized frequency plot for the space truss, detailed behavior in the close neighborhood of the second pair

from those in Table 1. The results of the finer FEM model and the CMM model are very close to the exact results that are obtained by the DSM model (with 2 degrees of freedom only, i.e., a (2×2) eigenvalue problem, and the problem is reduced to finding the zeros of the determinant of the dynamic stiffness matrix).

A 3 member space truss is used as the second example. The 3 members are built from steel

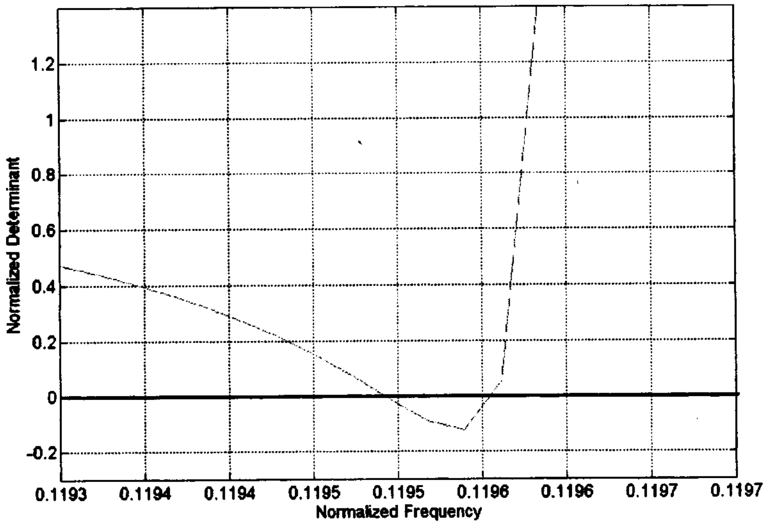

Fig. 8 Normalized determinant vs. normalized frequency plot for the space truss, detailed behavior in the close neighborhood of the third pair

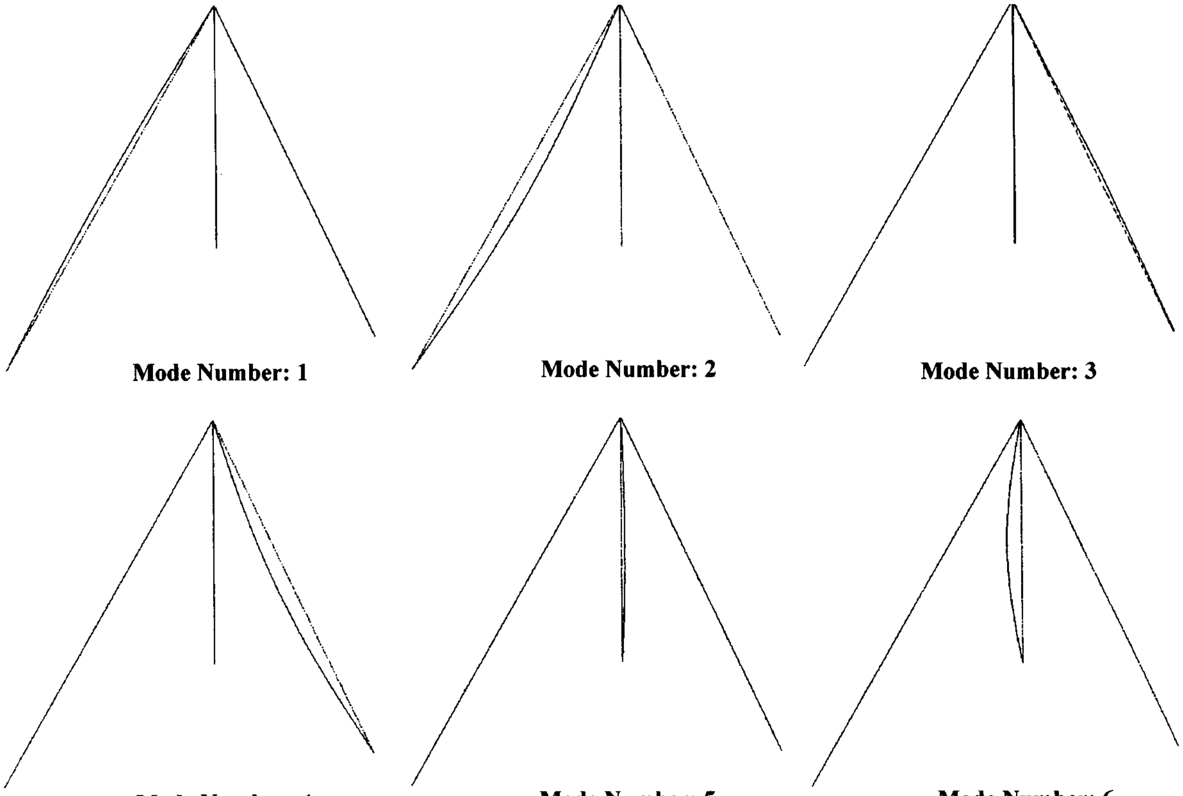

Fig. 9 The first 6 modes of the 3 member space truss

pipes (A=3.81e-4 m^2, I=1.12e-7 m^4), and are of different length as shown in Fig. 4. The symmetric nature of the members result in similar local behavior of each member in its own

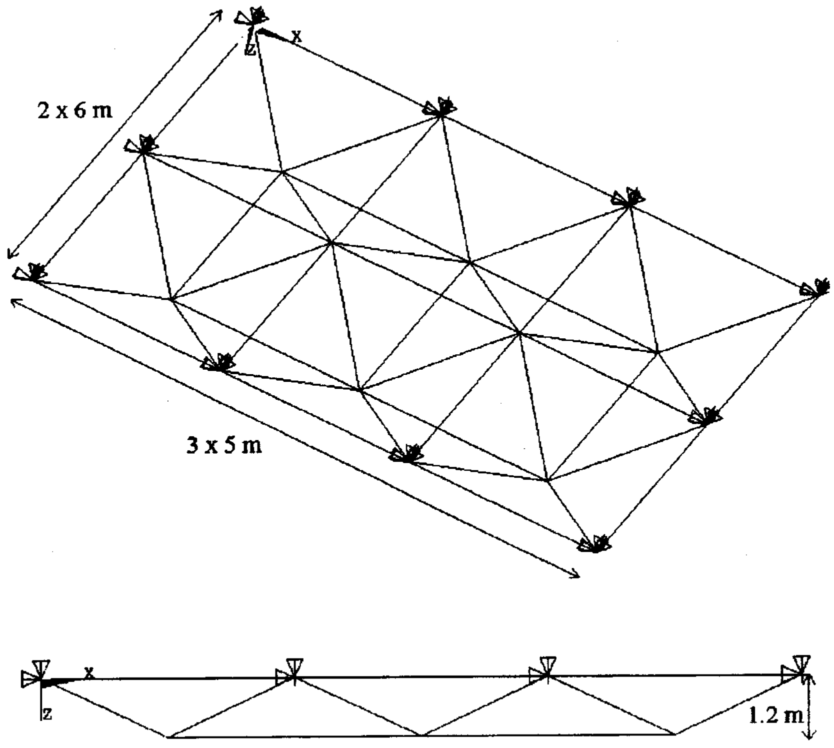

Fig. 10 Example of 48 member space truss

Table 5 Exact and FEM results for the 48 member space truss example

Exact	F.E. Consistent	Condensed dynamic stiffness method		F.E.	
Mode number	Mode number	Exact		Consistent mass matrix	
		rad/s	normalized	rad/s	normalized
1-22		27.4860-27.5431	0.1571-0.1574		
23-48		39.4464-39.6621	0.2254-0.2266		
49-96		58.3417-59.4100	0.3334-0.3395		
97-118		109.5497-110.1725	0.6260-0.6295		
119-144		157.3020-158.6484	0.8989-0.9065		
	1			175.0044	1.0000
145-192		227.0551-237.6400	1.2982-1.3579		
193-214		244.5264-247.8882	1.3974-1.4165		
	2			289.2963	1.6531
215-240		322.8061-356.9590	2.0013-2.0397		
	3			364.0516	2.0802
	4			440.3656	2.5163

principal axis, and thus will result in repeated local modes. However, since the end conditions, i.e., restraints exerted on each member by the other two members, is not equal, in each direction, and

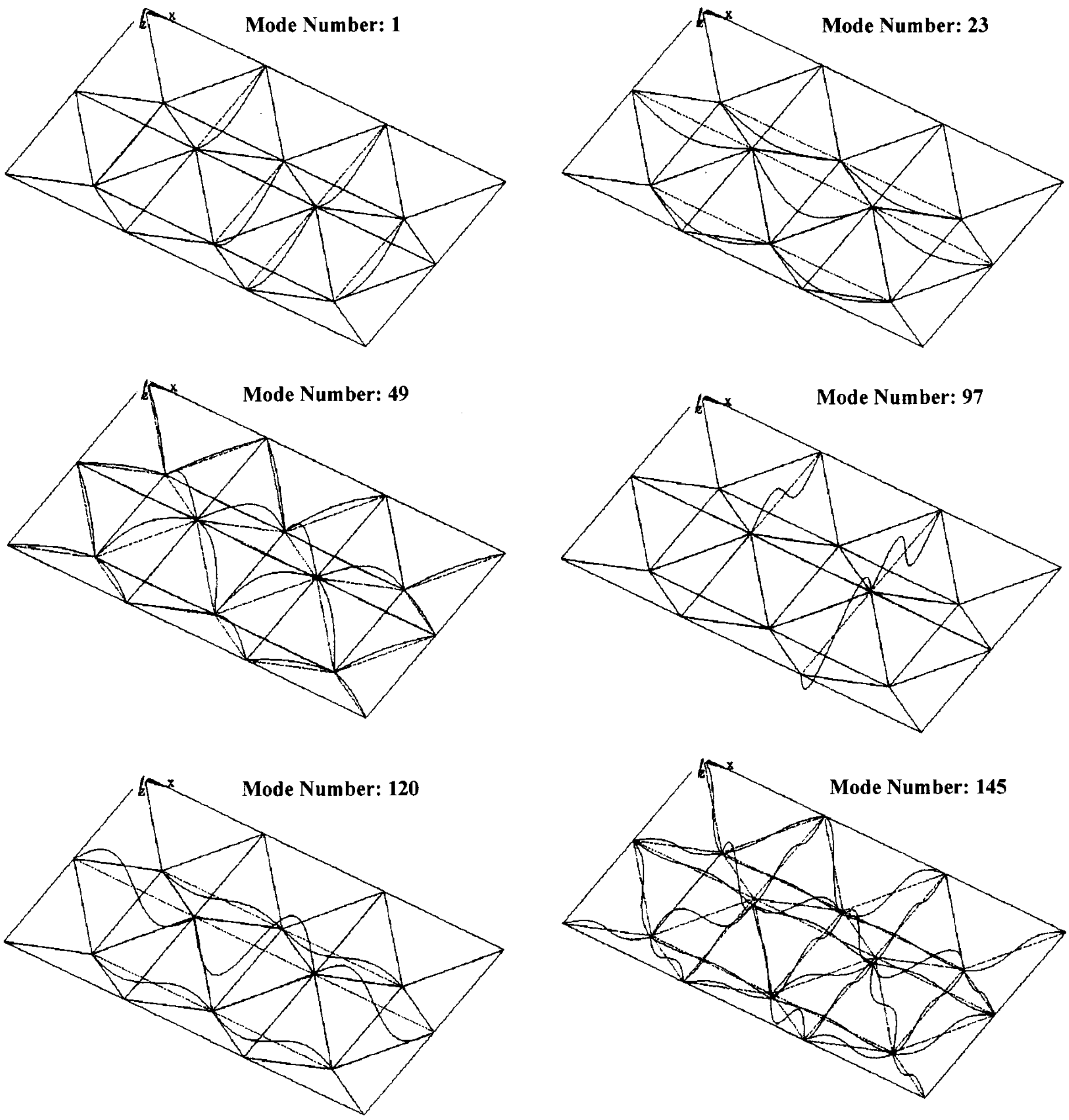

Fig. 11 The first 6 modes of the 48 member space truss

the repeated modes are slightly separated. This can be seen in Fig. 5 where the normalized determinant of the dynamic stiffness matrix is plotted vs. the frequency, and in the numerical results in Table 4. Figs. 6, 7, and 8 show the detailed behavior of the determinant in the close neighborhood of the three first pairs of the almost repeated modes as discussed above.

In Table 3 the results obtained from a 3 DOF FE model are given. Comparison of these values with the exact results in Table 4 (from DSM, CMM, and enhanced FEM) shows that 18 natural frequencies of the structure are lower than the first frequency from the simple FE analysis results in Table 3. The first 6 modes for this truss are shown in Fig. 9.

In the third example a 48 member space truss is analyzed. This is a double layer truss supported along the edges in all directions as shown in Fig. 10. The cross-sectional area and moment of inertia is taken as A=6.88e-4 m^2 and I=2.62e-7 m^4, respectively. Since there are groups of identical members and the sections are circular then one has many repeated local modes in this truss. However, since the end conditions of these members are not identical, but very close to being identical, we have separation of the frequencies as given in Table 5. We can see in the Table, that the regular finite element analysis fails to predict the frequencies of the truss completely. In Fig. 11 six modes, from the first six groups of very close frequencies are shown.

7. Conclusions

The use of the condensed dynamic stiffness matrix in the analysis of trusses overcomes the limitations of the finite element method in modeling the lateral inertias of the members. The main advantage of the Dynamic Stiffness Matrix is that the size of the model is kept very compact compared to other methods, (at least an order of magnitude size reduction), while the results are still exact, with significantly reduced computational effort. This advantage is more important for the analysis of large scale trusses, where model size is an obstacle for the other methods.

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