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A general closed-form solution to a Timoshenko beam on elastic foundation under moving harmonic line load

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Abstract. In this paper, a general closed-form solution for evaluating the dynamic behavior of a Timoshenko beam on elastic foundation under a moving harmonic line load is formulated in the frequency-wavenumber domain and in a moving coordinate system. It is found that the characteristic equation is quartic with real coefficients only, and its poles can be presented explicitly. This enables the substitution of these poles into Cauchy's residue theorem, leading to the general closed-form solution. The solution can be reduced to seven existing closed-form solutions to different sub-problems and a new closed-form solution to the subproblem of a Timoshenko beam on an elastic foundation subjected to a moving quasi-static line load. Two examples are included to verify the solution.

Keywords: closed-form solution; beam on elastic foundation; moving load; Timoshenko beam; Cauchy's residue theorem

1. Introduction

The classic problem of infinite beams on elastic foundation subjected to a moving load is of great theoretical and practical importance (Frýba 1999), which serves as an idealization of the behavior of railway tracks and pavement under wheel loads (Gan *et al*. 2015).

Over almost a century, a number of solutions have been proposed for this problem. However, only a few are closed-form solutions. Kenny gave the first closed-form solution of an Euler-Bernoulli (EB) beam on an elastic foundation subjected to a quasi-static moving load (Kenney 1954). Later, this solution was extended by Mathew (1958) to a similar model subjected to a moving load of harmonic amplitude variation, and by Frýba (1999) to a viscoelastic foundation instead of the original elastic foundation. Sun (2001) also derived a closed-form solution to an EB beam on a viscoelastic foundation under harmonic line loads. Subsequently, he formulated a closed-form solution to a beam on a viscoelastic foundation under moving loads (Sun 2002) and an explicit representation of the steady-state response of a beam on an elastic foundation under moving harmonic line loads (Sun 2003). A summary of these closed-form solutions is given in cases (1) to (6) of Table 1.

The beams in the aforementioned references are the EB type, and thus no shear deformation is considered. The shear deformation is important for a short beam, which has been used as an idealization of multilayered pavements (Luo *et al*. 2015) and embankments of a ballasted track (Galvin *et al*. 2010). To consider both flexural deformation

Table 1 Closed-form solutions of a beam-foundation system under dynamic excitations

Case No.	Beam type	Foundation type	Load type	Reference	Condition*
1	EB	Elastic	Moving quasi-static point load	(Kenney 1954)	$S=\infty$, $R=0$, $\omega=0$, and $l\to 0$
2		Elastic	Moving harmonic point load	(Mathews 1958)	$S=\infty$, $R=0$, and $l\to 0$
3		Viscoelastic	Moving harmonic point load	(Frýba 1999) with no damping	$S=\infty$, $R=0$, and $l\to 0$
4		Viscoelastic	Non-moving harmonic line load	(Sun 2001) with no damping	$S=\infty$, $R=0$, $v=0$, and $l\to 0$
5		Viscoelastic	Moving quasi-static line load	(Sun 2002) with no damping	$S=\infty$, $R=0$, and $\omega=0$
6		Elastic	Moving harmonic line load	(Sun 2003)	$S=\infty$, and $R=0$
7	Timoshenko	Viscoelastic	Non-moving harmonic line load	(Luo *et al*. 2016) with no damping	$v=0$
8		Elastic	Moving quasi-static line load	Section 3 of the present paper	$\omega=0$

*Note: S and R are the shear rigidity and the radius of gyration of the beam respectively; ω is the circular frequency of the load, l is the half-length of the load, and v is the load speed

and shear deformation effects, the Timoshenko beam rather than the EB beam needs to be studied. However, adding shear deformation poses a great challenge in formulating an explicit solution. This challenge is attributed to the so-called characteristic equation, the denominator of an integral representation of the beam deflection, is generally a fourth- or fifth-order polynomial with complex coefficients and thus its poles (or roots) usually cannot be explicitly expressed (Luo *et al*. 2015). This feature prevents the use of Cauchy's residue theorem to evaluate line integrals of analytic functions over closed curves. Consequently, the problem has no closed-form solution in general. There is an exception for a viscoelastically supported Timoshenko

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beam subjected to a non-moving harmonic line load, in which a closed-form solution exists (Luo *et al.* 2016). In that study, the characteristic equation is biquadratic with complex coefficients, and fourteen kinds of poles can be explicitly expressed with elementary functions given different combinations of viscous damping, frequency, and other sub-conditions. A closed-form solution was formulated by the use of the Cauchy's residue theorem.

If it is not restrictive to closed-form solutions, more advanced beam-foundation models can be adopted by using such as semi-analytical and numerical methods, which are better representations of the soil-pavement/track interaction problems. For example, Kargarnovin and Younesian *et al.* (2004) proposed a semi-anlaytical solution to a laminated composite beam resting on a Pasternak foundation subjected to an arbitrary moving load, in which those real poles were determined by using Gaussian quadrature method. Calim (2016) employed the Durbin's algorithm to convert transform domain results of curved Timoshenko beams resting on a viscoelastic foundation back into real space. Luo *et al.* (2017) employed a Fast Fourier transformation (FFT) to study the steady-state response of a beam on a Pasternak foundation under vehicular loads. Karahan and Pakdemirli (2017) used the classical multiple scales and the multiple scales Lindstedt Poincare methods to investigate nonlinear vibrations of an EB beam resting on a nonlinear elastic foundation.

This paper presents a general closed-form solution to a Timoshenko beam on an elastic foundation subjected to a moving harmonic line load. As the damping is excluded, the characteristic equation, as will be demonstrated later, is a quartic equation with real coefficients only. This kind of equation is the highest degree polynomial that can be analytically solved by radicals with no iterative techniques (Irving 2003). Several algorithms have been proposed for this purpose (Shmakov 2011). In this paper, we follow Ferrari algorithm to solve the quartic equation. Subsequently, a general closed-form solution is obtained by applying the Cauchy's residue theorem. The present solution is general in the sense that it can be reduced to eight closed-form solutions with specific conditions summarized in the last column of Table 1. The first seven are available in the literature; the eighth is a byproduct of the present general solution and is reported for the first time.

The paper is organized as follows. First, the deflection of a Timoshenko beam on an elastic foundation is derived in an integral representation, which is subsequently written in a general closed-form by applying the Cauchy's residue theorem on poles of a quartic characteristic equation. The solution is then reduced to seven existing different sub-problems and one new closed-form solution under zero speed. Finally, the solution is verified through two examples.

2. Problem formulation

2.1 Integral representation of beam deflection

An infinite Timoshenko beam rested on an elastic foundation is shown in Fig. 1. The Timoshenko beam is characterized by a mass per unit length m, Young's modulus E, cross-sectional moment of inertia I, shear rigidity S, and radius of gyration R of the beam. The elastic foundation has a stiffness per unit length k. The governing equations of deflection $w(y, t)$ and flexural rotation $\theta(y, t)$ of the beam in a fixed global Cartesian coordinate system (y) are given as follows (Luo *et al.* 2015).

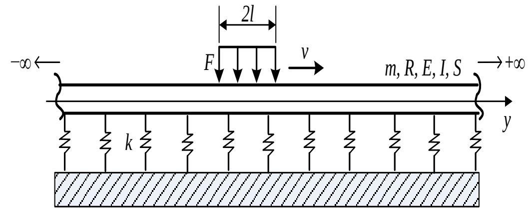


Fig. 1 An infinite Timoshenko beam on an elastic foundation

$$m\ddot{w}(y,t) + S[\theta'(y,t) - w''(y,t)] + kw(y,t) = F(y,t) \tag{1}$$

$$EI\theta''(y,t) + S[w'(y,t) - \theta(y,t)] = mR^2\ddot{\theta}(y,t) \tag{2}$$

The beam's boundary conditions at infinity are

$$\lim_{y\to\pm\infty} w(y,t) = 0,\ \lim_{y\to\pm\infty} w'(y,t) = 0,\ \lim_{y\to\pm\infty} \theta(y,t) = 0,\ \lim_{y\to\pm\infty} \theta'(y,t) = 0 \tag{3}$$

where $i = \sqrt{-1}$, the dot and the prime over a variable denote the differentiation with respect to t and y respectively, and $F(y, t)$ is a harmonic line load that moves rightwards with a constant speed v

$$F(y,t) = f[H(y - vt - l) - H(y - vt + l)]\exp(i\omega t) \tag{4}$$

where H is the Heaviside function, f is the amplitude of the load per unit of length, l is the half length of the load, and ω is the circular frequency of the load.

We introduce a local coordinate system y_1 that attaches to the center of the load and moves with the load at the same speed v. The relationship between the two coordinate systems is

$$y_1 = y - vt \tag{5}$$

where subscript "1" indicates that the variable is in the local moving coordinate system.

In a similar way, other variables are written in the local moving coordinate system as follows (Luo and Xia 2017)

$$\begin{gathered} \eta(y,t) = \eta_1(y_1,t) \\ \eta'(y,t) = \eta_1'(y_1,t) \\ \eta''(y,t) = \eta_1''(y_1,t) \\ \dot{\eta}(y,t) = \dot{\eta}_1(y_1,t) - v\eta_1'(y_1,t) \\ \ddot{\eta}(y,t) = \ddot{\eta}_1(y_1,t) - 2v\dot{\eta}_1'(y_1,t) + v^2\eta_1''(y_1,t) \end{gathered} \tag{6}$$

where η represents w or θ. Note that the chain rule of differentiation is used in Eq. (6), and the prime now denotes the differentiation with respect to y_1.

Substituting Eq. (6) into Eqs. (1)-(2) gives the following governing equations of w_1 and θ_1 in the local moving coordinate system

$$m(\ddot{w}_1 - 2v\dot{w}_1' + v^2 w_1'') + S(\theta_1' - w_1'') + kw_1 = F_1 \exp(i\omega t) \tag{7}$$

$$EI\theta_1'' + S(w_1' - \theta_1) = mR^2(\ddot{\theta}_1 - 2v\dot{\theta}_1' + v^2\theta_1'') \tag{8}$$

where variables y_1 and t are not shown for the sake of brevity.

The solutions of Eqs. (7)-(8) can be assumed in a harmonic form with respect to y_1 and t as follows

$$\begin{aligned} w_1(y_1, t) &= \tilde{w}(\xi, \omega)\exp(i\omega t)\exp(i\xi y_1), \\ \theta_1(y_1, t) &= \tilde{\theta}(\xi, \omega)\exp(i\omega t)\exp(i\xi y_1) \end{aligned} \tag{9}$$

where ξ is the wavenumber with respect to y_1, and it is generally a complex value with its real and imaginary parts representing the wavelength and the attenuation factor of a propagating wave respectively (Chen and Huang 2000, Chen *et al.* 2001). The tilde over a variable indicates its representation in the frequency-wavenumber domain.

Substituting Eq. (9) into Eqs. (7)-(8) yields two algebraic equations in the frequency-wavenumber domain

$$m(-\omega^2\tilde{w} + 2\omega\xi v\tilde{w} - v^2\xi^2\tilde{w}) + S(i\xi\tilde{\theta} + \xi^2\tilde{w}) + k\tilde{w} = 2f\sin(l\xi)/\xi \tag{10}$$

$$-EI\xi^2\tilde{\theta} + S(i\xi\tilde{w} - \tilde{\theta}) = mR^2(-\omega^2\tilde{\theta} + 2\omega\xi v\tilde{\theta} - v^2\xi^2\tilde{\theta}) \tag{11}$$

Eliminating $\tilde{\theta}$ in Eqs. (10)-(11) yields the deflection of the beam in the frequency–wavenumber domain

$$\tilde{w} = \frac{2f\sin(l\xi)[(EI - mR^2v^2)\xi^2 + 2mR^2v\omega\xi + S - mR^2\omega^2]}{\xi(A\xi^4 + B\xi^3 + C\xi^2 + D\xi + E)} \tag{12}$$

where A, B, C, D, and E are real-valued and given by $A = (EI - mR^2v^2)(S - mv^2)$, $B = 2mv\omega[EI + R^2(S - 2mv^2)]$, $C = (S - mR^2\omega^2)(S - mv^2) + (EI - mR^2v^2)(k - m\omega^2) - S^2 + 4v^2m^2\omega^2R^2$, $D = 2mv\omega(kR^2 + S - 2mR^2\omega^2)$, and $E = (k - m\omega^2)(S - mR^2\omega^2)$. Then the deflection of the beam can be obtained from an inverse Fourier transform of Eq. (12) as

$$w_1 = \int_{-\infty}^{\infty} \frac{2f\sin(l\xi)[(EI - mR^2v^2)\xi^2 + 2mR^2v\omega\xi + S - mR^2\omega^2]}{\xi(A\xi^4 + B\xi^3 + C\xi^2 + D\xi + E)}\exp(i\xi y_1)d\xi \tag{13}$$

The denominator of Eq. (13) is known as a characteristic equation of the beam-foundation system.

2.2 Poles of the characteristic equation

The trivial pole of the characteristic equation is $\xi_0 = 0$, and the other four are solutions of the following quartic equation with real coefficients only

$$A\xi^4 + B\xi^3 + C\xi^2 + D\xi + E = 0 \tag{14}$$

All physical parameters are real and positive. The shear rigidity is assumed to be considerably large such that $S > mv^2$. Furthermore, an undamped beam cannot sustain any wave with its velocity higher than the wave velocity of the axial wave, that is, $v < v_a = \sqrt{E/\rho} = \sqrt{EI/mR^2}$. In other words, A and B are positive, and C, D, and E may be positive, zero, or negative.

Dividing Eq. (14) by A derives the following

$$\xi^4 + c_1\xi^3 + c_2\xi^2 + c_3\xi + c_4 = 0 \tag{15}$$

where $c_1 = B/A$, $c_2 = C/A$, $c_3 = D/A$, and $c_4 = E/A$. With $\xi = x - c_1/4$, Eq. (15) can be reduced to the following form

$$x^4 + px^2 + qx + r = 0 \tag{16}$$

where $p = -6(c_1/4)^2 + c_2$, $q = 8(c_1/4)^3 - c_1c_2/2 + c_3$, and $r = -3(c_1/4)^4 + (c_1/4)^2c_2 - c_1c_3/4 + c_4$. This equation is usually called a depressed equation of Eq. (15), in which the cubic item of x is eliminated.

The Ferrari algorithm is followed to solve Eq. (16), and its procedure is presented in Appendix A for the sake of completeness of the solution. Finally, the four poles of the quartic equation Eq. (14) are set as $\xi = x - c_1/4$ and are explicitly written as

$$\xi_{1,2} = \frac{-\sqrt{2s} \pm \sqrt{\Delta_1}}{2} - \frac{c_1}{4} \tag{17}$$

$$\xi_{3,4} = \frac{\sqrt{2s} \pm \sqrt{\Delta_2}}{2} - \frac{c_1}{4} \tag{18}$$

where s is given in Eq. (A.15), a real root of the resolvent cubic Eq. (A.2); $\Delta_1 = -2(p+s) + \sqrt{2}q/\sqrt{s}$ and $\Delta_2 = -2(p+s) - \sqrt{2}q/\sqrt{s}$ are given in Eqs. (A.16)-(A.17) respectively.

The characteristics of the four non-trivial poles are discussed based on the value of s. $s = 0$ is excluded according to the definition of determinants Δ_1 and Δ_2. If $s < 0$, then both determinants Δ_1 and Δ_2 are complex, and thus the four poles are distinct and complex. If $s > 0$, then the admissible poles can be categorized as follows:

(a) If $\Delta_1 < 0$ and $\Delta_2 < 0$, then all four poles are distinct and complex.

(b) If $\Delta_1 > 0$ and $\Delta_2 < 0$, or $\Delta_1 < 0$ and $\Delta_2 > 0$, then two poles are distinct and real and the other two are distinct and complex.

(c) If $\Delta_1 > 0$ and $\Delta_2 > 0$, then all four poles are distinct and real.

For the condition of $\Delta_1 = 0$ or $\Delta_2 = 0$, at least two of the four poles are real and identical. These kinds of poles are not admissible because the poles cause singularity in Cauchy's residue theorem and the deflection of the beam is infinite (Frýba 1999). The condition gives a set of equations to determine the critical velocity and resonant frequency of the beam-foundation system. However, these equations are high-order polynomial functions of v and ω and cannot be solved analytically, except for $\omega = 0$, in which the critical velocity can be given in an explicit form [Eq. (26)]. For a general case, a search algorithm is used with the following objective function

$$\frac{|\Delta_1(v,\omega)|}{|\Delta_1(0,0)|} < U \quad \text{or} \quad \frac{|\Delta_2(v,\omega)|}{|\Delta_2(0,0)|} < U \tag{19}$$

subject to $s \neq 0$, $v > 0$, and $\omega > 0$

where U is a user-defined constant tolerance (10^{-4} in the present study). At a given frequency, a range of moving speeds is searched, and the one that satisfies Eq. (19) is the critical velocity. At a given moving speed, a range of frequencies is searched and the one that satisfies Eq. (19) is the resonant frequency.

2.3 A general explicit solution to beam deflection

The identification of five poles of the characteristic equation allows the deflection of the beam in Eq. (13) to be written in the following form

$$w_1 = \int_{-\infty}^{\infty} \frac{2f\sin(l\xi)[(EI - mR^2v^2)\xi^2 + 2mR^2v\omega\xi + S - mR^2\omega^2]}{A\xi(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)}\exp(i\xi y_1)d\xi \tag{20}$$

A distinction is made between the response in front of the moving load ($y_1 \geq 0$) and the response behind the load ($y_1 < 0$). Two of four non-trivial poles contribute to the former and the other two contribute to the latter. The response in front of the load will be discussed as follows. The response behind the load can be treated similarly and is thus not given for conciseness.

Apart from the trivial pole $\xi_0 = 0$, suppose ε_1 and ε_2 are the two poles that contribute to the responses in front of the load. Eq. (20) can be evaluated in the sense of Cauchy's residue theorem

$$w_1 = \frac{i2f}{A}\{\mathrm{Res}[W(\varepsilon_1)] + \mathrm{Res}[W(\varepsilon_2)]\} \tag{21}$$

where Res[.] represents Cauchy's residue of the function (Onu 2008)

$$\mathrm{Res}[W(\varepsilon)] = \lim_{\xi\to\varepsilon}(\xi-\varepsilon)\frac{(EI - mR^2v^2)\xi^2 + 2mR^2v\omega\xi + S - mR^2\omega^2}{(\xi-\xi_0)(\xi-\xi_1)(\xi-\xi_2)(\xi-\xi_3)(\xi-\xi_4)}\sin(l\xi)\exp(i\xi y_1) \tag{22}$$

The two poles are selected based on the following criteria:

(a) a pole has a positive imagery part if the pole is complex;

(b) a pole has a larger absolute value than the other one if only two poles are real; and

(c) the largest and the smallest poles are chosen if all four poles are real.

The first criterion is interpreted as a propagating wave with attenuation in an undamped beam that must have a finite deflection; the other two describe a propagating wave with no decay in an undamped beam that must have the Doppler effect due to a moving load, in which the resulting wavelength ahead of the load is shorter than behind (Frýba 1999).

3. Closed-form solution to reduced cases

Eq. (20) is a general closed solution to a Timoshenko beam on an elastic foundation subjected to a moving harmonic line load. This problem can be reduced to seven existing cases under the conditions listed in Table 1. For these seven existing cases, no further formulation is given here to avoid repetition.

Apart from the seven existing cases, a closed-form solution has not yet reported for the case under the condition of $\omega = 0$, which is an infinite Timoshenko beam resting on an elastic subgrade subjected to a moving quasi-static line load. This implies $B = 0$, $C = k(EI - mR^2v^2) - mSv^2$, $D = 0$, and $E = Sk$ and the characteristic equation (Eq. (14)) becomes a biquadratic equation with real coefficients only

Table 2 Formula of the critical velocity of a beam on an elastic foundation subjected to a moving quasi-static load

Case No.	Beam type	Formula	Reference	Condition
1	Shear	$v_{cr} = \sqrt{\frac{-EIk + 2S\sqrt{kEI}}{mS}}$	Eq. (16) in Kim and Cho (2006) with no axial force	$R = 0$
2	Rayleigh	$v_{cr} = \sqrt{\frac{-2kR^2 + 2\sqrt{kEI + k^2R^4}}{m}}$	Eq. (12) in Kim (2005) with no axial force	$S = \infty$
3	EB	$v_{cr} = \sqrt{\frac{2\sqrt{kEI}}{m}}$	Eq (13.11) in Frýba (1999)	$S = \infty$ and $R = 0$

$$A\xi^4 + C\xi^2 + E = 0 \tag{23}$$

Let $X = \xi^2$. Eq. (23) then becomes a quadratic equation of X

$$AX^2 + CX + E = 0 \tag{24}$$

with determinant $\Delta_3 = \Delta_1 = \Delta_2 = C^2 - 4AE$.

Letting $\Delta_3 = 0$ gives rise to the critical velocity, v_{cr}, of the infinite Timoshenko beam on the elastic foundation, which is equivalent to solving another biquadratic equation with respect to v

$$Lv^4 + Mv^2 + N = 0 \tag{25}$$

where $L = m^2(S - kR^2)^2$, $M = 2km[2R^2S^2 + EI(S - kR^2)]$, and $N = EI^2k^2 - 4EIkS^2$. Solving Eq. (25) (see Appendix B) gives the critical velocity as follows

$$v_{cr} = \sqrt{\frac{-EIk(S - kR^2) - 2kR^2S^2 + 2\sqrt{kS^3[EI(S - kR^2) + kR^4S]}}{m(S - kR^2)^2}} \tag{26}$$

This expression is general for a beam on elastic foundation subjected to a moving quasi-static line load, which can be simplified to three cases under specific conditions, as summarized in Table 2.

According to the value of Δ_3, three cases will be discussed as follows:

(a) When $\Delta_3 < 0$, that is, $v < v_{cr}$, Eq. (24) has two conjugate complex roots

$$X_{1,2} = \frac{-C \pm i\sqrt{-\Delta_3}}{2A} \tag{27}$$

This equation corresponds to $\Delta_1 < 0$ and $\Delta_2 < 0$, case (a) in Section 2.2, and then all four poles of Eq. (23) are distinct and complex

$$\xi_1 = j + id, \xi_2 = -j - id, \xi_3 = j - id, \xi_4 = -j + id \tag{28}$$

where both j and d are real and positive, and given by $j = \sqrt{\frac{-C+\sqrt{C^2-\Delta_3}}{4A}}$ and $d = \sqrt{\frac{C+\sqrt{C^2-\Delta_3}}{4A}}$. These complex poles imply that the resulting wave, generated by a moving quasi-static line load at a speed lower than the critical velocity, is a propagating wave with a wavelength of $\lambda = 2\pi/j$ and decays exponentially in a rate of $\exp(-dy_1)$. The two poles that contribute to the response in front of the moving load are chosen based on criterion (a), that is, $\varepsilon_1 = \xi_1 =$

$j + id$ and $\varepsilon_2 = \xi_4 = -j + id$. Substituting these two poles and the trivial pole $\xi_0 = 0$ into Eq. (20) yields the following closed-form result

$$w_1 = \sum_{j=1}^{2} \frac{if\sin(l\varepsilon_j)\left[(EI - mR^2v^2)\varepsilon_j^2 + S\right]\exp(i\varepsilon_j y_1)}{4A(G + H_j i)}, \quad y_1 \geq 0 \text{ and } v < v_{cr} \tag{29}$$

where $G = -2j^2d^2$, $H_1 = jd(j^2 - d^2)$, and $H_2 = -jd(j^2 - d^2)$. The other two poles for the response behind the load are $\varepsilon_1 = \xi_2 = -j - id$ and $\varepsilon_2 = \xi_3 = j - id$. The resulting closed-form solution can be verified as being identical with that ahead of the load, Eq. (29). In other words, the deflection of the beam is symmetric with respect to the load center if the speed of the load does not exceed the critical velocity.

(b) When $\Delta_3 > 0$, that is, $v > v_{cr}$, Eq. (24) has two distinct real roots

$$X_{1,2} = (-C \pm \sqrt{\Delta_3})/2A \tag{30}$$

The equation corresponds to $\Delta_1 > 0$ and $\Delta_2 > 0$, case (c) in Section 2.2, in which all four poles of Eq. (23) are distinct and real

$$\xi_{1,2} = \pm\sqrt{(-C + \sqrt{\Delta_3})/2A}, \quad \xi_{3,4} = \pm\sqrt{(-C - \sqrt{\Delta_3})/2A} \tag{31}$$

These real poles indicate that the resulting wave in the undamped beam subjected to a moving load at a speed higher than the critical velocity could only be a propagating wave with no attenuation. $\varepsilon_1 = \xi_1$ and $\varepsilon_2 = \xi_2$ are chosen based on criterion (c) for the response ahead of the moving load. Substituting these two poles and the trivial pole $\xi_0 = 0$ into Eq. (20) yields the following closed-form solution

$$w_1 = \sum_{j=1}^{2} \frac{i2Af\sin(l\varepsilon_j)\left[(EI - mR^2v^2)\varepsilon_j^2 + S\right]\exp(i\varepsilon_j y_1)}{-B\sqrt{\Delta_3} + \Delta_3}, \quad y_1 \geq 0 \text{ and } v > v_{cr} \tag{32}$$

For the response behind the load, the two poles are $\varepsilon_1 = \xi_3$ and $\varepsilon_2 = \xi_4$. Substituting them and the trivial pole $\xi_0 = 0$ into Eq. (20) gives the following closed form-solution

$$w_1 = \sum_{j=1}^{2} \frac{i2Af\sin(l\varepsilon_j)\left[(EI - mR^2v^2)\varepsilon_j^2 + S\right]\exp(i\varepsilon_j y_1)}{C\sqrt{\Delta_3} + \Delta_3}, \quad y_1 < 0 \text{ and } v > v_{cr} \tag{33}$$

This solution is not the same as Eq. (32), implying that the beam's deflection is not symmetric with respect to the center of the load when the speed of the load exceeds the critical velocity.

(c) When $\Delta_3 = 0$, that is, $v = v_{cr}$, Eq. (24) has two equal roots $X_{1,2} = -C/2A$. This leads to an infinite deflection (Frýba 1999), which is impractical.

Table 3 summarizes the closed-form solution to a Timoshenko beam on an elastic foundation when it is subjected to a moving quasi-static line load.

4. Verification

This section verifies the present solution through two examples. One is to reduce a classical closed-form solution of an EB beam on a Winkler foundation, and the other is to compare results of the present solution and the existing numerical example of a Timoshenko beam subjected to a moving axle load.

Table 3 Closed-form solution of an infinite Timoshenko beam on an elastic subgrade subjected to a moving quasi-static line load

Case	Condition	$y_1 \geq 0$	$y_1 < 0$
1	$v < v_{cr}$	Eq. (29)	Eq. (29)
2	$v > v_{cr}$	Eq. (32)	Eq. (33)

4.1 Example 1: An EB beam on Winkler foundation subjected to moving point load

This section demonstrates a reduction of the closed-form solution of Section 3 into a classical closed-form solution of an EB beam on a Winkler foundation, which has been well documented in Frýba (1999) and Mathews (1958) and commonly used as a benchmark study (Sun 2001, Luo *et al*. 2015, Luo and Xia 2017, Onu 2008). The reduction is realized by setting the shear rigidity to infinity and the radius of gyration to zero. In this regard, Eq. (23) is reduced to the characteristic equation of the reduced system

$$EI\xi^4 - mv^2\xi^2 + k = 0 \tag{34}$$

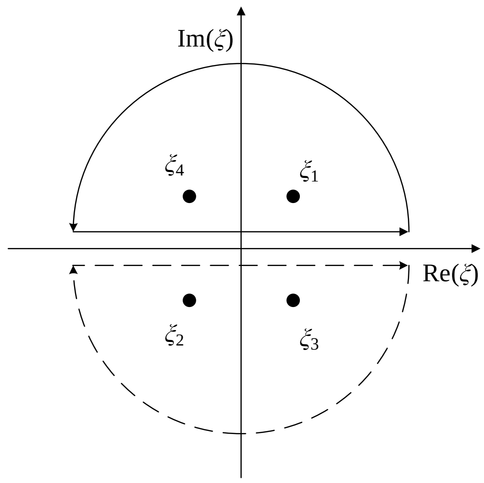
with a determinant of $m^2v^4 - 4kEI$. Setting this determinant equal to zero yields the critical velocity of the reduced system $v_{cr} = \sqrt[4]{4EIk/m^2}$, which is exactly the case (3) in Table 2.

Define a dimensionless speed of the load$\alpha = v/v_{cr}$, a static deflection $\lambda = \sqrt[4]{k/4EI}$, $a = \sqrt{1+\alpha^2}$, and $b = \sqrt{1-\alpha^2}$. In the case of $\alpha<1$, $j = \lambda a$ and $d = \lambda b$, leading to a dimensionless form of four non-zero poles $\xi_{1,3}/\lambda = a \pm ib$ and $\xi_{2,4}/\lambda = -a \mp ib$ according to Eq. (28). These expressions are consistent with Eqs. (13.18)-(13.25) in Frýba (1999) and are shown in Fig. 2(a). Substituting these poles into Eq. (29) yields

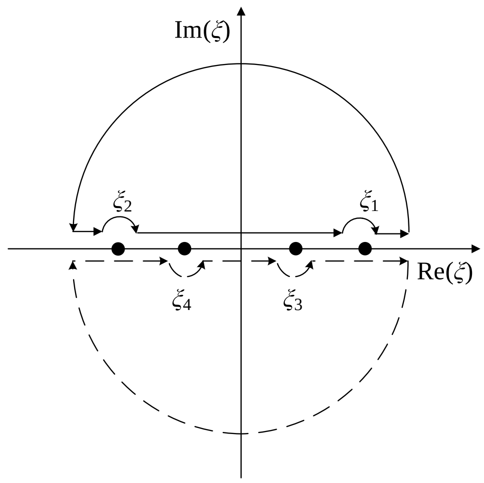
$$w_1 = \frac{\lambda\exp(-b\lambda|y_1|)}{2kab}\left[a\cos(a\lambda y_1) + b\sin(a\lambda|y_1|)\right], \alpha < 1 \tag{35}$$

This equation is the same as the available closed-form solution (see Eq. (13.54) in Frýba (1999)). Note that the terms $2f\sin(l\xi)/\xi$ and the trivial pole ξ_0 have been dropped in the general explicit solution (Eq. (20)) because a unit point load instead of a distributed line one is under consideration.

In the case of $\alpha>1$, the four non-zero poles in the dimensionless form are $\xi_{1,2}/\lambda = \pm a$ and $\xi_{3,4}/\lambda = \pm b$ according to a square root of Eq. (30). Again, these poles are identical with Eqs. (13.18)-(13.27) in Frýba (1999) and are shown in Fig. 2b. Substituting the first two poles into Eq. (32) and the latter two into Eq. (33) yields the response of the beam ahead and behind the load respectively



(a) $v < v_c$



(b) $v > v_{cr}$

Fig. 2 Poles of the characteristic equation (ξ_1 to ξ_4) of an infinite EB beam on a Winkler foundation

$$w_1 = \frac{-2\lambda\sin[(a-b)\lambda y_1]}{2k(a-b)\sqrt{\alpha^4-1}}, \alpha > 1 \text{ and} y_1 > 0 \quad (36)$$

$$w_1 = \frac{-2\lambda\sin[(a+b)\lambda y_1]}{2k(a+b)\sqrt{\alpha^4-1}}, \alpha > 1 \text{ and} y_1 < 0 \quad (37)$$

Again, these two solutions are consistent with Eq. (13.55) in Frýba (1999).

4.2 Example 2: A Timoshenko beam subjected to moving axle load

In the second example, the present explicit solution is compared with the existing numerical solution of a Timoshenko beam on an elastic foundation subjected to a moving axle load, which has been studied in Kim and Cho (2006) and Kim (2005). The system is a concrete pavement with the following mechanical properties: EI = 2.3 kNm^2, m = 48.2 kg/m, k = 68.9 MPa, R = 0.1 m, and S = 20 MN. The axle load is a uniformly distributed line load of $2l$ = 0.1524 m and f = 262.5 kN/m, representing a design equivalent

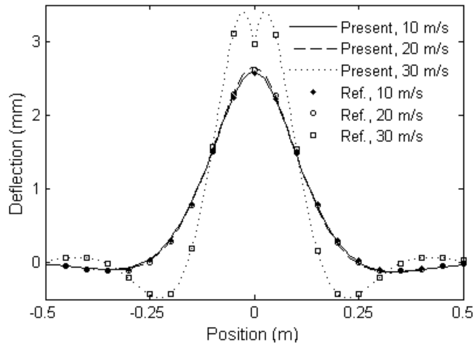


Fig. 3 Deflected shape of a Timoshenko beam on an elastic foundation subjected to a moving axle load of 2 Hz

axle load of 40 kN in a contact area of a typical tyre. Two load frequencies, 2 and 10 Hz, are considered, which correspond to the vehicle bounce frequency and the wheel hop frequency respectively (Hao and Ang 1998).

At a given moving load of 2 Hz, the critical velocity is identified by Eq. (19) for a range of load speeds from 1 m/s to 100 m/s with a step of 0.01 m/s. Two critical velocities, v_{cr1} = 66.04 m/s and v_{cr2} = 67.02 m/s, are found under the conditions of $\Delta_1 \rightarrow 0$ and $\Delta_2 \rightarrow 0$ respectively.

Assume the load moves at a speed of 10 m/s. In the case, Δ_1 = – 350.54 and Δ_2 = – 351.73, resulting in four distinct and complex poles $\xi_{1,2}$ = – 9.35 ± 9.36i and $\xi_{3,4}$ = 9.32 ± 9.38i according to Eqs. (17)-(18). The first and third poles are selected to compute the response in front of the moving load according to criterion (a); the other two poles are for the response behind the load. These poles are substituted into Eq. (21), and the deflection of the beam is obtained and shown in Fig. 3(a). The deflection pattern is similar to that under a non-moving harmonic load, that is, it is the maximum at the center of the load and symmetric with respect to the center. The result is compared with a reference computed by the exponential window method proposed for computing the forced responses of undamped structures (Kausel and Roësset 1992). Both results agree very well. The agreement is also observed for other load speeds lower than the first critical velocity. As the load speed increases, the deflection pattern is wider with a larger fluctuation. Meanwhile, the maximum deflection increases and not necessarily locates at the center of the load.

The load speed higher than the first critical velocity is impractical in the present example, but it would be interesting from a theoretical point of view because it allows us to see how the deflection of the beam will become for such a high load speed. As an example of v = 66.5 m/s, Δ_1 = 141.15 and Δ_2 = – 140.34, thus yielding two distinct and real poles ξ_1 = – 32.48 and ξ_2 = – 20.60 and two distinct and complex poles $\xi_{3,4}$ = 24.14 ± 5.92i according to Eqs. (17)-(18). The two real poles represent a propagating wave along the beam with no attenuation, shown as "Wave 1" in Fig. 4. The wave ahead of the load has a shorter wavelength $\lambda_1 = 2\pi/|\xi_1|$ = 0.19 m and a higher amplitude than behind ($\lambda_2 = 2\pi/|\xi_2|$ = 0.31 m). The two complex poles represent the other propagating wave with a wavelength $\lambda_2 = 2\pi/24.14$ = 0.26 m and exponential attenuation of exp(– 5.92 y_1). The shape of the wave is shown as "Wave 2" in

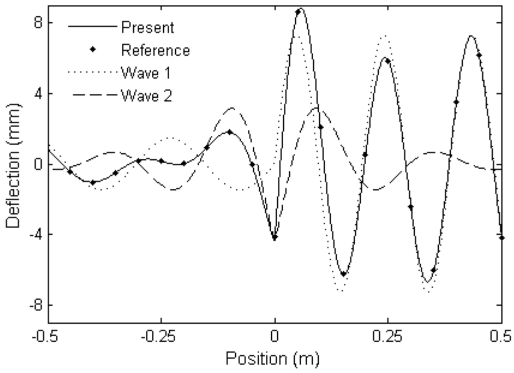


Fig. 4 Deflected shape of a Timoshenko beam on an elastic foundation subjected to a moving axle load of 2 Hz at a speed of 66.5 m/s

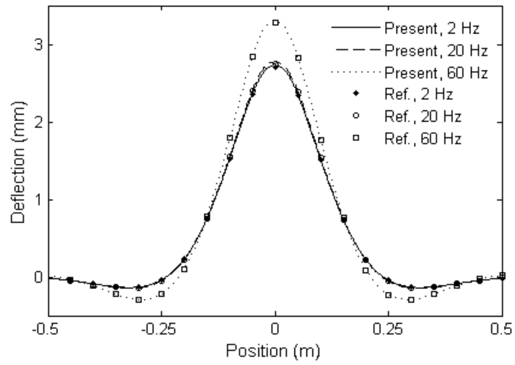
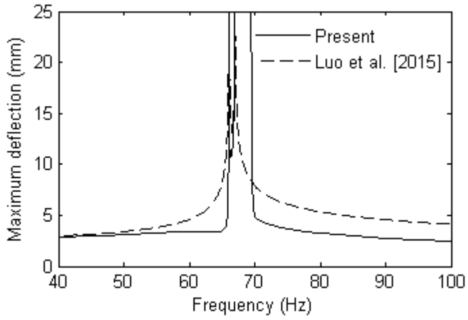
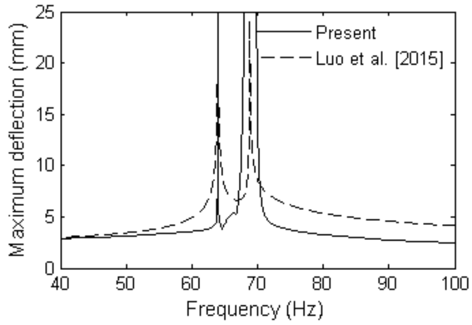


Fig. 6 Deflected shape of a Timoshenko beam on an elastic foundation subjected to a moving axle load at a speed of 30 m/s

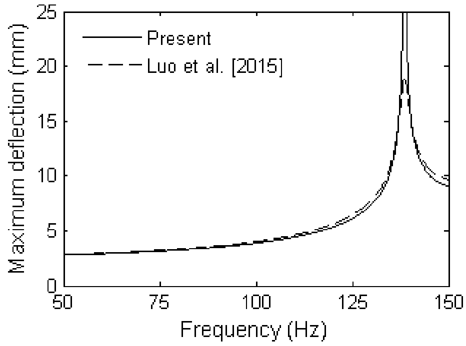


(a) $\omega = 4\pi$ Hz

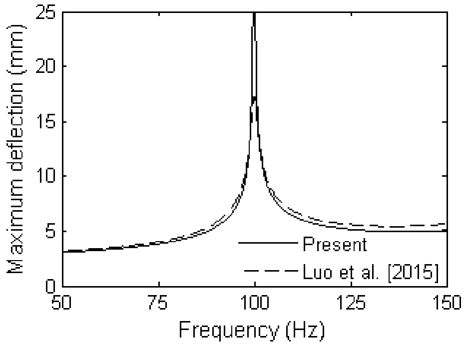


(b) $\omega = 20\pi$ Hz

Fig. 5 Maximum deflection versus load speed



(a) $v = 10$ m/s



(b) $v = 30$ m/s

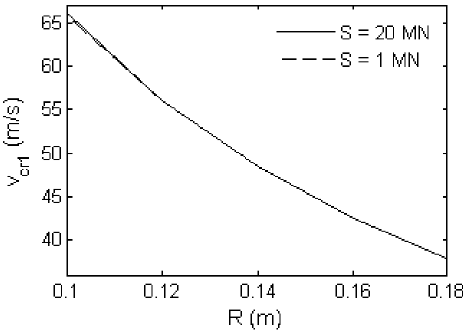
Fig. 7 Maximum deflection versus load frequency

Fig. 4. The wave is symmetric with respect to the center of the load and has the maximum deflection at the center. The combination of these two propagating waves gives rise to the deflection of the beam in Fig. 4. It is no longer symmetric, and it tends to coincide with the shape of "Wave 1" as the distance is further away from the load center. A reference result is also superimposed in the figure, agreeing very well with the present one.

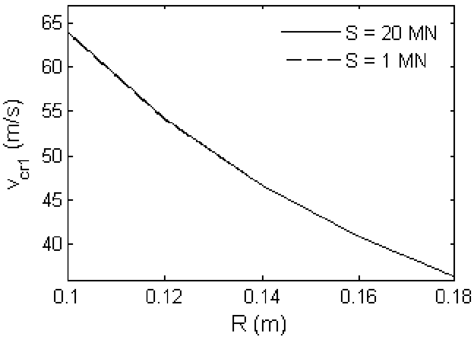
The maximum deflections due to a moving axle load of 2 Hz are shown in Fig. 5(a) for a range of load speeds of 40-100 m/s. As the load speed increases, the maximum deflection increases and reaches infinity at the first critical velocity; it then drops down sharply and increases again to infinity at the second critical velocity. A reference maximum reflection in Luo *et al*. (2015) is also superimposed, which is obtained through an evaluation of Eq. (20) by a FFT with hysteretic damping of 1%. Given that the damping is included, the peaks of the reference maximum deflection are finite in amplitude and are less sharp. From both curves of the maximum deflection, the first critical velocity can be picked and is almost the same as the identified one ($v_{cr1} = 66.04$ m/s).

The maximum deflections for the load at a frequency of 10 Hz are shown in Fig. 5(b). As a reference, the two critical velocities obtained by Eq. (19) are 63.91 and 68.81 m/s for a range of load speeds of 1-100 m/s with a step of 0.001 m/s, corresponding to $\Delta_1 \rightarrow 0$ and $\Delta_2 \rightarrow 0$ respectively. The same critical velocities can be found in both curves of the maximum deflections. The first critical velocity is lower than that for the load of 2 Hz, but the second one is higher.

Two load speeds, 10 and 30 m/s, are considered, which represent the low and high speeds of a vehicle respectively.

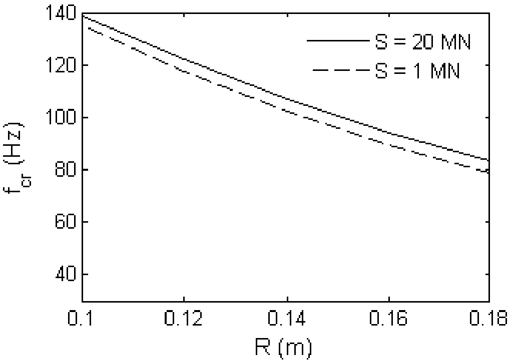


(a) $\omega = 4\pi$ Hz

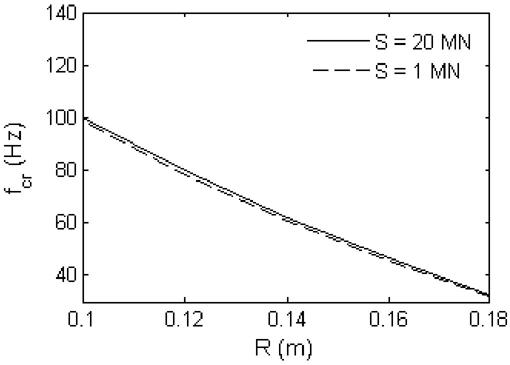


(b) $\omega = 20\pi$ Hz

Fig. 8 Effects of the radius of gyration and the shear rigidity on the first critical velocity



(a) $v = 10$ m/s



(b) $v = 30$ m/s

Fig. 9 Effects of the radius of gyration and the shear rigidity on the resonant frequency

The corresponding resonant frequencies are identified by Eq. (19) as 120.52 and 99.96 Hz for a range of frequencies 1-150 Hz with a step of 0.001 Hz. Both appear under a condition of $\Delta_1 \to 0$. In this particular example, the resonant frequency is rather high, while the normal axle load of a vehicle will never reach the same degree of frequency, thus only the deflection at a frequency lower than the resonant frequency is investigated. Fig. 6 shows the deflection of the beam due to the load with a speed of 30 m/s for three representative frequencies. The deflection is symmetric with respect to the load center and reaches the maximum at the center. As the load frequency increases, the maximum deflection is higher and the deflection is more fluctuated. In the three cases, the agreement between the present explicit solution and the reference one is excellent. Similar conclusions can be drawn from the cases with a load speed of 10 m/s; thus, these cases are not given for conciseness.

The maximum deflections due to the two load speeds are shown in Fig. 7 for a range of frequencies 50–150 Hz. As the load frequency increases, the maximum deflection increases steadily at first, reaches infinity at the resonant frequency, and then decreases sharply. The resonant frequency picked from the maximum deflection curves is the same as that computed by Eq. (19), and its value is lower as the load speed increases.

The first critical velocity is investigated in Fig. 8 for a range of radii of gyration (0.1-0.18 m) and two shear rigidity values (20 and 1 MN). The critical velocity decreases almost linearly with the increasing radius of gyration and is insensitive to the variation of shear rigidity regardless of the load frequency. It decreases slightly from the load of 2 Hz to that of 10 Hz. These observations are consistent with Luo *et al*. (2015).

The resonant frequency at a given load speed is also discussed in Fig. 9 for various radii of gyration and shear rigidity. A linear decrease in the resonant frequency is observed with the increasing radius of gyration. Meanwhile, a constant value of decrease is found as the shear rigidity decreases and the value is narrowed down as the load speed is higher. A large drop is observed in the resonant frequency from the load speed of 10 m/s to that of 30 m/s. Similar observations have been found in Luo *et al*. (2015).

5. Conclusions

In this paper, a general closed-form solution to a Timoshenko beam on elastic foundation is formulated when it is subjected to a moving harmonic line load. The formulation is based on the feature that the characteristic equation of the problem is a quartic equation with real coefficients only. The poles of the equation can be written in an explicit form. With these poles, the general closed-form solution of the beam's deflection is finally deduced by applying Cauchy's residue theorem. The solution can be reduced to seven existing closed-form solutions under specific conditions and to a new closed-form solution to a Timoshenko beam on an elastic foundation subjected to a moving quasi-static line. The solution is consistent with the classical closed-form solution to an EB beam on an elastic foundation under a point moving load, and is verified by a comparison with the existing numerical solutions.

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Appendix A

This section solves Eq. (16) by the classic Ferrari algorithm.

We introduce an arbitrary variable s and regroup Eq. (16) as a perfect square in the left-hand side

$$\left(x^2+\frac{p}{2}+s\right)^2=2sx^2-qx+s^2+sp+\frac{p^2}{4}-r \quad (A.1)$$

The value of s can be chosen, such that the right-hand side of Eq. (17) is a perfect square. This implies that the determinant in x of this quadratic equation is zero, that is, s is a root of the equation

$$s^3+ps^2+(p^2/4-r)s-q^2/8=0 \quad (A.2)$$

This is the resolvent cubic of Eq. (16). According to the intermediate value theorem, Eq. (18) at least has a real root (Irving 2003). With the real root, Eq. (17) becomes

$$\left(x^2+\frac{p}{2}+s\right)^2=\left(\sqrt{2s}x-\frac{q}{2\sqrt{2s}}\right)^2 \quad (A.3)$$

This results in two quadratic equations

$$x^2+\sqrt{2s}x+\frac{p}{2}+s-\frac{q}{2\sqrt{2s}}=0 \quad (A.4)$$

$$x^2-\sqrt{2s}x+\frac{p}{2}+s+\frac{q}{2\sqrt{2s}}=0 \quad (A.5)$$

Thus far, solving the quartic equation Eq. (15) is equivalent to seeking a real root of the resolvent cubic equation Eq. (A.2) and then solving two quadratic equations Eqs. (A.4)-(A.5).

In the first step, a depressed equation of Eq. (A.2) is obtained by a change of variable $s=z-p/3$

$$z^3+gz+h=0 \quad (A.6)$$

where $g=-r-p^2/12$, and $h=-p^3/108+pr/3-q^2/8$. Let $z=u+v$, then Eq. (A.6) becomes

$$u^3+v^3+h+(3uv+g)(u+v)=0 \quad (A.7)$$

As one more variable is introduced, another condition $3uv+g=0$ is imposed. This gives the following

$$u^3v^3=-\frac{g^3}{27} \quad (A.8)$$

Then Eq. (A.7) can be rewritten as follows

$$u^3+v^3=-h \quad (A.9)$$

The combination of Eqs. (A.8)-(A.9) leads to a quadratic equation of Z with its roots as u^3 and v^3

$$Z^2+hZ-\frac{g^3}{27}=0 \quad (A.10)$$

The determinant of the above equation is $\Delta=\frac{h^2}{4}+\frac{g^3}{27}$. Solving this quadratic equation results in

$$u^3=-\frac{h}{2}+\sqrt{\Delta},\ v^3=-\frac{h}{2}-\sqrt{\Delta} \quad (A.11)$$

A cubic root of Eq. (A.11) gives three roots of Eq. (A.6) (Irving 2003)

$$z_1=u_0+v_0 \quad (A.12)$$

$$z_2=-\frac{1}{2}(u_0+v_0)+\frac{i\sqrt{3}}{2}(u_0-v_0) \quad (A.13)$$

$$z_3=-\frac{1}{2}(u_0+v_0)-\frac{i\sqrt{3}}{2}(u_0-v_0) \quad (A.14)$$

where $u_0=\sqrt[3]{-\frac{h}{2}+\sqrt{\Delta}}$ and $v_0=\sqrt[3]{-\frac{h}{2}-\sqrt{\Delta}}$.

The characteristics of the three roots are determined by the value of Δ.

(a) If $\Delta\geq 0$, then all three roots are real numbers. Without loss of generality, we may choose z_1 as an input real root for Eqs. (A.4)-(A.5).

(b) If $\Delta<0$, then z_1 is a real number because $u_0=\sqrt[3]{-\frac{h}{2}+i\sqrt{-\Delta}}$ and $v_0=\sqrt[3]{-\frac{h}{2}-i\sqrt{-\Delta}}$ are a complex conjugate pair; the other two are complex conjugate roots.

Thus, the real root of Eq. (A.2) is $s=z_1-p/3$ and is explicitly expressed as

$$s=\sqrt[3]{-\frac{h}{2}+\sqrt{\frac{h^2}{4}+\frac{g^3}{27}}}+\sqrt[3]{-\frac{h}{2}-\sqrt{\frac{h^2}{4}+\frac{g^3}{27}}}-\frac{p}{3} \quad (A.15)$$

With this root, we can then start to solve Eqs. (A.4)-(A.5). The two determinants are defined as follows

$$\Delta_1=-2(p+s)+\frac{\sqrt{2}q}{\sqrt{s}} \quad (A.16)$$

$$\Delta_2=-2(p+s)-\frac{\sqrt{2}q}{\sqrt{s}} \quad (A.17)$$

The corresponding two roots of the quadratic equation are

$$x_{1,2}=\frac{-\sqrt{2s}\pm\sqrt{\Delta_1}}{2} \quad (A.18)$$

$$x_{3,4}=\frac{\sqrt{2s}\pm\sqrt{\Delta_2}}{2} \quad (A.19)$$

Appendix B

This section demonstrates the procedure to obtain the critical velocity of a system with an infinite Timoshenko beam on an elastic foundation subjected to a moving quasi-static line load.

Let $Y = v^2$. Thus, Eq. (25) becomes a quadratic equation of Y

$$LY^2 + MY + N = 0 \tag{B.1}$$

The determinant Δ_4 is

$$\Delta_4 = 16km^2S^3[kR^4S + EI(S - kR^2)] \tag{B.2}$$

As the shear rigidity is assumed to be considerably large, the determinant is positive. Thus, Eq. (B.1) has two distinct real roots

$$Y_{1,2} = \frac{-EIk(S - kR^2) - 2kR^2S^2 \pm 2\sqrt{kS^3[EI(S - kR^2) + kR^4S]}}{m(S - kR^2)^2} \tag{B.3}$$

As

$$Y_1Y_2 = \frac{EIk(EIk - 4S^2)}{m^2(S - kR^2)^2} < 0 \tag{B.4}$$

One real root is positive and the other is negative. Letting $Y_1 > 0$ and $Y_2 < 0$, the solutions of Eq. (25) are

$$v_{1,2} = \pm\sqrt{Y_1}, v_{3,4} = \pm i\sqrt{-Y_2} \tag{B.5}$$

Given that a real-valued root is of interest, v_1 is the critical velocity of the present beam-foundation system and its explicit form is given in Eq. (26).