Structural Engineering and Mechanics, *Vol. 65, No. 4 (2018) 415-422*
DOI: https://doi.org/10.12989/sem.2018.65.4.415

Conjugate finite-step length method for efficient and robust structural reliability analysis

Behrooz Keshtegar*

Department of Civil Engineering, Faculty of Engineering, University of Zabol, P.B. 9861335856 Zabol, Iran

(Received August 23, 2016, Revised November 28, 2017, Accepted November 29, 2017)

Abstract. The Conjugate Finite-Step Length" (CFSL) algorithm is proposed to improve the efficiency and robustness of first order reliability method (FORM) for reliability analysis of highly nonlinear problems. The conjugate FORM-based CFSL is formulated using the adaptive conjugate search direction based on the finite-step size with simple adjusting condition, gradient vector of performance function and previous iterative results including the conjugate gradient vector and converged point. The efficiency and robustness of the CFSL algorithm are compared through several nonlinear mathematical and structural/mechanical examples with the HL-RF and "Finite-Step-Length" (FSL) algorithms. Numerical results illustrated that the CFSL algorithm performs better than the HL-RF for both robust and efficient results while the CFLS is as robust as the FSL for structural reliability analysis but is more efficient.

Keywords: reliability analysis; conjugate search direction; conjugate finite-step length; failure probability

1. Introduction

Engineering problems are included various uncertainties, which may be encountered with them in the design and implementation phases of structural system such as the material and geometric properties, and external loads. The probabilistic models can be used to consider these uncertainties (Keshtegar and Miri 2013) using the reliability analysis based on the analytical or simulation approaches (Keshtegar and Chakraborty 2018), such as the Mean-Value First-Order Second Moment (MVFOSM) (Liu and Peng 2012), the First-Order Reliability Method (FORM) (Santos *et al.* 2012, Keshtegar and Miri 2014), Second-Order Reliability Method (SORM) (Kiureghian and Stefano 1991), and Monte Carlo Simulation (MCS) (Keshtegar and Kisi 2017). The main effort of the structural reliability analysis is estimated the failure probability (P_f) by a multi-dimensional integral as (Keshtegar 2017a)

$$P_f\left[g(\boldsymbol{X})\le 0\right]= \int\limits_{g(\boldsymbol{X})\le 0}\!\!\dots\!\int f_{\boldsymbol{X}}(x_1,x_2,\dots x_n)d\boldsymbol{X} \tag{1}$$

where, $g(\ )$ is the Limit State Function (LSF), which separates the design domain into failure ($g(\boldsymbol{X})<0$) and safe ($g(\boldsymbol{X})>0$) regions with respect to basic random variables $\boldsymbol{X}=(x_1,x_2,\dots,x_n)^T$. $f_{\boldsymbol{X}}$ is the joint Probability Density Function (PDF) of random variables $\boldsymbol{X}$. A closed form solution of the above integral is not available for general cases of nonlinear LSF with many random variables (Santosh *et al.* 2006). The MCS is generally provided accurate solution for integral in Eq. (1) when a sufficient number of simulations are used. However, high-computational effort is often required based on the MCS for complex engineering problems due to the expensive computations of LSF (Liu and Peng 2012, Meng *et al.* 2017, Periçaro *et al.* 2015). FORM (Koduru and Haukaas 2010, Liu and Der Kiureghian 1991) and SORM (Kiureghian and Stefano 1991) are the analytical methods to estimate the failure probability. The FORM method is typically used for the good balance between accuracy and efficiency for the reliability assessment (Xiao *et al.* 2011, Keshtegar and Meng 2017). In FORM, structural failure probability is estimated based on the reliability index (β) by linearizing the LSF on the failure surface at the "most probable point" (MPP) i.e., $\boldsymbol{U}^*$ i.e., $P_f\approx\Phi(-\beta)$) (Santosh *et al.* 2006). The MPP search is computed by the following constrained optimization problem (Keshtegar 2017a)

$$\begin{aligned}&Minimize\ \beta=(\boldsymbol{U}^T\boldsymbol{U})^{1/2}\\&subjected\ \ to\ g(\boldsymbol{U})=0\end{aligned} \tag{2}$$

Generally, the main goal of FORM is the MPP search ($\beta=\|\boldsymbol{U}^*\|$). The optimization schemes such as the gradient projection, the augmented Lagrangian and the sequential quadratic programming method were applied to search the MPP by Liu and Kiureghian (1991). Hasofer and Lind (1974) proposed a general iterative method for computing the reliability index. Later, Rackwitz and Fiessler (1987) extended the Hasofer and Lind iterative scheme using distribution information of the random variables (called as HL-RF method). Unlike the optimization methods (Liu and Kiureghian 1991), the HL-RF algorithm is widely utilized for structural reliability analysis (Gong and Yi 2011, Keshtegar 2017b). The iterative HL-RF scheme may lead to unstable results as periodic and chaotic solutions (Yang

*Corresponding author, Ph.D.
E-mail: bkeshtegar@uoz.ac.ir

ISSN: 1225-4568 (Print), 1598-6217 (Online)

2010, Meng *et al.* 2017, Keshtegar 2016a) or may converge very slowly in highly nonlinear problems with concave limit state function (Wang and Grandhi 1996, Keshtegar 2017c). There were suggested several modifications of FORM formula to improve the numerical instability and efficiency of the HL-RF method. Liu and Kiureghian (1991) introduced a merit function to improve the convergence performance of the HL-RF method. Santosh *et al.* (2006), Santos *et al.* (2012) and Periçaro *et al.* (2015) improved the HL-RF method by selecting a step size based on Armijo rule and Wolfe conditions. The BFGS approach was used to extract the search direction vector for reliability analysis-based FORM by Periçaro *et al.* (2015). Wang and Grandhi (1994 and 1996) enhanced the HL-RF method using the intervening variables and considering the adaptive nonlinear two-point approximation. Yang (2010) applied the stability transformation method (STM) using chaos feedback control to improve the robustness of FORM formula. Gong and Yi (2011) a simple iterative algorithm proposed based on finite-step length (FSL). The FSL and STM algorithms are more robust than the HL-RF method. However, the small control factor in the STM and large step length in the FSL may be required more iterative numbers to obtain the stabilization in highly nonlinear LSFs. Recently, Meng *et al.* (2017) improved the STM using the directional formulation to obtain the search direction vector. The relaxed HL-RF method was proposed based an adaptive relaxed factor between 0 and 1 by Keshtegar and Meng (2017). Keshtegar and Meng (2017) showed that the directional STM may produce the unstable results or may slowly converge for some applicable engineering problems. Keshtegar and Miri (2014) introduced the conjugate gradient optimization method to improve the robustness of HL-RF with Wolfe conditions. The improved HL-RF methods which developed by Keshtegar and Miri (2014), Santos *et al.* (2012) and Periçaro *et al.* (2015) are more robust than the HL-RF but, are computational extensive approaches to determine the step size using Wolfe conditions. Recently, the FORM-based conjugate search direction was developed to search MPP using the Armijo rules (Keshtegar 2016a, b, Keshtegar 2017a) and sufficient descent condition (Keshtegar 2017b, c, Keshtegar and Chakraborty 2018) to improve the convergence properties of FORM. The hybrid conjugate FORM (Keshtegar 2017a), adaptive conjugate search direction (Keshtegar and Chakraborty 2018), relaxed-finite step size using conjugate search direction (Keshtegar and Bagheri 2017), limited conjugate search direction (Keshtegar 2017b, c) and chaotic conjugate search direction (Keshtegar 2016b) were formulated using the FR conjugate scalar factor. Generally, the Armijo rule was applied to compute the finite-step size that the efficiency of the FORM is controlled using the conjugate search direction based on sufficient decent condition in limited (Keshtegar 2017b, c), and hybrid adaptive conjugate FORM (Keshtegar and Chakraborty 2018). The chaotic conjugate search direction using STM with complex formulations was proposed to control the instability of FORM using FR optimization approach (Keshtegar 2016a). The results by Keshtegar (2017a, b, c) indicated that the conjugate search direction can be used to improve the robustness and efficiency of FORM formula. However, the complex conjugate FORM formula with an adaptive finite-step size which is determine by Armijo rule, should be applied with huge computational burden to search the MPP of the structural problem.

In this paper, an iterative conjugate FORM algorithm called conjugate finite-step length (CFSL) is proposed to evaluate the failure probability without Armijo rule using the finite-step size which is simply adjusted using the descent condition. The CFSL is quickly converged with global convergence property in the nonlinear structural reliability problems. The results illustrate that the CFSL method is more efficient than the FSL and is more robust than the HL-RF.

This paper is organized as follows: The FORM is defined in Sec. 2. The proposed method is discussed in Sec. 3. Five nonlinear mathematical and structural/ mechanical examples are given in Sec. 4 to illustrate the performances of the CFSL method. Then the conclusions are provided in Sec. 5.

2. First-order reliability method

The basic process of FORM is the MPP search to compute the reliability index $\beta = \|\boldsymbol{U}^*\|$ (Keshtegar 2016a). Generally, the probability of failure ($P_f \approx \Phi(-\beta)$) is approximated using β by the three following steps;

Step 1: Transform the basic random variables form ***X***-space (***X*** is the original random variable vector) to ***U***-space (***U*** is standard normal vector with zero means, unit variance and independent components) by Rosenblatt transformation i.e., $u = \Phi^{-1}\{F_X(x)\}$ as follows

$$u = \frac{x - \mu_x^e}{\sigma_x^e} \tag{3}$$

where, μ_x^e and σ_x^e are the equivalent mean and the standard deviation of the basic random variable x, respectively. The μ_x^e and σ_x^e of non-normal random variables are determined as follows (Santosh *et al.* 2006, Keshtegar 2017b)

$$\sigma_x^e = \frac{1}{f_X(x)}\phi[\Phi^{-1}\{F_X(x)\}] \tag{4}$$

$$\mu_x^e = x - \sigma_x^e \Phi^{-1}[F_X(x)] \tag{5}$$

where $f_X(x)$ and $F_X(x)$ are the PDF and Cumulative Distribution Function (CDF) of variable x, respectively. Φ^{-1} is the inverse standard normal CDF and ϕ is the standard normal PDF.

Step 2: Search the MPP ($U^* = (u_1^*, u_2^*, ..u_n^*)^T$) using an iterative process as:

2.1 The HL-RF method

The HL-RF iterative formula is written as (Gong and Yi 2011, Keshtegar 2017a)

$$\boldsymbol{U}_{k+1}=\frac{\nabla^T g(\boldsymbol{U}_k)\boldsymbol{U}_k-g(\boldsymbol{U}_k)}{\nabla^T g(\boldsymbol{U}_k)\boldsymbol{\alpha}_{k+1}}\boldsymbol{\alpha}_{k+1} \tag{6}$$

where, $\boldsymbol{\alpha}_{k+1}$ is negative unit normal vector at point $\boldsymbol{U}_k$, which is computed as

$$\boldsymbol{\alpha}_{k+1}=-\frac{\nabla^T g(\boldsymbol{U}_k)}{\left\|\nabla^T g(\boldsymbol{U}_k)\right\|} \tag{7}$$

where, $\nabla g(\boldsymbol{U})=[\partial g/\partial u_1,\partial g/\partial u_2,\ldots,\partial g/\partial u_n]^T$ is the gradient vector of the LSF. The HL-RF method may fail to converge with the oscillating points for highly nonlinear LSF.

2.2 Finite-step length algorithm

Gong and Yi (2011) proposed a robust iterative algorithm using finite-step length based on the improved negative unit normal vector in Eq. (6) as follows

$$\boldsymbol{\alpha}_{k+1}^{\lambda}=\frac{\boldsymbol{U}_{k+1}^{\lambda}}{\left\|\boldsymbol{U}_{k+1}^{\lambda}\right\|} \tag{8}$$

in which, point $\boldsymbol{U}_{k+1}^{\lambda}$ is point whose is calculated as

$$\boldsymbol{U}_{k+1}^{\lambda}=\boldsymbol{U}_k-\lambda\nabla g(\boldsymbol{U}_k) \tag{9}$$

where, $\lambda>0$ is the finite-step length. It can be concluded from Eq. (9) that $\boldsymbol{U}_{k+1}^{\lambda}=\boldsymbol{U}_k$ when the step length is given as $\lambda=0$ thus $\boldsymbol{\alpha}_{k+1}^{\lambda}=\boldsymbol{\alpha}_k^{\lambda}$. This means that point $\boldsymbol{U}_{k+1}$ is a fixed point. Thus, if $\left\|\boldsymbol{U}_{k+2}-\boldsymbol{U}_{k+1}\right\|>\left\|\boldsymbol{U}_{k+1}-\boldsymbol{U}_k\right\|$, set $\lambda=\lambda/c$ where, 1.2≤*c*≤1.5 is the adjusting coefficient.

Step 3: Estimate the probability of failure.

Based on MPP i.e., $\boldsymbol{U}^*$, it is approximated the failure probability as $P_f\approx\Phi(-\left\|\boldsymbol{U}^*\right\|)$.

3. Conjugate finite-step length algorithm

Concavity degree of the LSF is an essential factor for convergence performances including robustness and efficiency of the HL-RF and FSL algorithms. The HL-RF and FSL methods may be inefficiently converged for highly nonlinear performance function. In order to reduce the parallel risk of the unit normal vector $\boldsymbol{\alpha}^{\lambda}$ with the normalized search direction vector $\boldsymbol{\alpha}_k$, a conjugate unit normal vector is proposed based on as follows

$$\boldsymbol{U}_{k+1}^{C\lambda}=\boldsymbol{U}_k+\lambda_k\boldsymbol{d}_k \tag{10}$$

where, point $\boldsymbol{U}_{k+1}^{C\lambda}$ is along the conjugate search direction at point $\boldsymbol{U}_k$ and $\boldsymbol{d}_k$ is the conjugate gradient vector which is computed as follows

$$\boldsymbol{d}_k=-\nabla g(\boldsymbol{U}_k)+\frac{\left\|\nabla g(\boldsymbol{U}_k)\right\|^2-\nabla^T g(\boldsymbol{U}_k)\nabla g(\boldsymbol{U}_{k-1})}{\left\|\boldsymbol{d}_{k-1}\right\|^2}\boldsymbol{d}_{k-1} \tag{11}$$

The above conjugate gradient vector is formulated using the conjugate scalar factor of $\dfrac{\left\|\nabla g(\boldsymbol{U}_k)\right\|^2-\nabla^T g(\boldsymbol{U}_k)\nabla g(\boldsymbol{U}_{k-1})}{\left\|\boldsymbol{d}_{k-1}\right\|^2}$ in which the previous conjugate gradient i.e., $\boldsymbol{d}_{k-1}$, new i.e., $\nabla g(\boldsymbol{U}_k)$ and previous gradient i.e., $\nabla g(\boldsymbol{U}_{k-1})$ vectors are applied to adjust the new conjugate gradient vector i.e. $\boldsymbol{d}_k$. However, the conjugate scalar factors of the existing conjugate FORM formula including chaotic conjugate STM (Keshtegar 2016a), chaotic conjugate chaos control (Keshtegar 2016b), enriched FR (Keshtegar 2017b), limited FR (Keshtegar 2017c), relaxed-finite conjugate method (Keshtegar and Bagheri 2017) and hybrid adaptive conjugate method (Keshtegar and Chakraborty 2018) are extracted based on new and previous gradient vectors as $\dfrac{\left\|\nabla g(\boldsymbol{U}_k)\right\|^2}{\left\|\nabla g(\boldsymbol{U}_{k-1})\right\|^2}$. The participation factor of the previous conjugate gradient vector in Eq. (11) is directly computed without a condition while the limited (Keshtegar 2017b, c) and adaptive hybrid conjugate (Keshtegar and Chakraborty 2018) factors are applied to satisfy the sufficient descent condition in the conjugate FORM. The iterative formula (10) is used to compute the conjugate unite normal vector based on the conjugate gradient vector (11) as

$$\boldsymbol{\alpha}_{k+1}^{C\lambda}=\frac{\boldsymbol{U}_{k+1}^{C\lambda}}{\left\|\boldsymbol{U}_{k+1}^{C\lambda}\right\|} \tag{12}$$

The iterative formula of FORM (6) can be computed based on the conjugate finite-step length method (CFSL) in terms of Eq. (12). Fig. 1 shows a cycle of the conjugate search direction vector in two-dimension normal standard space, schematically. It is illustrated that conjugate gradient vector $\boldsymbol{d}_k$ is not located along direction the negative unit normal vector ($\boldsymbol{\alpha}_{k+1}$). This means that unit normal vector is not parallel to the normalized conjugate search direction $\boldsymbol{\alpha}_{k+1}^{C\lambda}$.

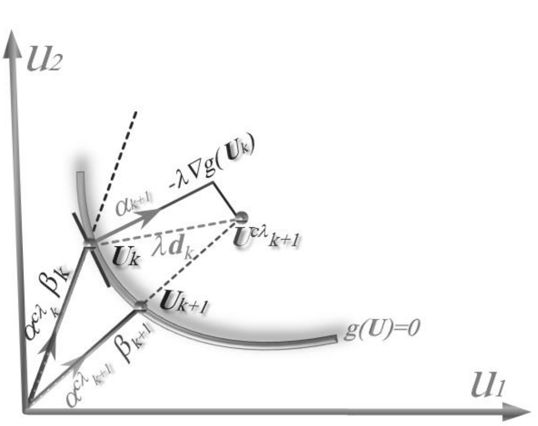

Fig. 1 Schematic the iterative procedure of the CFSL algorithm

If the step length λ is well-defined then the CFSL algorithm is efficiently converged. Thus, the descent condition is considered to adjust maximum step length as $\nabla^T g(\boldsymbol{U}_k)\boldsymbol{d}_k \le -c_1\|\nabla g(\boldsymbol{U}_k)\|^2$ and $0 < c_1 < 1$. Consequently, if $\|\boldsymbol{d}_{k+1}\| > \|\boldsymbol{d}_k\|$ then the step size is defined as follows

$$\lambda_k = \begin{cases} \lambda_k / c & \|\boldsymbol{d}_{k+1}\| > \|\boldsymbol{d}_k\| \\ \lambda_k & O.W \end{cases} \tag{13}$$

where, $1.2 \le c \le 1.5$. According to above equations, the steps of the CFSL scheme for MPP search are stated as follows.

Step 0	Define $g(\boldsymbol{U}) = 0$, Given $\boldsymbol{\mu}$, $\boldsymbol{\sigma}$ and constants $c \in [1.2, 1.5]$, step length $\lambda >> 0$ and stopping criterion ε, Let $k = 0$, Choose an initial point $\boldsymbol{X}_0 = \boldsymbol{\mu}$
Step 1	Normalize random variable in terms of Eqs. (3)-(5)
Step 2	Compute the LSF, gradient vector and conjugate gradient vector Adjust step size based on Eq. (13), Determine point $\boldsymbol{U}_{k+1}^{C\lambda}$ by Eq. (10) and normalized conjugate search direction vector using Eq. (12) Determine the new point as follows: $$\boldsymbol{U}_{k+1} = \frac{\nabla^T g(\boldsymbol{U}_k)\boldsymbol{U}_k - g(\boldsymbol{U}_k)}{\nabla^T g(\boldsymbol{U}_k)\boldsymbol{\alpha}_{k+1}^{C\lambda}}\boldsymbol{\alpha}_{k+1}^{C\lambda} \tag{14}$$
Step 3	If $\|\boldsymbol{U}_k - \boldsymbol{U}_{k-1}\| < \varepsilon$ then stop, print $\boldsymbol{X}^* = \boldsymbol{X}_{k+1}$, $\boldsymbol{U}^* = \boldsymbol{U}_{k+1}$, $\beta = \|\boldsymbol{U}^*\|$, and $P_f \approx \Phi(-\beta)$, else $k = k+1$ and Go to Step 1.

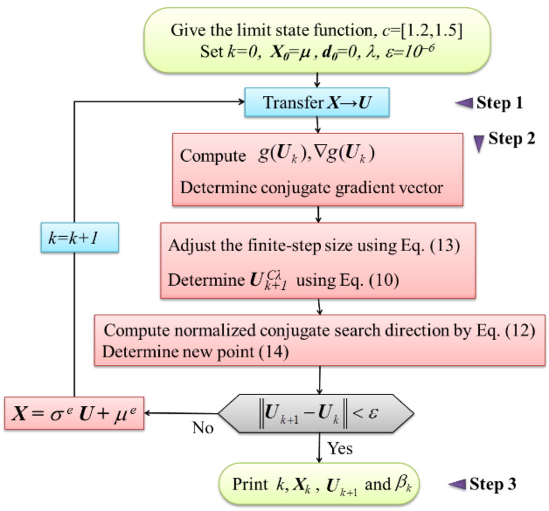

Fig. 2 framework of FORM for the CFSL method

The framework of the iterative procedure of the CFSL method is given in Fig. 2. The step size is adjusted using the conjugate gradient information while step sizes in chaotic conjugate gradient (Keshtegar 2016b) and the enriched FR (Keshtegar 2017b) methods are adapted using Armijo rule

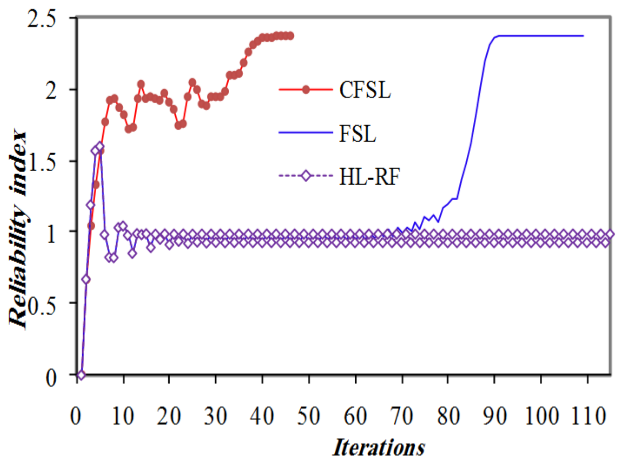

Fig. 3 Iterative history comparison of reliability index for Example 1

Table 1 Iterations of the FSL and CFSL algorithms respect to various λ for Example 1

Algorithm	λ: 50	30	20	15	10	5	1	0.5	0.2	0.1	0.05	0.01	0.005	0.001	β
FSL	119	114	110	108	107	101	90	86	74	72	68	56	50	39	2.36544
CFSL	51	50	49	45	42	36	37	34	34	33	29	29	23	21	

using new and previous points. The normalized conjugate vector is applied to compute the search direction in proposed CFSL while the search direction of HL-RF and FSL methods are extracted from the gradient vector.

4. Examples and comparative studies

Computational performance of the FSL and CFSL algorithms on a set of six nonlinear limit state functions are evaluated. Effectiveness and robustness of the CFSL are compared with the FSL and HL-RF methods that $\lambda_0 = 15$ and $c = 1.4$ are taken in FSL and CFSL. The gradient vector $\nabla g(\boldsymbol{U})$ is computed using central finite difference and also, all algorithms are implemented the same stopping criterion i.e., $\varepsilon = 10^{-6}$.

Example 1: highly nonlinear performance function as (Wang and Grandhi 1994)

$$g_1(X) = x_1^4 + 2x_2^4 - 20 \tag{15}$$

where, x_1 and x_2 are independent normal random variables with $\mu_1 = \mu_2 = 10$ and $\sigma_1 = \sigma_2 = 5$. The reliability index is extracted from Wang and Grandhi (1994) and Santos *et al.* (2012) which is equal to 2.3654 and 2.3655, respectively. The convergence history of reliability index is shown in Fig. 3 on the basis of the HL-RF, FSL and CFSL algorithms. It shown that the stable solution is captured based on the CFSL, i.e., $\beta = 2.3654347$ more efficient than the FSL. Nevertheless, the HL-RF algorithm yields the periodic-2 solutions i.e., {0.9267, 0.9863}. The convergent results of the FSL and CFSL algorithms are obtained after 108 and 45 iterations as $\boldsymbol{U}^* =$ [-1.63679, -1.70771] and

Table 2 Statistical properties of the random variables for rotating disk example

Random variable	Mean	Standard deviation	Distribution
α_m	0.9378	0.04655	Weibull*
S_u (lb/in^2)	220000	5000	Normal
ρ (lb/in^3)	0.29	0.00577	Uniform**
ω (rpm)	21000	1000	Normal
R_o (in)	24	0.5	Normal
R_i (in)	8	0.3	Normal

*Scale parameter=25.508; shape parameter=0.958
**Uniformly distributed in interval (0.28-0.3)

$\boldsymbol{U}^*$ = [-1.63687, -1.70763], respectively. The CFSL algorithm is required much less number of iterations to attain stable results than FSL.

Iterations of the FSL and CFSL algorithms to obtain stable results with respect to various step lengths are tabulated in Table 1 for Example 1. As seen, the FSL and CFSL schemes are converged to stable results. The result implies that iterations are depended on the search direction vector in FORM. The CFSL scheme is more efficient than FSL algorithm and the CFSL is converged almost twice faster than the FSL algorithm. Therefore, the conjugate search direction based on Eq. (11) can be improved the efficiency of the FORM compared to FSL method.

Example 2: a burst margin of rotating disk with inner radius R_i and outer radius R_o is considered as follows (Rao and Chowdhury 2009)

$$g_2 = \sqrt{\alpha_m S_u [\frac{3(385.82)(R_o - R_i)}{\rho(\frac{2\omega\pi}{60})^2 (R_o^3 - R_i^3)}]} - 0.37473 \tag{16}$$

The disk is subject to an angular velocity ω about an axis perpendicular to its plane at the center. S_u is the ultimate strength material, ρ is the density and α_m is the material utilization factor whose the statistical properties are listed in Table 2.

According to the results extracted from Rao and Chowdhury (2009) for this example, the reliability index is equal to 3.08728 ($P_f = 0.00101$) and 3.12336 ($P_f = 0.000894$) using the MCS and the FORM reliability analysis, respectively and also the MPP is extracted to be $\boldsymbol{X}^*$ = [0.81, 218989.90, 0.30, 21686.87, 24.51, 8.57]. Based on the reliability analysis undertaken using the CFSL algorithm, the converged reliability index and MPP are computed to be β = 3.123554 and $\boldsymbol{X}^*$ =[0.80851, 217479.4, 0.29397, 22913.87, 24.36376, 8.093455] after 9 iterations, respectively. The reliability index and iteration are computed to achieve stable result of this example based on the HL-RF and FSL as 3.123553 (11 iterations) and 3.123576 (52 iterations), respectively. It is obvious that the CFSL algorithm is efficiently converged to the stable solution with 3 iterations less than HL-RF and 43 iterations less than FSL approach for this example. Therefore, the CFSL is significantly efficient than the FSL method.

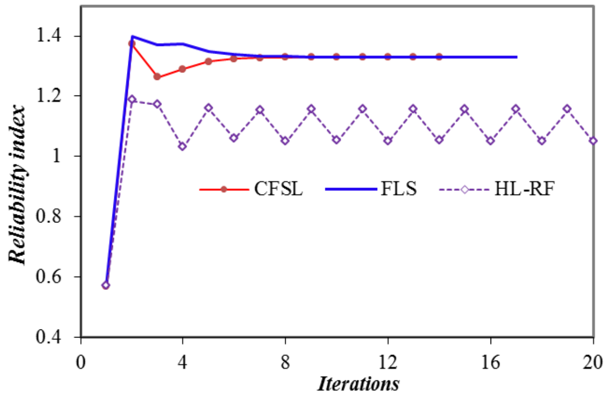

Fig. 4 Iterative history of reliability index for Example 3

Example 3: a non-linear performance function with non-normal variables that it is taken from response-surface fitting a pipeline for the reliability analysis by LSF as (Liu and Kiureghian 1991)

$$\begin{aligned} g_3 &= 1.1 - 0.00115\, x_1 x_2 + 0.00157\, x_2^2 + 0.00117\, x_1^2 + 0.0135\, x_2 x_3 \\ &- 0.0705\, x_2 - 0.00534\, x_1 - 0.0149\, x_1 x_3 - 0.0611\, x_2 x_4 + 0.0717\, x_1 x_4 \\ &- 0.226\, x_3 + 0.0333\, x_3^2 - 0.558\, x_3 x_4 + 0.998\, x_4 - 1.339\, x_4^2 \end{aligned} \tag{17}$$

where, x_1 to x_4 are statistically independent basic random variables. The random variable x_1 has the type-II largest value distribution with a mean of 10 and standard deviation of 5. x_2 and x_3 are the normally distributed random variables with means equal to 25 and 0.8, and standard deviations equal to 5 and 0.2 respectively; and the random variable x_4 follows a lognormal distribution density function with a mean 0.0625 and standard deviation 0.0625.

According to the results extracted from Liu and Kiureghian (1991), the reliability index and the MPP are equal to β =1.36 and $\boldsymbol{X}^*$ = [15.09, 25.027, 0.8653, 0.03582], respectively. Yang (2010) analyzed this example using the STM that the converged results are as the safety index of 2.3482 and the MPP of [14.906, 25.067, 0.8995, 0.04606]. The reliability indexes are obtained by Gong and Yi (2011) and Santos *et al.* (2012) as 1.3304 and 1.3653, respectively. Based on the reliability analysis undertaken using the proposed CFSL algorithm, the safety index β =1.3305032 and MPP $\boldsymbol{X}^*$ = [14.9043, 25.0673, 0.8596, 0.04609] are attained after 14 iterations. The iterative histories of reliability index for different reliability methods are shown in Fig. 4 for nonlinear problem of Example 3. As seen from Fig. 4, the HL-RF produces the unstable results as periodic-2 solutions i.e. β = {1.0496, 1.1536}. However, the FSL and CFSL methods are converged to stable results, more efficiently. The CFSL is converged with three iterations less than the FSL.

Example 4: a two degree of freedom primary-secondary dynamic system is shown in Fig. 5, which is a type of nonlinear problem with following LSF (Kiureghian and

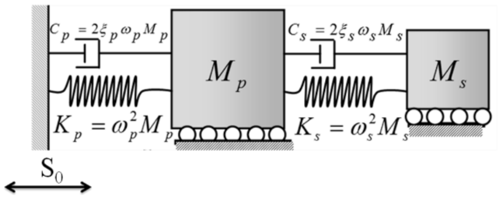

Fig. 5 Two-degree of freedom dynamic system

Table 3 Statistical basic random variables for Example 5

Random variable	Mean	Standard deviation	Random variable	Mean	Standard deviation
M_p	1	0.1	ξ_p	0.05	0.02
M_s	0.01	0.001	ξ_s	0.02	0.01
K_p	1	0.2	F_s	15	1.5
K_s	0.01	0.002	S_0	100	10

Stefano 1991)

$$g_4 = Fs - Ks \times P(E[x_s^2])^{1/2} \tag{18}$$

where, F_s denotes force capacity, P is the peak factor that is considered as 3 for simplicity and $E[x_s^2]$ is mean-square relative displacement of the secondary spring which is given as follows

$$E[x_s^2] = \frac{\pi S_0}{4\xi_s\omega_s}\left[\frac{\xi_a\xi_s}{\xi_p\xi_s(4\xi_a^2+\theta^2)+\gamma\xi_a^2}\times \frac{(\xi_p\omega_p^3+\xi_s\omega_s^3)\omega_p}{4\xi_a\omega_a^4}\right] \tag{19}$$

in which, $\gamma = \dfrac{M_s}{M_p}$ is mass ratio, $\omega_a = \dfrac{\omega_p+\omega_s}{2}$ and $\xi_a = \dfrac{\xi_p+\xi_s}{2}$ are average frequency and damping ratio of the two systems. $\theta = \dfrac{\omega_p-\omega_s}{\omega_a}$ is a tuning parameter and S_0 is intensity of the white noise. The means and standard deviations of eight random variables with Lognormal distributions are presented in Table 3.

Safety index is extracted from Kiureghian and Stefano (1991) which is equal to 2.01. Based on the results extracted from Keshtegar and Miri (2014), the reliability index is equal to 2.01645. Recently, Keshtegar (2016a, 2017a) used this problem to compare the performances of iterative FORM formula using chaotic conjugate STM (Keshtegar 2016a) and hybrid conjugate search direction (Keshtegar 2017a) that the reliability indexes are extracted as β= 2.016348 and β=2.01644, respectively. The convergence history of this example is shown in Fig. 6. As seen, the HL-RF method is converged to the periodic-2 solution of the reliability index as {4.9807, 4.2170}. However, converged reliability analysis of FSL i.e., β= 2.016459and CFSL i.e., β= 2.016445 are shown stable convergence, which are more agreement with the results from literatures. Furthermore, this example clearly indicated robustness of the FSL and CFSL iterative approaches in compared with the HL-RF algorithm in complex limit state function. It is obvious from

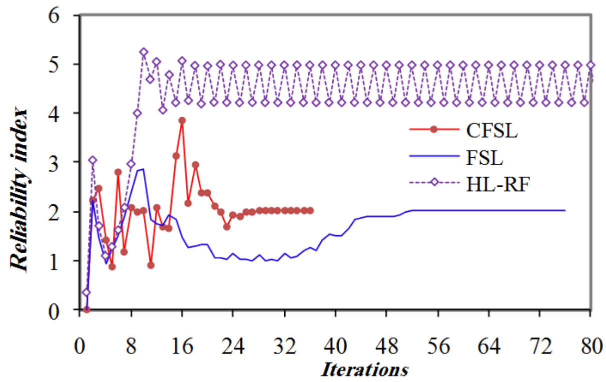

Fig. 6 Iterative history of reliability index for Example 4

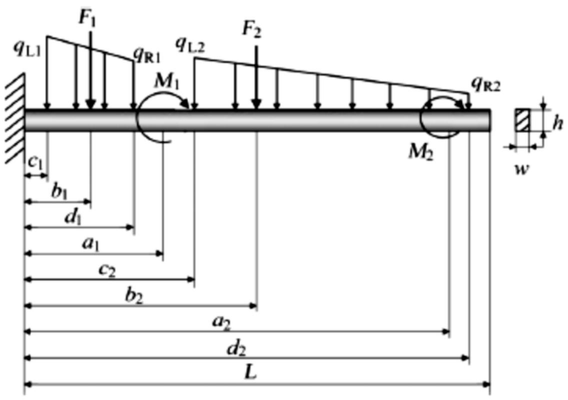

Fig. 7 Schematic view of cantilever beam

Table 4 Statistical properties of the random variables for cantilever beam example

Random variable	Mean	Standard deviation	Distribution	Random variable	Mean	Standard deviation	Distribution
M_1 (N-m)	50×10³	5×10³	Normal	b_1 (m)	0.75	0.001	Normal
M_2 (N-m)	30×10³	3×10³	Normal	b_2 (m)	2.5	0.001	Normal
F_1 (N)	18×10³	4×10³	Extreme value type I	c_1 (m)	0.25	0.0005	Normal
F_2 (N)	30×10³	3×10³	Normal	c_2 (m)	1.75	0.001	Normal
qL_1 (N/m)	30×10³	1×10³	Normal	d_1 (m)	1.25	0.001	Normal
qR_1 (N/m)	20×10³	1×10³	Normal	d_2 (m)	4.75	0.001	Normal
qL_2 (N/m)	20×10³	1×10³	Normal	w (m)	0.2	0.0001	Normal
qR_2 (N/m)	1×10³	10	Normal	h (m)	0.4	0.0001	Normal
a_1 (m)	1.5	0.005	Normal	S (Pa)	80×10⁶	8×10⁶	Normal
a_2 (m)	4.5	0.005	Normal	τ_{max} (Pa)	3.5×10⁶	0.5×10⁶	Normal

Fig. 6 that the CFSL is more robust than the HL-RF algorithm and is slightly efficient algorithm compared to FSL with convergence rate about twice faster than the FSL algorithm.

Example 5: A cantilever beam is subjected to external forces F_1 and F_2, moments M_1 and M_2, and distributed loads represented by $[qL_1, qR_1]$ and $[qL_2, qR_2]$ which is shown in Fig. 7 (Du 2010).

Two LSFs are considered for this example as

$$g_{5\text{-}1} = S - 6\frac{M}{wh^2} \tag{20}$$

$$g_{5-2} = \tau_{\max} - \frac{3Q}{2wh} \tag{21}$$

The first LSF represents the difference between the maximum normal stress and the yield strength. The second LSF is defined as the difference between the maximum shear stress and the allowable shear stress. Where, the bending moment (M) is computed as

$$\begin{aligned} M = \sum_{i=1}^{2} M_i + \sum_{i=1}^{2} F_i b_i + \sum_{i=1}^{2} q_{Li}(d_i - c_i)(d_i + c_i)/2 \\ + \sum_{i=1}^{2} [(q_{Ri} - q_{Li})(d_i - c_i)/2][c_i + 2(d_i - c_i)/3] \end{aligned} \tag{22}$$

and the shear force (Q) is computed as

$$Q = \sum_{i=1}^{2} F_i + \sum_{i=1}^{2} q_{Li}(d_i - c_i) + \sum_{i=1}^{2} (q_{Ri} - q_{Li})(d_i - c_i)/2 \tag{23}$$

This example involves 20 random variables which are listed in Table 4. The numerical results from Liu and Peng (2012) were extracted that the reliability index are given by FORM, SORM, and MVFOSA to be 3.4793 (P_f=2.5133×10^{-4}), 3.4719 (P_f=2.584×10^{-4}), and 3.4414 (P_f=2.8933×10^{-4}) for first LSF (Eq. 20) and also, 3.0486 (P_f=1.1495×10^{-3}), 3.0200 (P_f=1.2639×10^{-3}), and 3.0262 (P_f=1.2383×10^{-3}) for second LSF (Eq. 21), respectively. The CFSL is converged to reliability indexes of 3.4686377 and 3.0485344 for first and second LSFs after 4 and 5 iterations, respectively. The FSL and HL-RF algorithms are converged to stable results as well as the similar reliability indexes of CFSL for LSFs in Eqs. (20) and (21).

5. Discussions

The number of gradient vector evaluations $\nabla g(\boldsymbol{U})$ (Iter), number of evaluating LSF (CF) and reliability index (β) are used to compare the HL-RF (Eqs. (6), (7)), the FSL (Eqs. (6), (8)), and CFSL (Eqs. (12), (13), (14)) algorithms. The converged results of these methods are presented in Table 5. It is clear from Table 5 that the FSL and CFSL algorithms are the robust FORM but the HL-RF scheme provides the unstable results (see Examples 1, 3, 4). The results of Example 3 showed that the HL-RF method is exhibited unstable solutions as periodic-2 solutions of the reliability indexes i.e., β = {1.04958, 1.15364}. However, the FSL and CFSL algorithm is robustly converged for all studied examples. The highly nonlinear LSFs in Examples 1, 3 and 4, can be analyzed using improved search directions using FSL and CFSL formulas to achieve the stabilization. The CFSL has the top speed convergence compared to the FSL algorithm. The proposed conjugate search direction in Eq. (11) performed better in terms of efficiency than FSL in the same case reliability examples.

The CFSL can be improved the convergence performances of FORM formula, more efficiently in the first stage and more robustly in the second stage for both concave and convex reliability problems. It is obvious from Table 5 and Figs. 4 and 6 that the CFSL is significantly more efficient than the FSL method but its iterations are shown the zigzag drawbacks at the initial iterations. Consequently, it may increase the iterations of the CFSL to achieve the stable results. In order to improve the efficiency of the CFSL method for highly nonlinear problems, the conjugate search direction of the conjugate FORM using CFLS may also enhanced in future.

Table 5 Comparison of the convergence for the FSL, CFSL, and HL-RF iterative algorithms

Example		FSL (λ=15) Iter \ CF \ β	CFSL (λ=15) Iter \ CF \ β	HL-RF (λ=∞) Iter \ CF \ β	MCS CF \
#1		108 \ 540 \ 2.365454	45 \ 225 \ 2.365435*	Failed periodic-2 {0.9267, 0.9863}	10^6\2.8861
#2		52 \ 676 \ 3.123576	9 \ 117 \ 3.123554*	12 \ 156 \ 3.123553	1×10^6\3.0873
#3		17 \ 153 \ 1.330537	14 \ 126 \ 1.330503*	Failed periodic-2 {1.0496, 1.1536}	2×10^5\1.4961
#4		75 \ 1275 \ 2.016459	36 \ 612 \ 2.016445*	Failed periodic-2 {4.2170, 4.9807}	5×10^5\2.6338
#5	g_1	6 \ 246 \ 3.468638	4 \ 164 \ 3.468638*	7 \ 287 \ 3.468638	8×10^6\3.4708
	g_2	7 \ 287 \ 3.048534	5 \ 205 \ 3.048534*	9 \ 369 \ 3.048534	8×10^6\3.0156

*Algorithm with minimum iteration or minimum cup-run time

6. Conclusions

A general idea to solve a variety of highly nonlinear reliability problems is proposed using conjugate search direction combined by finite-step size, which is called as conjugate finite-step length (CFSL). The conjugate FORM satisfied the sufficient descent condition i.e., $\nabla^T g(\boldsymbol{U}_k)\boldsymbol{d}_k < -c_1 \|\nabla g(\boldsymbol{U}_k)\|^2$ using a step size without line search rules. The performance of CFSL algorithm is compared with the HL-RF and the finite-step length approaches using five nonlinear mathematical and structural/mechanical problems. The results demonstrated that the CFSL algorithm is the top performer and is quickly converged in comparison with the FSL method. The CFSL is more robust than HL-RF in highly nonlinear performance function. The proposed conjugate FORM using CFSL formula is converged about twice faster than the FSL algorithm in highly nonlinear performance function. Generally, the CFSL method is a robust method with fast convergence performance.

Acknowledgments

The research described in this paper was financially supported by the University of Zabol under Grant No. UOZ-GR-9517-3.

References

Du, X., (2010), "System reliability analysis with Saddlepoint approximation", *Struct. Multidiscip. Optim.*, **42**(2), 193-208.

Gong, J.X. and Yi, P. (2011), "A robust iterative algorithm for structural reliability analysis", *Struct. Multidisc. Optim.*, **43**, 519-527.

Hasofer, A.M. and Lind, N.C. (1974), "Exact and invariant second-moment code format", *ASCE J. Eng. Mech. Div.*, **100**(1), 111-121.
Keshtegar, B. and Bagheri, M. (2017), "Fuzzy relaxed-finite step size method to enhance the instability of the fuzzy first order reliability method using conjugate discrete map", *Nonlin. Dyn.*
Keshtegar, B. and Chakraborty, S. (2018), "A hybrid self-adaptive conjugate first order reliability method for robust structural reliability analysis", *Appl. Math. Modell.*, **53**(1), 319-332.
Keshtegar, B. and Meng, Z. (2017), "A hybrid relaxed first-order reliability method for efficient structural reliability analysis", *Struct. Safety*, **66**(1), 84-93.
Keshtegar, B. (2016a), "Chaotic conjugate stability transformation method for structural reliability analysis", *Comput. Meth. Appl. Mech. Eng.*, **310**(1), 866-885.
Keshtegar, B. (2016b), "Stability iterative method for structural reliability analysis using a chaotic conjugate map", *Nonlin. Dyn.*, **84**(4), 2161-2174.
Keshtegar, B. (2017a), "A hybrid conjugate finite-step length method for robust and efficient reliability analysis", *Appl. Math. Modell.*, **45**(1), 226-237.
Keshtegar, B. (2017b), "Enriched FR conjugate search directions for robust and efficient structural reliability analysis", *Eng. Comput.*, 1-16.
Keshtegar, B. (2017c), "Limited conjugate gradient method for structural reliability analysis", *Eng. Comput.*, **33**(3), 621-709.
Keshtegar, B. and Kisi, O. (2017), "M5 model tree and Monte Carlo simulation for efficient structural reliability analysis", *Appl. Math. Modell.*, **48**(1), 899-910
Keshtegar, B. and Miri, M. (2014), "Introducing conjugate gradient optimization for modified HL-RF method", *Eng. Comput.*, **31**(4), 775-790.
Kiureghian, A.D. and Stefano, M.D. (1991), "Efficient algorithm for second-order reliability analysis", *J. Eng. Mech.*, **117**(12), 2904-2923.
Koduru, S.D. and Haukaas, T. (2010), "Feasibility of FORM in finite element reliability analysis", *Struct. Safety*, **32**(1), 145-153.
Liu, D. and Peng, Y. (2012), "Reliability analysis by mean-value second-order expansion", *J. Mech. Des.*, **134**(6), 1-8
Liu, P.L. and Der Kiureghian, A. (1991), "Optimization algorithms for structural reliability", *Struct. Safety*, **9**(3), 161-177.
Meng, Z., Li, G., Yang, D. and Zhan, L. (2017), "A new directional stability transformation method of chaos control for first order reliability analysis", *Struct. Multidiscipl. Optim.*, **55**(2), 601-612.
Periçaro, G.A., Santos, S.R., Ribeiro, A.A. and Matioli, L.C. (2015), "HLRF-BFGS optimization algorithm for structural reliability", *Appl. Math. Modell.*, **39**(7), 2025-2035.
Rackwitz, R. and Fiessler, B. (1978), "Structural reliability under combined load sequences", *Comput. Struct.*, **9**(8), 489-494.
Rao, B.N. and Chowdhury, R. (2009), "Enhanced high-dimensional model representation for reliability analysis", *J. Numer. Meth. Eng.*, **77**(5), 719-750.
Santos, S.R., Matioli, L.C. and Beck, A.T. (2012), "New optimization algorithms for structural reliability analysis", *Comput. Model. Eng. Sci.*, **83**(1), 23-56.
Santosh, T.V., Saraf, R.K., Ghosh, A.K. and Kushwaha, H.S. (2006), "Optimum step length selection rule in modified HL-RF method for structural reliability", *J. Press Vess. Pip.*, **83**, 742-748.
Wang, L.P. and Grandhi, R.V. (1994), "Efficient safety index calculation for structural reliability analysis", *Comput. Struct.*, **52**(1), 103-111.
Wang, L.P. and Grandhi, R.V. (1996), "Safety index calculation using intervening variables for structural reliability analysis", *Comput. Struct.*, **59**(6), 1139-1148.
Xiao, N.C., Huang, H.Z., Wang, Z., Pang, Y. and He, L. (2011), "Reliability sensitivity analysis for structural systems in interval probability form", *Struct. Multidiscipl. Optim.*, **44**, 691-705.
Yang, D. (2010), "Chaos control for numerical instability of first order reliability method", *Commun. Nonlin. Sci. Numer. Simulat.*, **5**, 3131-3141.

CC