Structural Engineering and Mechanics, *Vol. 64, No. 6 (2017) 695-701*
DOI: https://doi.org/10.12989/sem.2017.64.6.695

Some general properties in the degenerate scale problem of antiplane elasticity or Laplace equation

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(Received May 1, 2017, Revised June 23, 2017, Accepted June 27, 2017)

Abstract. This paper investigates some general properties in the degenerate scale problem of antiplane elasticity or Laplace equation. For a given configuration, the degenerate scale problem is solved by using conformal mapping technique, or by using the null field BIE (boundary integral equation) numerically. After solving the problem, we can define and evaluate the degenerate area which is defined by the area enclosed by the contour in the degenerate configuration. It is found that the degenerate area is an important parameter in the problem. After using the conformal mapping, the degenerate area can be easily evaluated. The degenerate area for many configurations, from triangle, quadrilles and *N*-gon configuration are evaluated numerically. Most properties studied can only be found by numerical computation. The investigated properties provide a deeper understanding for the degenerate scale problem.

Keywords: degenerate scale problem; degenerate area; maximum property for degenerate area; laplace equation; conformal mapping

1. Introduction

The degenerate scale problem in antiplane elasticity or Laplace equation attracts many researchers with considerable attention (Hu *et al*. 1996, Chen *et al*. 2001, Chen *et al*. 2000, Kuo *et al*. 2013ab, Chen 2013, 2016, Chen and Lin 2010, Corfdir and Bonnet 2013). The problem typically appears in the Dirichlet problem for an exterior region. For example, even vanishing boundary value is assumed on the contour, non-vanishing solution for the Laplace equation exists in the exterior region when the scale takes a critical value.

A simple convenient method was suggested to evaluate the degenerate scales (Hu *et al*. 1996). On the other hand, using the necessary and sufficient boundary integral formulation could eliminate the non-equivalence of the conventional boundary integral formulations. In the potential problem, the Dirichlet boundary value problem was shown to yield a nonunique solution when the configuration reaches the degenerate scale (Chen *et al*. 2001). The degenerate scale was studied using the degenerate kernels and circulants. The degenerate scale for BIE in plane elasticity and antiplane elasticity was evaluated by using the complex variable and the conformal mapping (Chen *et al*. 2000). Basic equations in the complex variable method of plane elasticity were compactly addressed. From the formulation of an exterior problem in plane elasticity, the background for existence of the degenerate scale was discussed in detail.

It was proved that the unit leading coefficient of the linear term in the conformal mapping resulted in an interior null field which matched well the BEM (boundary element method) result once the degenerate scale happened (Kuo *et al*. 2013a). Degenerate scale of a regular *N*-gon domain was studied by using the BEM and complex variables (Kuo *et al*. 2013b). It was found that the contour of nonzero exterior field for the degenerate scale using the BEM matched well with that of Schwarz-Christoffel transformation. A solution for the degenerate scale for *N*-gon configuration in antiplane elasticity was provided (Chen 2013). The solution depends on a conformal mapping function of *N*-gon configuration in an infinite region.

The degenerate scale problem for the Laplace equation in a multiply connected region with an outer elliptic boundary was studied (Chen and Lin 2010). Inside the elliptic boundary, there are many voids with arbitrary configurations. When the used scale coincides with the degenerate scale, a non-trivial solution is found.

The degenerate scale issue does not appear only for Dirichlet condition of the Laplace equation but also for Robin boundary condition and some other conditions (Corfdir and Bonnet 2013, 2017).

The problem of finding a degenerate scale for Laplace equation in a half-plane was studied (Corfdir and Bonnet 2013). It was shown that if the boundary condition on the line bounding the half-plane is of Dirichlet type, there was no degenerate scale. In the case of a boundary condition of Neumann type, there was a degenerate scale. A numerical solution for the degenerate scale in antiplane elasticity using the null field BIE (boundary integral equation) was suggested (Chen 2016).

Similarly, many researchers studied the degenerate scale problem in plane elasticity (Vodicka and Mantic 2008, Vodicka 2013, Vodicka and Petrik 2015). The solution of a Dirichlet boundary value problem of plane isotropic

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http://www.techno-press.com/journals/sem&subpage=7
ISSN: 1225-4568 (Print), 1598-6217 (Online)

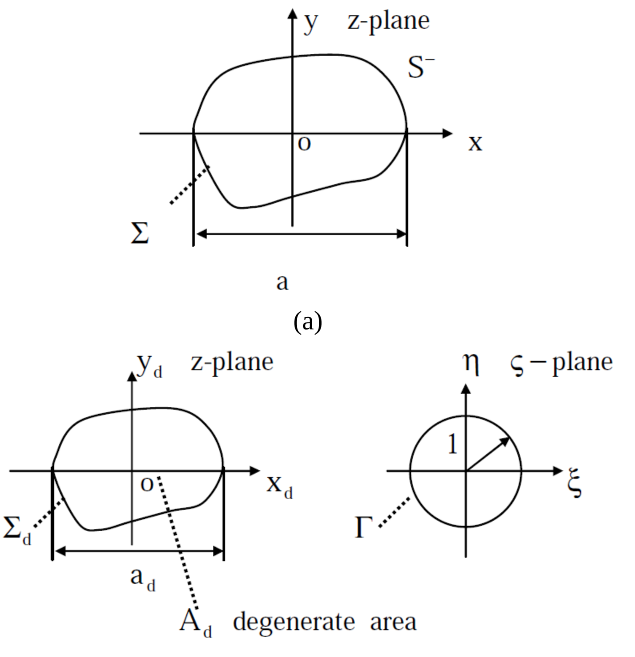

Fig. 1 (a) A contour Σ with the exterior region S^-, (b) A degenerate contour Σ_d, (c) The unit circle Γ in ς-plane

elasticity by the BIE of the first kind obtained from the Somigliana identity was considered (Vodicka and Mantic 2008). The logarithmic function appearing in the integral kernel leads to the possibility of this operator being non-invertible. A kind of the degenerate scale problem in plane elasticity was studied (Vodicka and 2013). It was pointed out that the logarithmic function appearing in the integral kernel may cause that the operator is non-invertible. In addition, the degenerate scales closely connected to the presence of a logarithmic function in the integral kernel of the single-layer potential operator (Vodicka and Petrik 2015).

An exact solution was proposed for the hypersingular boundary integral equation of two-dimensional elastostaticcs (Zhang and Zhang 2008). Properties of integral operators in complex variable boundary integral equation in plane elasticity were investigated (Chen and Wang 2013), which are derived from the Somigliana identity in the complex variable form.

This paper investigates some general properties in the degenerate scale problem of antiplane elasticity or Laplace equation. For a given configuration, the degenerate scale problem is solved by using the null field BIE (Chen 2016, Chen and Wu 2007). After solving the problem, we can define and evaluate the degenerate area which is defined by the area enclosed by the contour in the degenerate configuration. The degenerate area is defined from the conformal mapping of the contour configuration. After using the conformal mapping, the degenerate area can be easily evaluated. In the case of non-circular contour, the degenerate area takes the maximal value for the contour with the symmetric configuration. For example, the degenerate area for the square configuration is larger than that for the rectangular contour. The degenerate area for many configurations, from triangle, quadrilles and *N*-gon configuration are evaluated numerically. Note that those properties can only be found by numerical computation at present time. The investigated properties provide a deeper understanding for the degenerate scale problem.

2. Analysis

2.1 Formulation of the degenerate scale problem based on conformal mapping

After using complex potential $\phi_*(z)$ in antiplane elasticity, all the physical quantities can be expressed through a complex potential $\phi_*(z)$ (Chen *et al*. 2000)

$$\phi_*(z) = GW(x,y) + iF(x,y) \tag{1}$$

In Eq. (1), $z=x+iy$, G denotes the shear modulus of elasticity, $W(x,y)$ is the displacement in the longitudinal direction, $F(x,y)$ is the conjugate harmonic function with respect to the function $GW(x,y)$. Clearly, the displacement component $W(x,y)$ and $F(x,y)$ satisfies the following Laplace equation

$$\nabla^2 W(x,y) = 0\,,\quad \nabla^2 F(x,y) = 0\,,$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{2}$$

From Eq. (1), we can express the displacement $W(x,y)$ by

$$W(x,y) = \frac{1}{2G}(\phi_*(z) + \overline{\phi_*(z)}) \tag{3}$$

The degenerate scale or size is defined such that a non-trivial solution exists in the exterior Dirichlet boundary value problem of antiplane elasticity or Laplace equation even the vanishing displacement is assumed on the boundary. Now we will introduce a particular Dirichlet problem for exterior BVP (boundary value problem) as follows (Fig. 1)

$$W(x,y)\,\Big|_{\Sigma_d} = 0\,,\ \text{or}\ \ (\phi_*(z) + \overline{\phi_*(z)})\,\Big|_{\Sigma_d} = 0 \tag{4}$$

where Σ_d is a particular configuration to be investigated (Fig. 1).

In the formulation of the degenerate scale problem, the following mapping function is used

$$z = \omega(\varsigma) = R\chi(\varsigma)\,,\ \text{with}\ \ \chi(\varsigma) = (\varsigma + \sum_{k=1}^{\infty} q_k \varsigma^{-k})$$

$$(R\ \text{-positive real},\ \ q_k\ \text{-complex value}) \tag{5}$$

which maps the unit circle and its exterior region in ς-plane into some contour Σ and its exterior region in the z-plane (Fig. 1). After using the conformal mapping, we can define the following complex potential

$$\phi(\varsigma) = \phi_*(z)\,\Big|_{z=\omega(\varsigma)} \tag{6}$$

Clearly, from Eq. (6), the condition shown by Eq. (4) can be rewritten in the following form

$$\phi(\varsigma)+\overline{\phi(\varsigma)}\ \Big|_{\Gamma}=0\,,\ (\text{with}\ \ \varsigma=e^{i\theta}\,) \tag{7}$$

where Γ denotes the unit circle in the ς-plane (Fig. 1).

Since we have $z\approx R\varsigma$ at infinity, the investigated complex potential $\phi(\varsigma)$ can be expressed in the following form

$$\phi(\varsigma)=\ln(R\varsigma) \tag{8}$$

Substituting Eq. (8) into (7) yields

$$R=R_d=1 \tag{9}$$

where the subscript '*d*' denotes a critical value . Therefore, we can conclude that the following mapping function $z=\omega(\varsigma)\ |_{R=1}=\chi(\varsigma)$ maps the unit circle on the ς-plane into a critical contour, or the degenerate contour Σ_d, in the *z*-plane (Fig. 1).

In this case, we can evaluate the area bound by the contour Σ_d

$$A_d=-\oint_{\Sigma_d} ydx \tag{10}$$

The value A_d defined by Eq. (10) is called the degenerate area in the present study. Note that the integration in Eq. (10) is performed in the anti-clockwise direction (Fig. 1).

In Eq. (5), we make the following substitution $\varsigma=e^{i\theta}$, and obtain

$$x+iy=\chi(\varsigma)\ \Big|_{\varsigma=e^{i\theta}}=(e^{i\theta}+\sum_{k=1}^{\infty}q_k e^{-ki\theta}) \tag{11}$$

Substituting Eq. (11) into (10) yields

$$A_d=-\oint_{\Sigma_d} ydx=\pi(1-\sum_{k=1}^{\infty}k|q_k|^2)\,,$$

$$(\text{where}\ \ |q_k|^2=q_k\overline{q}_k\,) \tag{12}$$

This equation was proposed by other researcher.

From Eq. (12) we see that the degenerate area A_d solely depends on the coefficients q_k (k=1,2,..) involved in the mapping function. We also see that if $q_k\neq 0$ (k=1,2, …), the degenerate area A_d satisfies the following inequality

$$A_d<\pi \tag{13}$$

Clearly, in the condition of $q_k=0$ (k=1,2,3,..), or $\chi(\varsigma)=\varsigma$, we will obtain the maximum value for the degenerate area

$$A_{d,\max}=-\oint_{\Sigma} ydx\ \Big|_{q_k=0,k=1,2,..}=\pi \tag{14}$$

Therefore, The degenerate area A_d takes its maximum $A_{d,\max}=\pi$ in the case of circular boundary.

In a real computation using BIE, we must design a normal configuration to perform computation (see below section 2.2). In the case of an elliptic contour with two semi-axes "*a*" an "*b*", if choose *b*=5 and *a*=10, we have relevant area $A=\pi ab=50\pi$ which satisfies the condition

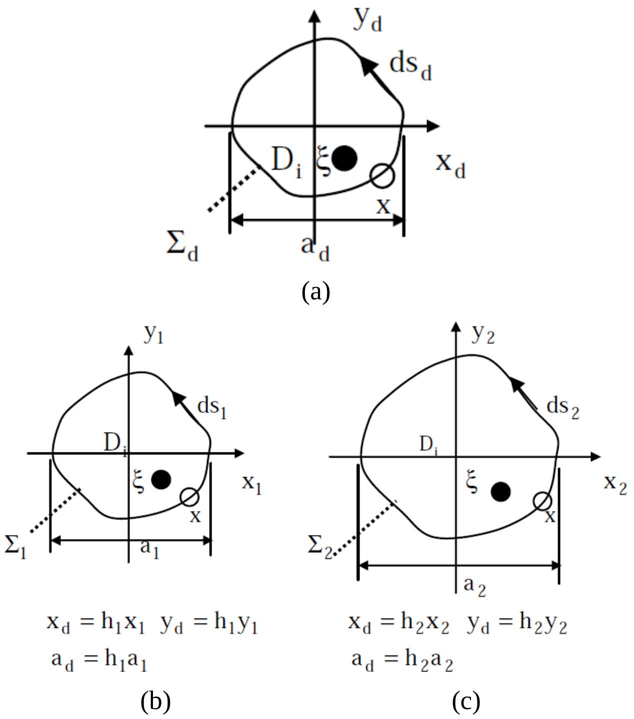

Fig. 2 Formulation of the degenerate scale problem, (a) the degenerate scale, (b) the first normal scale with the relation to the degenerate scale by $x_d=h_1x_1$, $y_d=h_1y_1$, $a_d=h_1a_1$, (b) the second normal scale with the relation to the degenerate scale by $x_d=h_2x_2$, $y_d=h_2y_2$, $a_d=h_2a_2$

$A>>A_{d,\max}$ ($A_{d,\max}=\pi$). Therefore, it (choosing *b*=5, *a*=10) is an appropriate option for the solution of the exterior BVP of Laplace equation in the case of elliptic contour.

In the meantime, if one does not know the conformal mapping function beforehand, one can evaluate the A_d from a numerical solution.

2.2 Formulation of the degenerate scale problem based on the null field boundary integral equation

In the following derivation, the formulation for the degenerate scale problem is based on the null field BIE (Chen 2016, Chen and Wu 2007) (Fig. 2)

It is assumed that for a particular scale or the degenerate scale Σ_d, there is a non-trivial solution for $p(x)$ to the following integral equation

$$\int_{\Sigma_d}\ln|x-\xi|p(x)ds_d(x)=0\,,$$

$$(\text{for}\ \ \xi\in D_i\,) \tag{15}$$

where $\xi\in D_i$ denotes the null nodes (Fig. 2) .

The proposed null field BIE shown by Eq. (15) may be solved in the normal scale with coordinates ox_jy_j (j=1,2). The two coordinates ox_jy_j (j=1,2) and ox_dy_d have the following relations (Fig. 2)

$$x_d=h_jx_j\,,\ \ y_d=h_jy_j\,,\ \ a_d=h_ja_j\ \ (j\text{=1,2}) \tag{16}$$

where h_j (j=1,2) denotes a magnified or a reduced factor. For solely evaluating the solution for the generate scale problem, one only needs to solve one of cases for j=1,2, for example, j=1. However, in order to prove the invariant poverty for the degenerate configuration, or for a_d, we have to solve and propose the problem for two cases, or j=1,2, simultaneously.

Since $\ln|x-\xi|_{in\, o\, x_d y_d} = \ln|x-\xi|_{in\, o\, x_j y_j} + \ln h_j$ (j=1,2), substituting this relation into Eq. (15) yields

$$\int_{\Sigma_j} \ln|x-\xi| p(x) ds_j(x) = -Q_j \ln h_j , \quad (\text{for } \xi \in D_i . j=1,2) \tag{17}$$

where

$$Q_j = \int_{\Sigma_j} p(x) ds_j(x) , \ (j=1,2) \tag{18}$$

Note that the integration contour Σ_j in Eq. (17) is a normal scale. In the case, the integral operator in the left hand side of Eq. (17) is invertible.

In order to find a non-trivial solution from Eq. (17), we propose the following basic solution for the BIE in the normal scale defined by

$$\int_{\Sigma_j} \ln|x-\xi| p(x) ds_j(x) = 1 , \quad (\text{for } \xi \in D_i , j=1,2) \tag{19}$$

Since the BIE shown by Eq. (19) is formulated in the normal scale, or Σ_j, it must have a definite solution. This solution for Eq. (19) is denoted by $p_{*j}(x)$. Thus, we have

$$\int_{\Sigma_j} \ln|x-\xi| p_{*j}(x) ds_j(x) = 1 , \quad (\text{for } \xi \in D_i , j=1,2) \tag{20}$$

From obtained solution $p_{*j}(x)$, we can evaluate

$$Q_{*j} = \int_{\Sigma_j} p_{*j}(x) ds_j(x) , \ (j=1,2) \tag{21}$$

From Eqs. (17) and (20), we see the following relation

$$p(x) = -Q_j \ln h_j \, p_{*j}(x) , \ (j=1,2) \tag{22}$$

After using the following operator $\int_{\Sigma_j} \{....\} ds_j(x)$ to both sides of Eq. (22), and using Eqs. (18) and (21) , we have

$$-Q_{*j} \ln h_j = 1, \text{ or } \ h_j = \exp(-\frac{1}{Q_{*j}}) , \quad (j=1,2) \tag{23}$$

Finally, we will find the following degenerate scale

$$a_{d,j} = h_j a_j , \ (j=1,2) \tag{24}$$

Therefore, the degenerate scales a_d is finally evaluated.

In the following derivation, we denote $\gamma=a_2/a_1$. To prove the invariant property of the degenerate scale a_d, it is equivalent to prove the following equalities

$$a_{d,1} = a_{d,2} , \text{ or } \ h_2 a_2 = h_1 a_1 , \text{ or } \ \frac{h_2}{h_1} = \frac{1}{\gamma} ,$$

$$\text{or } \ \ln(\frac{h_2}{h_1}) = -\ln\gamma ,$$

$$\text{or } \ -\frac{1}{Q_{*2}} + \frac{1}{Q_{*1}} = -\ln\gamma ,$$

$$\text{or } \ Q_{*2} = (1 - Q_{*2} \ln\gamma) Q_{*1} \tag{25}$$

In the first step of derivation, we want to find the relation between two functions $p_{*2}(x)$ and $p_{*1}(x)$. Clearly, from Eq. (20) we have the following integral equation

$$\int_{\Sigma_2} \ln|x-\xi| p_{*2}(x) ds_2(x) = 1 , \quad (\text{for } \xi \in D_i) \tag{26}$$

By adding $\int_{\Sigma_2} -\ln\gamma\, p_{*2}(x) ds_2(x) = -Q_{*2} \ln\gamma$ to Eq. (26), we have

$$\int_{\Sigma_2} (\ln|x-\xi| - \ln\gamma) p_{*2}(x) ds_2(x) = 1 - Q_{*2} \ln\gamma , \quad (\text{for } \xi \in D_i) \tag{27}$$

In addition, Eq. (27) can be converted into (Note $ds_2(x) = \gamma ds_1(x)$)

$$\int_{\Sigma_1} \ln|x-\xi| p_{*2}(x) ds_1(x) = \frac{1 - Q_{*2} \ln\gamma}{\gamma} , \quad (\text{for } \xi \in D_i) \tag{28}$$

Thus, comparing Eq. (19) with (28), we will find

$$p_{*2}(x) = \frac{1 - Q_{*2} \ln\gamma}{\gamma} p_{*1}(x) \tag{29}$$

In addition, from Eqs. (21) and (29), we have (Note $ds_2(x) = \gamma ds_1(x)$)

$$Q_{*2} = \int_{\Sigma_2} p_{*2}(x) ds_2(x) = \gamma \int_{\Sigma_1} p_{*2}(x) ds_1(x) = (1 - Q_{*2} \ln\gamma) Q_{*1} \tag{30}$$

Finally, the validity of Eq. (25) is finally proved.

Therefore, we find the first property in the degenerate scale problem. Without regarding the option of the normal scale, for example a_1 or a_2 (Fig. 2), we will obtain the invariant values for the degenerate scale, or $a_{d,1}=a_{d,2}$, or $h_2 a_2 = h_1 a_1$.

The merit of the derivation from Eq. (15) to (30) is compactly introduced below. In the transfer coordinate method, it is necessary to choose a normal scale for evaluating the degenerate scale. From the mentioned derivation, we can confirm that under the different normal scales, the obtained degenerate scale is unique

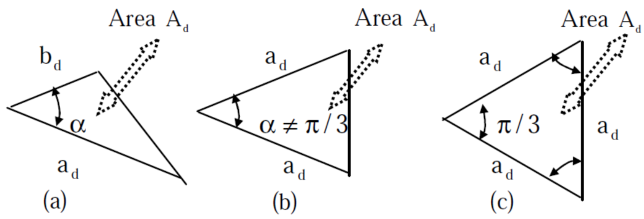

Fig. 3 Different configurations in the degenerate scale problem (a) a triangle, (b) an isosceles triangle, (c) an equilateral triangle, with the relation $A_d\,|_{case\ a}<A_d\,|_{case\ b}<A_d\,|_{case\ c}$

Table 1 The degenerate scale a_d (=$f_1(\alpha, \beta)$) and the degenerate area A_d (=$\pi g_1(\alpha, \beta)$) with $\beta=b_d/a_d$ (see Eqs. (31), (32) and Fig. 3(a))

$f_1(\alpha,\beta)$										
β=	0.6	0.7	0.8	0.9	1.0	1.1	1.1	1.3	1.4	1.5
α=										
$\pi/6$	3.3268	3.2284	3.1188	2.9919	2.8524	2.7122	2.5754	2.4424	2.3165	2.1999
$2\pi/6$	2.8981	2.7613	2.6295	2.5019	2.3786	2.2634	2.1561	2.0561	1.9629	1.8765
$3\pi/6$	2.6037	2.4635	2.3349	2.2165	2.1076	2.0071	1.9140	1.8286	1.7502	1.6778
$4\pi/6$	2.4397	2.3008	2.1765	2.0647	1.9635	1.8715	1.7876	1.7108	1.6401	1.5750
$g_1(\alpha,\beta)$										
β=	0.6	0.7	0.8	0.9	1.0	1.1	1.1	1.3	1.4	1.5
α=										
$\pi/6$	0.5285	0.5806	0.6192	0.6411	0.6475*	0.6439	0.6334	0.6171	0.5979	0.5777
$2\pi/6$	0.6946	0.7357	0.7624	0.7765	0.7798*	0.7767	0.7689	0.7575	0.7435	0.7280
$3\pi/6$	0.6474	0.6761	0.6941	0.7037	0.7070*	0.7052	0.6997	0.6919	0.6825	0.6720
$4\pi/6$	0.4922	0.5107	0.5223	0.5288	0.5314*	0.5310	0.5285	0.5244	0.5191	0.5129

*Taking maximum value

3. Two numerical examples

Example 1

In the first example, we evaluate the degenerate scale and the degenerate area for the configuration of the triangle (Fig. 3). The null field BIE technique suggested in section 2.2 or in (Chen 2016) is used to solve the problem numerically.

The triangle has an arbitrary α and two edges "a" and "b" with different lengths (Fig. 3). The computed degenerate scale for length "a" is expressed by

$$a_d = f_1(\alpha,\beta)\text{ , (with } \beta = b_d / a_d\text{)} \tag{31}$$

In addition, the degenerate area is expressed by

$$A_d = \pi g_1(\alpha,\beta)\text{ , (from } A_d = (\beta a_d^2 \sin\alpha)/2\text{)} \tag{32}$$

For the case of (a) $\alpha=\pi/6$, $2\pi/6$, $3\pi/6$ to $4\pi/6$, (b) β=0.6, 0.7,….1.0,…..1.5, the computed results for $f_1(\alpha, \beta)$ and $g_1(\alpha, \beta)$ are listed in Table 1.

For any given α, from Table 1 we find that $g_1(\alpha, \beta)$ takes a maximum value at $\beta=b_d/a_d=1$. For example, at $\alpha=3\pi/6$, $\beta=b_d/a_d=1$, $g_1(\alpha, \beta)$ takes a maximum value of 0.7070, which is listed in the third row for $g_1(\alpha, \beta)$ in Table 1. Generally, we have $g_1(\alpha, \beta)<1$.

From the computed results, we can find the second property as follows. For a given α, among all possibilities for $\beta=b_d/a_d$, the non-dimensional degenerate area $g_1(\alpha, \beta)$ (=A_d/π) takes a maximum value at the condition $\beta=b_d/a_d=1$ (see Fig. 3(a),(b) and Table 1, maximum value marked by *).

In a particular case, the triangle has an arbitrary α and two edges with equal lengths (Fig. 3(b)). The computed degenerate scale for length "a" is expressed by

$$a_d = f_2(\alpha) \tag{33}$$

In addition, the degenerate area is expressed by

$$A_d = \pi g_2(\alpha)\text{ ,(from } A_d = (a_d^2 \sin\alpha)/2\text{)} \tag{34}$$

For the case of $\alpha=\pi/12$, $2\pi/12$, $3\pi/12$ to $10\pi/12$ the computed results for $f_2(\alpha)$ and $g_2(\alpha)$ are listed in Table 2.

Table 2 The degenerate scale a_d (=$f_2(\alpha)$) and the degenerate area A_d (=$\pi g_2(\alpha)$) (see Fig. 3(b), (c))

α =	$\pi/12$	$2\pi/12$	$4\pi/12$	$4\pi/12$	$5\pi/12$	$6\pi/12$	$7\pi/12$	$8\pi/12$	$9\pi/12$	$10\pi/12$
$f_2(\alpha)$	3.2322	2.8524	2.5829	2.3786	2.2235	2.1076	2.0220	1.9635	1.9288	1.9159
$g_2(\alpha)$	0.4303	0.6475	0.7508	0.7798*	0.7601	0.7070	0.6285	0.5314	0.4187	0.2921

Note: $g_{2,\max}=g_2(\alpha)\,|_{\alpha=4\pi/12}$=0.7798, * taking maximum value

We see from Table 2 that, under the condition $\beta=b_d/a_d=1$ (or $b_d=a_d$), $g_2(\alpha)$ takes a maximum value of 0.7798 at $\alpha=\pi/3$, which is listed in the third row for $g_2(\alpha)$ in Table 2 (see Fig. 3(b), (c) and Table 2, maximum value marked by *)

We will find the third property as follows. Among all possibilities of α in the case of isosceles triangle (see Fig. 3(b), (c)), the non-dimensional degenerate area $g_2(\alpha)$ (=A_d/π) takes a maximum 0.7798 at the condition $\alpha=\pi/3$ (=60 degree), or the case of equilateral triangle (see Fig. 3(b), (c) and Table 2, maximum value indicated by *).

From the second and third properties for the triangle configuration, we will find the following result, for all possible cases of α, $\beta=b_d/a_d$, $g_1(\alpha, \beta)$ (=A_d/π) reaches its maximum 0.7798 (see the listed results . 0.7508, 0.7798 , 0.7601 in Table 2).

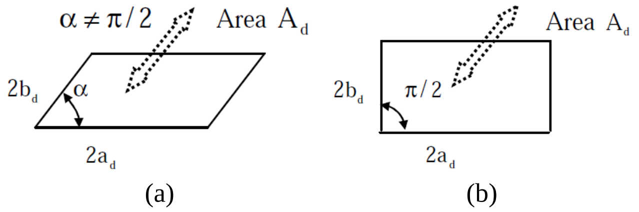

Fig. 4 Different configurations in the degenerate scale problem (a) a parallelogram, (b) a rectangle , with the relation $A_d\,|_{case\ a}<A_d\,|_{case\ b}$

Example 2

In the second example, we evaluate the degenerate scale and the degenerate area for the configuration of the parallelogram (Fig. 4). As in the first example, the null field BIE technique is suggested to solve the problem numerically mentioned in section 2.2 (Chen 2016).

The parallelogram has an arbitrary inclined angle α and two edges "2a" and "2b" with different lengths. The

Table 3 The degenerate scale a_d (=$f_3(\alpha, \beta)$) and the degenerate area A_d (=$\pi g_3(\alpha, \beta)$) with $\beta=b_d/a_d$ for the parallelogram configuration (see Eqs. (35), (36) and Fig. 4)

$f_3(\alpha,\beta)$										
$\beta=$	0.6	0.7	0.8	0.9	1.0	1.1	1.1	1.3	1.4	1.5
$\alpha=$										
$\pi/6$	1.1469	1.0774	1.0163	0.9622	0.9138	0.8704	0.8311	0.7954	0.7627	0.7328
$2\pi/6$	1.0861	1.0191	0.9607	0.9093	0.8636	0.8227	0.7857	0.7522	0.721 6	0.6935
$3\pi/6$	1.0670	1.0009	0.9434	0.8928	0.8479	0.8078	0.7715	0.7387	0.7087	0.6812
$4\pi/6$	1.0861	1.0191	0.9607	0.9093	0.8636	0.8227	0.7857	0.7522	0.7216	0.6935
$g_3(\alpha,\beta)$										
$\beta=$	0.6	0.7	0.8	0.9	1.0	1.1	1.1	1.3	1.4	1.5
$\alpha=$										
$\pi/6$	0.5024	0.5173	0.5260	0.5304	0.5316*	0.5305	0.5277	0.5236	0.5185	0.5128
$2\pi/6$	0.7805	0.8016	0.8142	0.8206	0.8224*	0.8209	0.8169	0.8110	0.8038	0.7956
$3\pi/6$	0.8698	0.8928	0.9065	0.9135	0.9155*	0.9138	0.9095	0.9031	0.8953	0.8863
$4\pi/6$	0.7805	0.8016	0.8142	0.8206	0.8224*	0.8209	0.8169	0.8110	0.8038	0.7956

Note: $g_{3,\max}=g_3(\alpha, \beta)\,|_{\alpha=3\pi/6\ \ \beta=1}$ =0.9155, * taking maximum value

computed degenerate scale for length “a” is expressed by

$$a_d = f_3(\alpha,\beta)\text{, (with } \beta = b_d / a_d\text{)} \qquad (35)$$

In addition, the degenerate area is expressed by

$$A_d = \pi g_3(\alpha,\beta)\text{, (from } A_d = 4\beta a_d^2 \sin\alpha\text{)} \qquad (36)$$

For the case of (a) $\alpha=\pi/6$, $2\pi/6$, $3\pi/6$ to $4\pi/6$ (b) β=0.6,0.7,….1.0,..1.5, the computed results for $f_3(\alpha, \beta)$ and $g_3(\alpha, \beta)$ are listed in Table 3.

From the computed results, we can find the following results. For a given α, among all possibilities for $\beta=b_d/a_d$, the non-dimensional degenerate area $g_3(\alpha, \beta)$ (=A_d/π) takes a maximum at the condition $\beta=b_d/a_d$=1 (Fig. 4(a)). For example, in the condition of $\alpha=2\pi/6$, $\beta=b_d/a_d$=1, $g_3(\alpha, \beta)$ takes a maximum value of 0.8224 (see second row for $g_3(\alpha, \beta)$results in Table 3).

From the computed results, we can find the fourth property as follows. Among all possibilities for $\beta=b_d/a_d$, the non-dimensional degenerate area $g_3(\alpha, \beta)$ (=A_d/π) takes a maximum value at the condition $\beta=b_d/a_d$=1 (see Fig. 4(a), (b) and Table 3, maximum value indicated by *).

Now we study the property for $g_3(\alpha,1) = g_3(\alpha,\beta)\,|_{\beta=1}$ (or for the case of $b_d=a_d$) for the different α values. From Table 3 we find that, at $\alpha=3\pi/6$, $g_3(\alpha, 1)$ takes a maximum value of 0.9155. Note that $\alpha=3\pi/6$, $\beta=b_d/a_d$=1 corresponds to a square configuration. Finally, $g_{3,\max} = g_3(\alpha,\beta)\,|_{\alpha=3\pi/6\ \ \beta=1}$=0.9155 is the largest value for $g_3(\alpha, \beta)$ in the studied range (see Table 3, maximum value indicated by *) .

In addition, for two cases $\alpha=\alpha_1$ and $\alpha=\alpha_2=\pi-\alpha_1$ (for example $\alpha=\alpha_1=2\pi/6$, $\alpha=\alpha_2=4\pi/6$, the functions $f_3(\alpha, \beta)$ or $g_3(\alpha, \beta)$ take the same values . Generally, we have $g_3(\alpha, \beta)<1$.

From the computed results, we can find the fifth property as follows. Among all possibilities for α, the non-dimensional degenerate area $g_3(\alpha, \beta)\,|_{\beta=1}$ (=A_d/π) takes a maximum value 0.9155 at the condition $\alpha=3\pi/6$ which corresponds to a square configuration (see Fig. 4(a), (b)and Table 3, maximum value indicated by *).

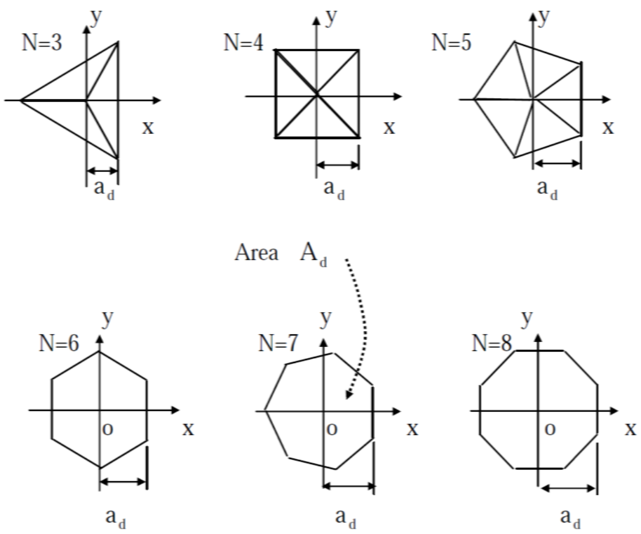

Fig. 5 N-gon configuration for N=3,4,5,6,7 and 8, and the definition for the degenerate scale a_d and the degenerate area A_d

4. Analysis for the N-gon configuration

In the third example, we evaluate the degenerate scale and the degenerate area for the N-gon configuration (Fig. 5). The mapping function was suggested previously (Kuo *et al*. 2013b, Chen 2013)

$$z = \omega(\varsigma) = R\chi(\varsigma)\,,$$

$$\text{with } \chi(\varsigma) = \varsigma - \frac{2}{N(N-1)}\frac{1}{\varsigma^{N-1}} - \sum_{k=2}^{\infty} c_k \frac{1}{\varsigma^{kN-1}}\,, \quad (N \ge 3) \qquad (37)$$

where

$$c_k = \frac{2(2-N)...(2-(k-1)N)}{k!N^k(kN-1)} \qquad (38)$$

The computed degenerate scale for length “a_d” is expressed by

$$a_d = f_4(N)\text{, (}N\text{=3,4,…)} \qquad (39)$$

where

$$f_4(N) = \chi(\varsigma)\,|_{\varsigma=1} = 1 - \frac{2}{N(N-1)} - \sum_{k=2}^{\infty} c_k\,, \quad (N \ge 3) \qquad (40)$$

In addition, the degenerate area A_d is expressed by

Table 4 The degenerate scale a_d (=$f_4(N)$) and the degenerate area A_d (=$\pi g_4(N)$) with (see Eqs. (39), (40), (41) and Fig. 5)

N=	3	4	5	6	7	8	9	10
$f_4(N)$	0.6845	0.8472	0.9102	0.9410	0.9583	0.9689	0.9760	0.9809
$g_4(N)$	0.7749	0.9139	0.9579	0.9763	0.9853	0.9903	0.9932	0.9951

$$A_d = \pi g_4(N)\text{, (from } A_d = Na_d^2 \tan(\pi / N)\text{)} \quad (N=3,4,..) \tag{41}$$

For the case of N=3,4,…10, the computed results for $f_4(N)$ and $g_4(N)$ are listed in Table 4. From computed results we find the following results. Generally, we have $f_4(N)< f_4(N+1)$ and $g_4(N)<g_4(N+1)$. Particularly, when $N\to\infty$, we have the following limitation $\lim_{N\to\infty} f_4(N)=1$ and $\lim_{N\to\infty} g_4(N)=1$.

From above mentioned results, we can propose the fourth property in the degenerate scale problem for the N-gon configuration. If the N-gon configuration is more nearly to the circular configuration, or $N\to\infty$, the degenerate scale $f_4(N)(=a_d)$ and the moralized degenerate area $g_4(N)$ (=A_d/π) will approach unity, respectively.

5. Conclusions

The concept for degenerate area, or A_d, in the degenerate scale problem is introduced. A conjecture for the degenerate area is proposed in the present study. For the triangle case, the conjecture is as follows. Among all possibilities for the triangle, the non-dimensional degenerate area A_d/π takes a maximum value for the configuration of equilateral triangle, (see Fig. 3(a), (b), (c)). This conjecture has been proved numerically in the first example. However, this conjecture is not easy to prove analytically. Clearly, this conjecture is also valid for the N- gon configuration with different edge lengths.

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