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DOProC-based reliability analysis of structures

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Abstract. Probabilistic methods are used in engineering where a computational model contains random variables. The proposed method under development: Direct Optimized Probabilistic Calculation (DOProC) is highly efficient in terms of computation time and solution accuracy and is mostly faster than in case of other standard probabilistic methods. The novelty of the DOProC lies in an optimized numerical integration that easily handles both correlated and statistically independent random variables and does not require any simulation or approximation technique. DOProC is demonstrated by a collection of deliberately selected simple examples (i) to illustrate the efficiency of individual optimization levels and (ii) to verify it against other highly regarded probabilistic methods (e.g., Monte Carlo). Efficiency and other benefits of the proposed method are grounded on a comparative case study carried out using both the DOProC and MC techniques. The algorithm has been implemented in mentioned software applications, and has been used effectively several times in solving probabilistic tasks and in probabilistic reliability assessment of structures. The article summarizes the principles of this method and demonstrates its basic possibilities on simple examples. The paper presents unpublished details of probabilistic computations based on this method, including a reliability assessment, which provides the user with the probability of failure affected by statistically dependent input random variables. The study also mentions the potential of the optimization procedures under development, including an analysis of their effectiveness on the example of the reliability assessment of a slender column.

Keywords: Direct Optimized Probabilistic Calculation; DOProC; probabilistic method; random variable; reliability assessment; probability of failure; Monte Carlo

1. Introduction

When designing a structure, various reliability criteria should be met in accordance with the relevant standards (Vrouwenvelder 2002, Ang and Cornell 1974). Each random variable in the probabilistic computations contains uncertainties (Falsone and Settineri 2014, Kiureghian and Ditlevsen 2009) which may be broadly classified into two main categories:

• *aleatoric* (stochastic) *uncertainties* of a random nature and

• *epistemic* (state of knowledge) *uncertainties* that arise owing to imperfect knowledge on the part of the analyst.

Typical sources of aleatoric uncertainties are material properties and production and/or assembly inaccuracies in the geometry or the environment where the structure should be located (Spackova *et al.* 2013, Sejnoha *et al.* 2007).

The final reliability of the structure is also affected by epistemic uncertainties which depend on the computational model used, statistical processing of input data, which also involves a human factor in the design process, and/or construction and use of the structure.

Recently, two different approaches have been developed to address uncertainties. On the one hand, there are formal mathematical languages that work with degrees of truth (“fuzzy logic”) or degrees of membership of a set (“fuzzy set theory”) (Tao *et al.* 2017, Jafari and Jahani 2016, Antucheviciene *et al.* 2015). On the other hand here are methods based on probabilistic formalism (Ditlevsen and Madsen 1996). The dominant opinion is that probability calculus is the most convenient mathematical basis for evaluations of structural reliability and that probability-based methods deserve to be further developed.

These methods are known as:

• *analytical methods* and

• *simulation methods*.

Analytical methods for the computations of failure probabilities and generalized reliability indices are often denoted under the abbreviation FORM (Hohenbichler and Rackwitz 1983). Methods that include the curvature correction of the limit state surface are known as SORM (Fiessler *et al.* 1979). Among methods for efficiently solving very complex problems, response surface and its modifications (Fang and Tee 2017, Zhao *et al.* 2017, Goswami *et al.* 2016) should not be omitted. There is a

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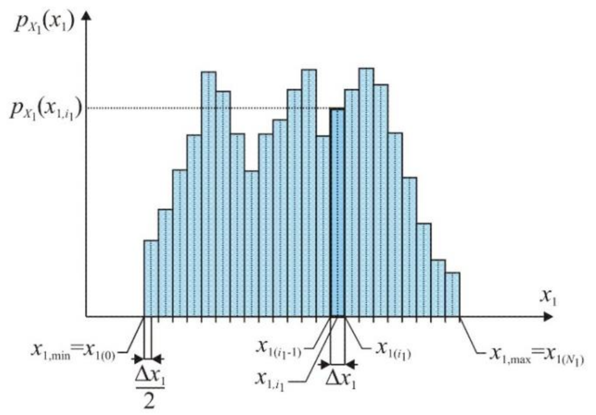

Fig. 1 Histogram of a continuous random quantity X_1 after discretization

variety of software products based on direct Monte Carlo sampling, as well as on stratified methods. Beyond these approaches, there are a number of advanced Monte Carlo simulation techniques, see e.g., (de Angelis *et al.* 2015, Bhattacharjya *et al.* 2015, Roudsari and Gordini 2015, Neil 2003, Ditlevsen and Madsen 1996, Saliby 1990, Melchers 1989, Bucher 1988, Bjerager 1988). The advent of Markov chain Monte Carlo (MCMC) simulation techniques (Vargas *et al.* 2015) gave rise to a new approach called subset simulation, which is suitable for computing very small probabilities (Au and Beck 2001).

This paper presents a new method, currently known as Direct Optimized Probabilistic Calculation (DOProC), which is based on the theorem of overall probability, and, as such, in principle belongs to the first category and primarily deals with aleatoric uncertainties. In this regard, the Point Estimate Method (PEM) should be mentioned here for DOProC might be compared to it or at least put into the category of PE methods. Recall with reference to (Rosenblueth 1981, He and Sallfors 1994) that PEM is a statistical method with which continuous probabilistic distributions are replaced by several discrete points. Their locations and weighting factors are optimized by matching, or approximately matching, the lowest three or five statistical moments. In case of n random variables the optimal PEM schemes are expressed in (He and Sallfors 1994) for 2^n, $2n$, and 2^n+2n points, which are referred to as the optimal ones.

In DOProC, both statistically independent as well as correlated random variables are expressed by means of histograms. They are characterized either by a class (interval) of a given variable or by a set of correlated variables. If continuous random distributions (including parametric ones) are available, the continuous quantity is subdivided into intervals; see Fig. 1. All points representing these intervals are used in a special form of optimized numerical integration, without reducing their number as is typical of PEM.

The novelty of the proposed method lies in its optimized numerical approach to the demanding nature of numerical integration. It is our belief that, despite tremendous achievements in the development of simulation techniques, the proposed method provides advantages in terms of effectiveness, variability, and competitiveness.

This paper is organized as follows: the main concepts and relationships relevant to reliability computations are briefly recalled in Sec. 2. Sec. 3 describes the principles of Direct Optimized Probabilistic Calculation (DOProC). A few deliberately selected simple application examples are demonstrated in Sec. 4, followed by our conclusions (Sec. 5).

The method is expounded in an expressively succinct manner in a few short international conference papers, and in detail in monograph (Janas *et al.* 2015) available in Czech language only. The DOProC method was introduced in the context of other probability methods in (Krejsa *et al.* 2016a), along with a brief description of the principle of basic computational algorithm with the possibility of reducing computational steps using optimization procedures. The conference paper (Krejsa *et al.* 2016b) outlines further possibilities and the potential of the method developed in the context of application useful in engineering practice.

2. Basic concept and formulae in reliability computations

The load effect, E, is random in both time and space. Histograms are typically used to describe random variables in the loading process, but the load duration curves are also often applied. The structural resistance, R, depends on the computational model. The probability of failure p_f is defined as

$$p_f = P(Z \le 0) = P(R - E \le 0), \tag{1}$$

where

$$Z = R - E > 0 \tag{2}$$

is the safety margin.

In general, the p_f can be expressed as

$$p_f = \int_{D_f} f_{\mathbf{X}}(x_1, x_2, \ldots, x_n) \mathrm{d}x_1, \mathrm{d}x_2, \ldots, \mathrm{d}x_n \,, \tag{2}$$

where D_f is the domain of failure, $Z(\boldsymbol{X})\le 0$. The Z contains random input variables $\boldsymbol{X} = X_1, X_2, \ldots, X_n$, and $f_{\mathbf{X}}(x_1, x_2, \ldots, x_n)$ is the corresponding joint density function.

It is often difficult to determine the probability p_f on the basis of the explicit calculation of the integral in Eq. (3). Unlike simulation methods, DOProC solves the integral of Eq. (3) directly and numerically, utilizing either the univariate histograms of statistically independent input variables X_i or the multivariate histograms of the vector $\mathbf{X}$ for correlated variables.

3. Direct Optimized Probabilistic Calculation (DOProC)

This section presents the fundamentals of DOProC. To begin with, Sec. 3.1 addresses the algorithms transforming (merging) histograms of input variables into the histogram of a function of these variables. Sec. 3.2 then converts the

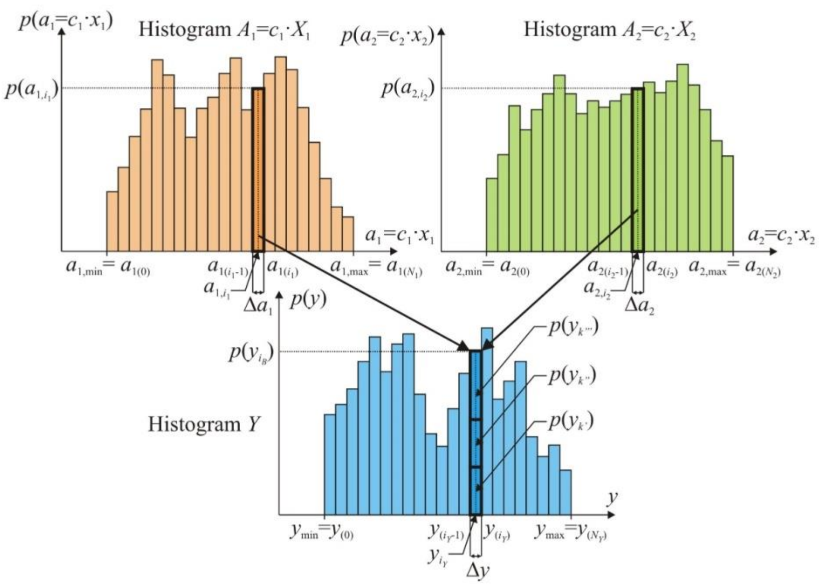

Fig. 2 Principles of numerical operations with two histograms for computations with Eq. (8)

derived formulae for calculating the p_f. To make standard procedures efficient and able to compete, a sequence of optimization strategies is proposed in Sec. 3.4. The fundamental elements of the optimization process include: grouping of input/output variables, interval optimization, zonal optimization, trend optimization, parallelization, and combinations of these.

3.1 Operations with histograms

Each interval of the histogram of an input random variable X_j is characterized by its representative value x_{j,i_j} and the probability of occurrence $p_{X_j}(x_{j,i_j})$, where the number of intervals is i_j=1, 2, …, N_j. As x_{j,i_j}, one may select the midpoint, the local mean, or the local mode of the subset of random values (the outcomes of X_j) falling inside the interval. This allows us to use the probability mass functions of discrete variables as surrogates for histograms.

For a combination of n random variables, both statistically independent and/or correlated, i.e.

$$Y = g(X_1, \ldots, X_n) = \sum_{j=1}^{n} c_j \cdot X_j , \tag{4}$$

we obtain

$$\begin{aligned} p_Y(y) &= \sum_{i_1}\sum_{i_2}\cdots\sum_{i_j}\cdots \\ &\cdots\sum_{i_n} p_{X_1 X_2 \ldots X_j \ldots X_n}\left(x_{1,i_1} x_{2,i_2} \ldots x_{j,i_j} \ldots x_{n,i_n}\right) \end{aligned} \tag{5}$$

for all x_{j,i_j} such that $\sum_{j=1}^{n} c_j \cdot x_{j,i_j} = y$.

Similarly, for the cumulative distribution function it holds

$$\begin{aligned} &F_Y(y) = \\ &= \sum_{\substack{i_1, i_2, \ldots, i_j, \ldots, i_n \\ \sum_{j=1}^{n} c_j \cdot x_{j,i_j} \le y}} \!\!\!\!\!\!\!\! \sum \ldots \sum p_{X_1 X_2 \ldots X_j \ldots X_n}\left(x_{1,i_1} x_{2,i_2} \ldots x_{j,i_j} \ldots x_{n,i_n}\right). \end{aligned} \tag{6}$$

Eventually, for independent variables X_j, j=1…n, Eq. (6) becomes

$$\begin{aligned} F_Y(y) = &\sum_{i_2}\cdots\sum_{i_j}\cdots\sum_{i_n} F_{X_1}\left[\frac{1}{c_1}\left(y - \sum_{j=2}^{n} c_j \cdot x_{j,i_j}\right)\right]\cdot \\ &\cdot p_{X_2}\left(x_{2,i_2}\right)\ldots p_{X_j}\left(x_{j,i_j}\right)\ldots p_{X_n}\left(x_{n,i_n}\right). \end{aligned} \tag{7}$$

These formulae can be easily modified for two discrete random variables X_1, X_2. In this case

$$Y = g(X_1, X_2) = c_1 \cdot X_1 + c_2 \cdot X_2 , \tag{8}$$

c_1, c_2 being deterministic constants. Evidently, Eq. (5) can be rewritten as

$$\begin{aligned} p_Y(y) &= \sum_{\substack{i_1, i_2 \\ c_1 \cdot x_{1,i_1} + c_2 \cdot x_{2,i_2} = y}} \!\!\!\!\!\! \sum p_{X_1 X_2}\left(x_{1,i_1} x_{2,i_2}\right) = \\ &= \sum_{i_2} p_{X_1 X_2}\left[\frac{1}{c_1}\cdot\left(y - c_2 \cdot x_{2,i_2}\right), x_{2,i_2}\right] \end{aligned} \tag{9}$$

and Eq. (6) will take the following form:

$$\begin{aligned} F_Y(y) &= \sum_{\substack{i_1, i_2 \\ c_1 \cdot x_{1,i_1} + c_2 \cdot x_{2,i_2} \le y}} \!\!\!\!\!\! \sum p_{X_1 X_2}\left(x_{1,i_1} x_{2,i_2}\right) = \\ &= \sum_{\substack{i_2 \\ x_{1,i_1} \le \frac{1}{c_1}\left(y - c_2 \cdot x_{2,i_2}\right)}} p_{X_1 X_2}\left[\frac{1}{c_1}\cdot\left(y - c_2 \cdot x_{2,i_2}\right), x_{2,i_2}\right]. \end{aligned} \tag{10}$$

Evidently, $p_Y(y)$ expresses the probability of a single value y from all possible combinations x_{1,i_1} and x_{2,i_2} of the random variables X_1 and X_2, which correspond to the specification in Eq. (8).

If the independent random variable X_1 is represented by N_1 values x_{1,i_1}, and if the independent random variable X_2 is represented by N_2 values x_{2,i_2}, then Eq. (8) can be used for the computation of $k_{\max}$=$N_1 \cdot N_2$ values for y_k, where some values can be identical. Index k sets the order of possible combinations of x_{1,i_1} and x_{2,i_2}, which pertain in accordance

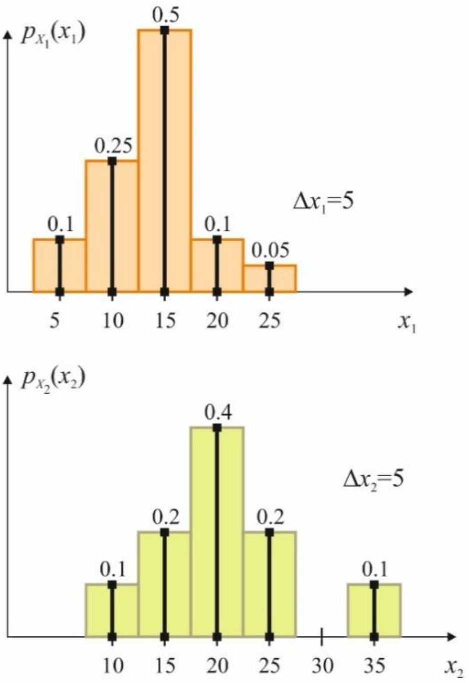

Fig. 3 Histograms of two independent variables X_1, X_2.

with Eq. (8) to y_k. When making up a histogram of the random variable Y with N_Y classes, then the class y_{i_Y} involving y_k is specified by its limits fulfilling the following condition

$$y_{(i_Y-1)} = y_{i_Y} - \frac{\Delta y}{2} \le y_k < y_{i_Y} + \frac{\Delta y}{2} = y_{(i_Y)}, \tag{11}$$

where $\Delta y = y_{(i_Y)} - y_{(i_Y-1)}$ is the width of the class (the interval) in the histogram of Y. The probability of y_k for the statistically independent X_1 and X_2 is

$$p(y_k) = \left[p(x_{1,i_1}) \cdot p(x_{2,i_2})\right]_k \tag{12}$$

and the probability of the class y_{i_Y} is

$$p(y_{i_Y}) = \sum_k p(y_k) \tag{13}$$

for each y_k.

The above procedure is graphically displayed in Fig. 2.

In view of mathematical formalism, we state that the DOProC method is underlined by Eqs. (9) and (10). In case of two statistically independent variables X_1, X_2, Eqs. (7) and (10) are converted to

$$\begin{aligned} F_Y(y) &= \sum_{\substack{i_1,i_2 \\ x_{1,i_1} \le \frac{1}{c_1}(y - c_2 \cdot x_{2,i_2})}} \sum p_{X_1}(x_{1,i_1}) \cdot p_{X_2}(x_{2,i_2}) = \\ &= \sum_{i_2} F_{X_1}\left[\frac{1}{c_1} \cdot \left(y - c_2 \cdot x_{2,i_2}\right)\right] \cdot p_{X_2}(x_{2,i_2}). \end{aligned} \tag{14}$$

The nature of this method is demonstrated via the quotient of two random variables

$$Y = g(X_1, X_2) = \frac{X_2}{X_1}. \tag{15}$$

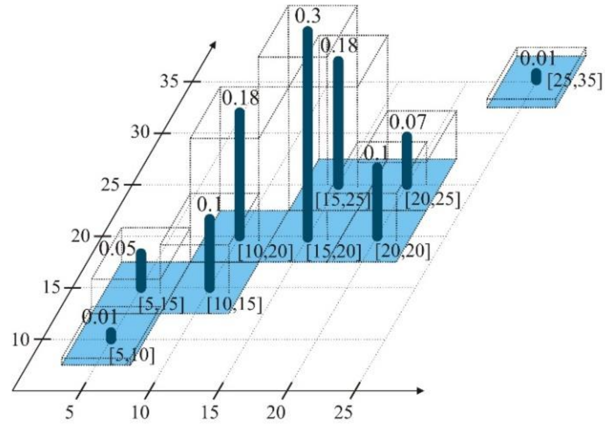

Fig. 4 Bivariate histogram of two correlated variables X_{3_d}, X_{4_d}.

The resulting histogram is deliberately derived in two steps:

(i) In the first step, we derive the probability mass function of Y which, in accordance with properly modified Eq. (9), takes the form

$$\begin{aligned} p_Y(y) &= \sum_{\substack{i_1,i_2 \\ \frac{x_{2,i_2}}{x_{1,i_1}} = y}} \sum p_{X_1X_2}(x_{1,i_1} x_{2,i_2}) = \\ &= \sum_{x_{1,i_1}} p_{X_1X_2}\left[x_{1,i_1}, y \cdot x_{1,i_1}\right]. \end{aligned} \tag{16}$$

The use of Eq. (16) is illustrated through the following example, which allows us to present the algorithm alternatively in the matrix form. The univariate histograms of input variables are displayed in Fig. 3 for two statistically independent variables X_1 and X_2, and the bivariate histogram in Fig. 4 for statistically dependent (correlated) variables X_{3_d} and X_{4_d}.

Division of two discrete variables is formally denoted as

$$Y = \frac{X_2}{X_1} = \left(\frac{10,15,20,25,30,35}{5,10,15,20,25}\right). \tag{17}$$

The possible combinations satisfying Eq. (16) are put down in the form of quotients localized as the entries of the combination matrix; see Table 1. The first row stores the resulting probabilities $p_Y(y)$ calculated from Eq. (16) and modified for independent variables in the following form

$$\begin{aligned} p_Y(y) &= \sum_{\substack{i_1,i_2 \\ \frac{x_{2,i_2}}{x_{1,i_1}} = y}} \sum p_{X_1}(x_{1,i_1}) \cdot p_{X_2}(x_{2,i_2}) = \\ &= \sum_{x_{1,i_1}} p_{X_1}(x_{1,i_1}) \cdot p_{X_2}(y \cdot x_{1,i_1}). \end{aligned} \tag{18}$$

The entries in the second row in Table 1 are all the possible values (outcomes) of the discrete variable $Y=X_2/X_1$. The entries in the matrix beneath this row (the 3rd to 7th rows) illustrate the possible combinations of these values.

In particular, for y=2 and with reference to Fig. 3, we arrive at

Table 1 Matrix of probabilities $p_Y(y)$: row 1 includes realizations of the division X_2/X_1; row 2 includes the view outlining all possible combinations

$p_Y(y)$	1/200=0.005	1/100=0.01	1/100=0.01	1/20=0.05	1/50=0.02	1/50=0.02	7/40=0.175	0	1/50=0.02	1/5=0.2	1/200=0.005	1/20=0.05	1/10=0.1	1/100=0.01	11/100=0.11	1/20=0.05	1/20=0.05	1/50=0.02	1/40=0.025	1/25=0.04	1/50=0.02	0	1/100=0.01
$y=\frac{x_2}{x_1}$	0.4	0.5	0.6	$0.\bar{6}$	0.75	0.8	1	1.2	1.25	$1.\bar{3}$	1.4	1.5	$1.\bar{6}$	1.75	2	$2.\bar{3}$	2.5	3	3.5	4	5	6	7
$y=\frac{x_2}{5}$															$\frac{10}{5}$			$\frac{15}{5}$		$\frac{20}{5}$	$\frac{25}{5}$	$\frac{30}{5}$	$\frac{35}{5}$
$y=\frac{x_2}{10}$							$\frac{10}{10}$					$\frac{15}{10}$			$\frac{20}{10}$		$\frac{25}{10}$	$\frac{30}{10}$	$\frac{35}{10}$				
$y=\frac{x_2}{15}$				$\frac{10}{15}$			$\frac{15}{15}$			$\frac{20}{15}$			$\frac{25}{15}$		$\frac{30}{15}$	$\frac{35}{15}$							
$y=\frac{x_2}{20}$		$\frac{10}{20}$			$\frac{15}{20}$		$\frac{20}{20}$		$\frac{25}{20}$			$\frac{30}{20}$		$\frac{35}{20}$									
$y=\frac{x_2}{25}$	$\frac{10}{25}$		$\frac{15}{25}$			$\frac{20}{25}$	$\frac{25}{25}$	$\frac{30}{25}$			$\frac{35}{25}$												

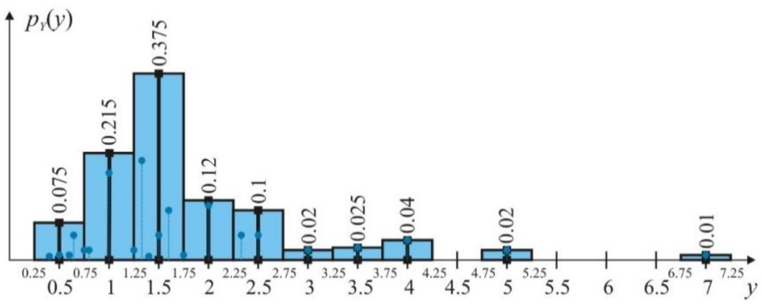

Fig. 5 Histogram of $Y=X_2/X_1$, $\Delta y=0.5$

$$
\begin{aligned}
&p_Y(y=2)=p_{X_1}(5)\cdot p_{X_2}(10)+\\
&+p_{X_1}(10)\cdot p_{X_2}(20)+p_{X_1}(15)\cdot p_{X_2}(30)= \\
&=0.1\cdot 0.1+0.25\cdot 0.4+0.5\cdot 0=0.11.
\end{aligned}
\tag{19}
$$

(ii) In the second step, the transition from the probability mass function of Y to the respective histogram of this variable is carried out in the standard way by allocating the points from the second row of the matrix (see Table 1), along with the corresponding probabilities, to the appropriate interval.

In Table 1 there are thirty non-zero values which correspond to $k_{\max}=N_1\cdot N_2=5\cdot 6=30$. Considering the interval's width $\Delta y=0.5$ yields the resulting histogram depicted in Fig. 5. If Eqs. (11) through (13) are used, it is possible to obtain y_k and $p(y_k)$, which are listed in Table 2. For instance, for the class $y_{i_Y}=2$, where y_k for $k=1$, 12, 23 and 29 are included in accordance with Eq. (11), the probability of that class is

$$
\begin{aligned}
&p(y_{i_Y}=2)=p(y_{k'})+p(y_{k''})+p(y_{k'''})+p(y_{k''''})=\\
&=p(y_1)+p(y_{12})+p(y_{23})+p(y_{29})=\\
&=0.01+0.10+0+0.01=0.12.
\end{aligned}
\tag{20}
$$

Using the values in Table 2, the characteristic values (the midpoint, mean, and mode) are determined for each class.

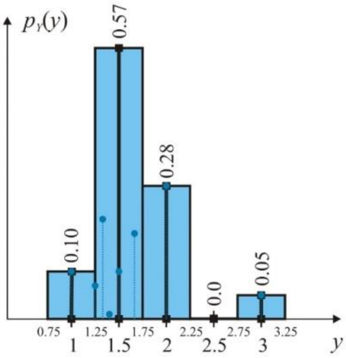

Fig. 6 Histogram of $Y=X_{4_d}/X_{3_d}$ for two correlated variables X_{3_d}, X_{4_d}, $\Delta y=0.5$

A detailed computation will be now performed for two statistically correlated variables X_{3_d}, X_{4_d} for which $Y=X_{4_d}/X_{3_d}$ is expressed by a bivariate histogram in Fig. 4. Similarly to Eq. (19), for the correlated variables it immediately follows with reference to Fig. 4 that

$$
\begin{aligned}
&p_Y(y=2)=p_{X_{3_d}X_{4_d}}(5{,}10)+p_{X_{3_d}X_{4_d}}(10{,}20)+\\
&+p_{X_{3_d}X_{4_d}}(15{,}30)=0.1+0.18+0=0.28.
\end{aligned}
\tag{21}
$$

The result of the probabilistic computation is a histogram with $\Delta y=0.5$ in Fig. 6.

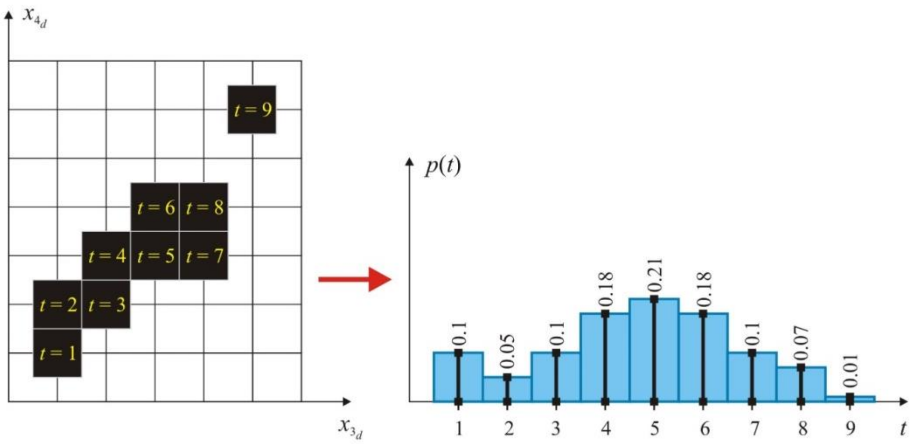

Fig. 7 Transformation of the bivariate histogram in Fig. 4 into the condensed histogram for two correlated random variables X_{3_d}, X_{4_d}

Table 2 Values y_k=$(x_2/x_1)_k$ and $p(y_k)$ used for creation of the histogram Y=X_2/X_1

k	x_1	$p(x_1)$	x_2	$p(x_2)$	y_k=$(x_2/x_1)_k$	$p(y_k)$
1	5	0.10	10	0.10	2.000	0.010
2	10	0.25	10	0.10	1.000	0.025
3	15	0.50	10	0.10	0.667	0.050
4	20	0.10	10	0.10	0.500	0.010
5	25	0.05	10	0.10	0.400	0.005
6	5	0.10	15	0.20	3.000	0.020
7	10	0.25	15	0.20	1.500	0.050
8	15	0.50	15	0.20	1.000	0.100
9	20	0.10	15	0.20	0.750	0.020
10	25	0.05	15	0.20	0.600	0.010
11	5	0.10	20	0.40	4.000	0.040
12	10	0.25	20	0.40	2.000	0.100
13	15	0.50	20	0.40	1.333	0.200
14	20	0.10	20	0.40	1.000	0.040
15	25	0.05	20	0.40	0.800	0.020
16	5	0.10	25	0.20	5.000	0.020
17	10	0.25	25	0.20	2.500	0.050
18	15	0.50	25	0.20	1.667	0.100
19	20	0.10	25	0.20	1.250	0.020
20	25	0.05	25	0.20	1.000	0.010
21	5	0.10	30	0.00	6.000	0.000
22	10	0.25	30	0.00	3.000	0.000
23	15	0.50	30	0.00	2.000	0.000
24	20	0.10	30	0.00	1.500	0.000
25	25	0.05	30	0.00	1.200	0.000
26	5	0.10	35	0.10	7.000	0.010
27	10	0.25	35	0.10	3.500	0.025
28	15	0.50	35	0.10	2.333	0.050
29	20	0.10	35	0.10	1.750	0.010
30	25	0.05	35	0.10	1.400	0.005

As for the computational algorithm, it is advisable to transform the bivariate histogram into the so-called condensed histogram. This conversion is evident from Fig. 7. Obviously, each class with a non-zero probability of the bivariate histogram corresponds to the class t of the condensed histogram with the probability $p(t)$, in compliance with Table 3. Application of condensed histograms will be further discussed in Sec. 3.3.

Table 3 Description of the condensed histogram in Fig. 7 for two correlated variables X_{3_d}, X_{4_d}

t	x_{3_d}	x_{4_d}	$p(t)$=$p(x_{3_d},x_{4_d})_t$
1	5	10	0.10
2	5	15	0.05
3	10	15	0.10
4	10	20	0.18
5	15	20	0.21
6	20	20	0.10
7	15	25	0.18
8	20	25	0.07
9	25	35	0.10
			$\sum$ = 1.0

3.2 Computing the probability of failure for two random variables

The probability of failure for two continuous random input variables can be expressed by the joint probability density function $f_{R,E}(r,e)$ for the load effect, E, and structural resistance, R.

If R does not depend statistically on E, then $f_{R,E}(r,e)=f_R(r)\cdot f_E(e)$, and Eq. (3) assumes the following well-known form

$$p_f = P(R-E \le 0) = = \int_{-\infty}^{\infty}\int_{-\infty}^{r\le e} f_R(r)\cdot f_E(e)\,\mathrm{d}r\,\mathrm{d}e = \int_{-\infty}^{\infty} F_R(e)\cdot f_E(e)\,\mathrm{d}x\,, \tag{22}$$

where $F_R(e)$ is a distribution function of structural resistance.

Passing from discretized continuous functions to their representative values, i.e., from probability density functions to histograms as their surrogates, we formally obtain the same result as that described in Eq. (22). To that end, recall Eq. (8). Setting c_1=1, c_2=-1 and further setting

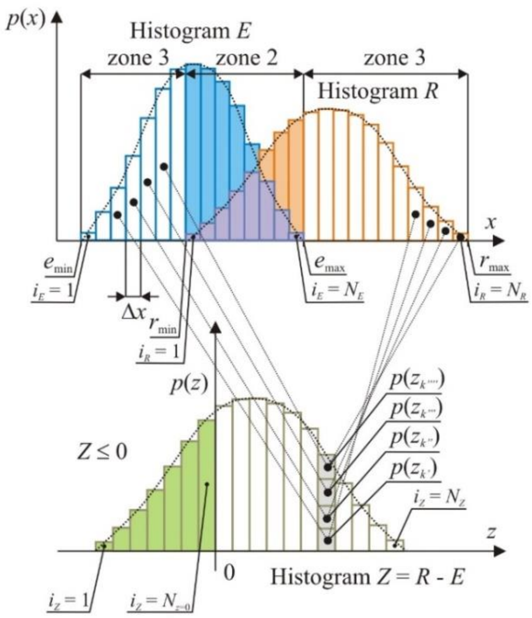

Fig. 8 Basic computation of the reliability function histogram, Z, of two random variables using DOProC

$X_1=R$, $X_2=E$, and finally $Y=Z$ with reference to Eq. (14), the probability becomes

$$p_f = P(R-E \le 0) = F_Z(0) = \sum_j F_R(e_j) \cdot p_E(e_j) \cdot \qquad (23)$$

The *original and most straightforward method* used in DOProC follows Eq. (1) and the computation is performed as outlined in Fig. 8

$$p_f = \sum_{i_Z=1}^{N_{z=0}} p(z_{i_Z}), \qquad (24)$$

where $p(z_{i_Z})$ is the probability of z_{i_Z} in the class $i_Z \in \langle 1, N_{z=0} \rangle$. The class $N_{z=0}$ is limited from above by z=0. In this process, it is necessary to calculate the histogram of Z and all classes for the use of Eqs. (12) and (13), and then p_f from Eq. (24). The number of operations is proportional to the product $k_{\max}=N_E \cdot N_R$, where N_E (N_R) is the total number of classes in the histogram of E (R).

To guarantee that DOProC becomes competitive and the numerical integration runs as effectively as possible, a variety of optimization strategies as been proposed in the sequel. To that end, the histogram of each input variable is split into zones 1, 2 and 3, see Figs. 9 and 10. Zone 1 consists of the intervals which always give for any values of the remaining input variables the result $Z \le 0$. The intervals falling in zone 3 always result for any values of the remaining input variables into $Z>0$. Zone 2, on the other hand, consists of the intervals which for the values of the remaining input variables either may or may not participate in a failure. Hence $Z \le 0$ or $Z>0$. In Fig. 8 zones 2 and 3 are identified in the histograms for E and R. Zone 2 in both histograms R and E can be involved, but not necessarily, in the development of a failure. Zone 3 does not affect the failure. When calculating p_f, the intervals of both histograms in zone 3 can be disregarded. Then the number of operations in this *second approach* is proportional to the product $k^*=(N_{z=0})^2$ if the width of the classes in both histograms is identical (this condition, however, does not need to be fulfilled - this then requires certain modifications in the computation.)

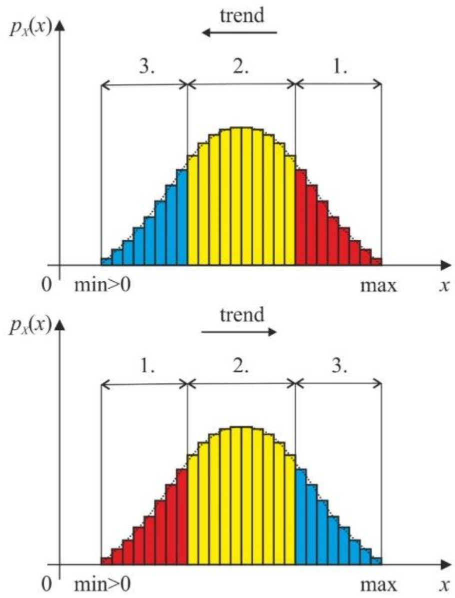

Fig. 9 Histograms of input quantities with monotonic effects on the probability of failure

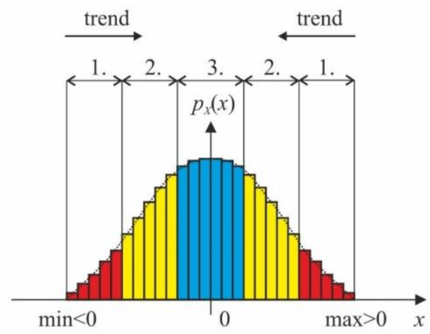

Fig. 10 Histogram of input quantity with non-monotonic effects on the probability of failure

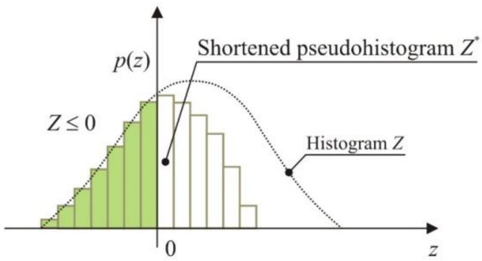

Fig. 11 The final histogram of the reliability function Z obtained by DOProC after zonal optimization-the shortened pseudohistogram Z^*

If the computation is carried out based on Eqs. (12) and (13) for only k^* combinations, the result is an incomplete (shortened) pseudohistogram Z^* as outlined in Fig. 11. The number of classes in this pseudohistogram is less than the number of classes in the histogram Z. The

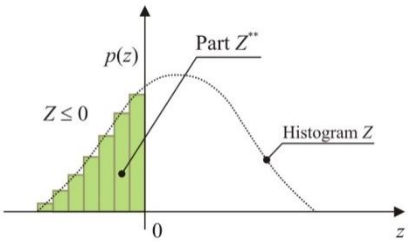

Fig. 12 The final histogram of the safety margin, Z, after trend optimization-the Z^{**} histogram contains only the negative part of the Z histogram

obtained p_f however, complies with Eq. (24) because negative part of the histogram Z is identical with that shortened pseudohistogram Z^*. The *second method* can be used for the computation of p_f only if zones 2 and 3 are specified. This is simple for two random variables because zone 2 in both histograms is limited by the values $r_{\min}$ and $e_{\max}$.

The *third method* for the computation of p_f making use of Eq. (24) consists in evaluating the part Z^{**} of Z, which meets the condition $Z \le 0$; see Fig. 12. In this case, Eq. (24) is modified as follows

$$p_f = \sum_{i_Z=1}^{N_{z=0}} p\left(z_{i_Z}\right) = \sum_{i_f=1}^{N_f} \left[p_E\left(x_{i_f}\right) \cdot \sum_{i_f=1}^{N_f} p_R\left(x_{i_f}\right) \right] \cdot \qquad (25)$$

Evidently, the identical formula emerges from Eq. (23).

The failure can be analyzed explicitly only for two random variables. If there are several random variables, it is recommended to use another approach hereafter termed the trend optimization.

This algorithm requires not only specification of the zones but also determination of a trend in numbering of the classes for histograms of E and R in order to minimize the number of operations. This is because the values z_{i_Z} in Z are not considered if $z_{i_Z}>0$. The trend in the numbering (or the computation method) should be chosen such that the starting zone for each random variable is the one with the lowest number while the final zone is the zone with the highest number; see Figs. 9 and 10.

If a part of the histogram of $Z \le 0$, referred to as Z^{**}, is analyzed, then the number of computational steps is approximately half the number of steps in the second approach described above.

3.3 Statistically non-correlated random variables and a group of statistically correlated random variables

Let Y be a random variable that is a function of non-correlated random variables X_j, where $j \in \langle 1, n \rangle$, and let X_{j_d} be a group of correlated random variables, where $j_d \in \langle 1, m \rangle$. Any constant c_i is regarded as a statistically non-correlated variable with only one class and the probability $p(c_i)=1$.

Consquently

$$Y = f\begin{pmatrix} X_1, X_2, \ldots, X_j, \ldots, X_n, \\ X_{1_d}, X_{2_d}, \ldots, X_{j_d}, \ldots, X_{m_d} \end{pmatrix} \cdot \qquad (26)$$

The statistically correlated variables X_{j_d} enter the computation in an ingenious manner utilizing only classes with a non-zero probability, i.e. incorporating the condensed histograms, see Fig. 7, for which it holds

$$\sum_{t=1}^{T} p(t) = 1 \cdot \qquad (27)$$

The outcome of variable Y for any k^{th} combination of non-correlated variables X_{j,i_j} and for the class t in the group of correlated variables $X_{j_d,t}$ is obtained from

$$y_k = f\begin{pmatrix} x_{1,i_1}, x_{2,i_2}, \ldots, x_{j,i_i}, \ldots, x_{n,i_n}, \\ x_{1_d,t}, x_{2_d,t}, \ldots, x_{j_d,t}, \ldots, x_{m_d,t} \end{pmatrix}_k, \qquad (28)$$

where k is the serial number of classes in the combination of histograms. This number ranges in the interval $k \in \langle 1, k_{\max} \rangle$, where $k_{\max}$ is equal to

$$k_{\max} = N_1 \cdot N_2 \cdot \ldots \cdot N_j \cdot \ldots \cdot N_n \cdot T, \qquad (29)$$

where N_j is the number of classes (intervals) of the non-correlated variable X_j. The probability of occurrence of y_k for the k^{th} combination of classes of non-correlated random variables X_j and for the order t of the group of correlated random variables X_{j_d} is equal to the product

$$p(y_k) = \begin{bmatrix} p\left(x_{1,i_1}\right) \cdot p\left(x_{2,i_2}\right) \cdot \ldots \\ \ldots \cdot p\left(x_{j,i_j}\right) \cdot \ldots \cdot p\left(x_{n,i_n}\right) \cdot p(t) \end{bmatrix}_k \cdot \qquad (30)$$

Any value y_k from the interval y_{i_Y} has the probability $p(y_k)$. This value belongs to the class i_Y if the inequality in Eq. (11) is met.

Probability $p(y_{i_Y})$ is the sum of probabilities of all values y_k falling into the class y_{i_Y} of the random variable Y as stipulated in Eq. (13). Then

$$\sum_{i_Y=1}^{N_Y} p\left(y_{i_Y}\right) = 1 \cdot \qquad (31)$$

The procedure with statistically non-correlated random variables and a group of statistically correlated random variables will be elucidated in the following simple example

$$Y = g\left(X_1, X_2, X_{3_d}, X_{4_d}\right) = \frac{X_2 + X_{4_d}}{X_1 + X_{3_d}} \cdot \qquad (32)$$

The final histogram for Y is obtained by the approach based on Eq. (30); see Fig. 13. Fig. 14 shows the results for the class width Δy=0.5 and for non-correlated X_1, X_2 from Fig. 3, and for the correlated X_{3_d}, X_{4_d} expressed by a bivariate histogram/condensed histogram in Figs. 4 and 7.

It follows from Eq. (29) that the number of operations needed for computation of Y is proportional to the number of non-correlated variables n, to the number of classes N_j of each non-correlated variable, and to the number T of non-zero classes in the group of random correlated variables X_{j_d}. For instance, if n=10, N_1=N_2=... ...=N_j=...=N_n=N=256 and T=16, then, in compliance with Eq. (29), $k_{\max}=256^{10} \cdot 10 = 1.2089 \cdot 10^{25}$. This number is rather high, and the computation would be demanding even with high-performance computers. In order to get over this obstacle DOProC makes use of optimization strategies already

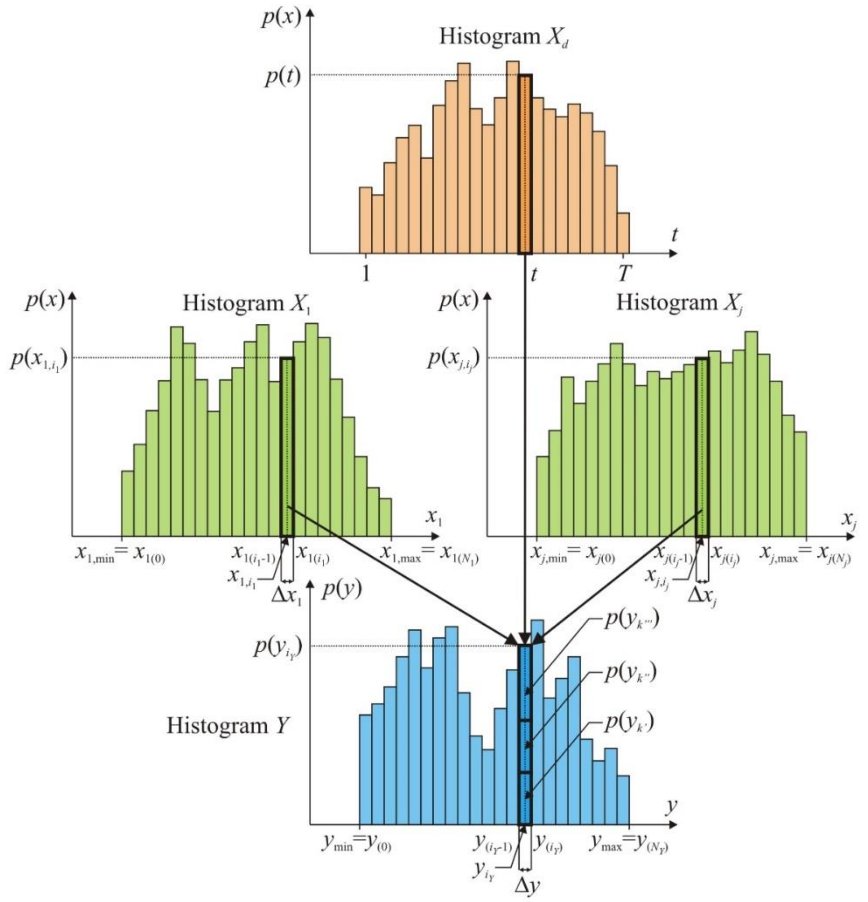

Fig. 13 Principles of numerical operations with n statistically independent random variables X_1, X_2, ..., X_i, ..., X_n and with the group of m statistically correlated random variables X_{1_d}, X_{2_d}, ..., X_{j_d}, ..., X_{m_d} expressed using the so-called condensed histogram

outlined in Sec. 3.2 and summarized in the following Sec. 3.4.

3.4 Optimization strategies of probabilistic computations

The purpose of the optimization strategies are to minimize the number of computational operations. The following overview contains a brief description of the nature already developed optimization, while all possibilities are not still completely exhausted. The basic principles of the optimization techniques have been outlined only in (Janas *et al*. 2015, in Czech).

3.4.1 Operation with random variables

• Interval optimization means that the width of intervals N_j for random variables X_j are optimized in order to reduce the number of operations (the computer time needed) and to acquire the specified accuracy. Each random variable may otherwise affect the result, which is differently sensitive to it. This means that for more sensitive variables the number of classes is generally higher, for less sensitive ones it is lower. A limit value is a constant that has only one value (class). Reducing the number of classes of each variable is limited by the required accuracy of the calculation. A supporting tool is a sensitivity analysis.

Detailed interval optimization has not been published yet.

• Grouping - the purpose is to reduce the number of random variables n. It means that equivalent histograms are created for random variables as functions of the input random variables. In DOProC, any input random variable can be included in one group only and not in other groups.

In the example above, let the number of statistically independent random variables be reduced using grouping from n=10 to n'=6 and the number of classes N=256 by interval optimization on average by 40 %, i.e., N'=154 (the decrease will not be the same in each class N_j in the practical example, because each random input variable affects the result differently). Then, the number of computational operations $k_{\max}'=2.134\cdot10^{14}$ will be approx $5.665\cdot10^{10}$times less than the original value, what is very important for the calculation time.

3.4.2 Optimization operations for p_f

• Zonal optimization takes into account the contribution of zones of the input random variables X_j and X_{j_d} with respect to p_f. Individual intervals are properly placed in the zones of all respective histograms as described in Sec. 3.2.

• Trend optimization is based on the zonal optimization. It is necessary to specify for each input random variable the direction of changes, i.e., to predict the trend, which yield to

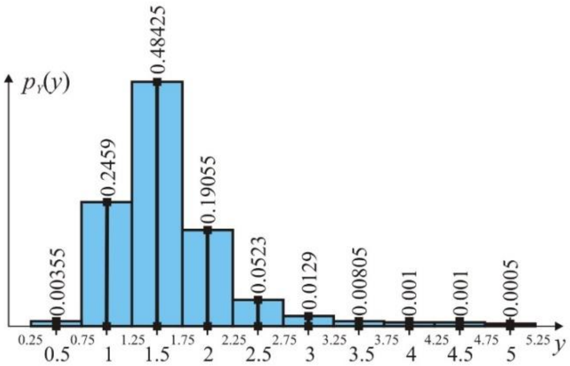

Fig. 14 Histogram of $Y = (X_2 + X_{4_d})/(X_1 + X_{3_d})$ for X_1, X_2 and two correlated variables X_{3_d}, X_{4_d}, $\Delta y = 0.5$

failures. The direction of the trend is given by the zone numbers 1, 2, or 3 for each histogram of the input random variables (see Figs. 9 and 10 and the text in Sec. 3.2). It is recommended to head from the boundary of zone 2 for zone 3. In other words, from the classes of histograms substantially affecting the probability of failure, p_f, to those with negligible or zero impact on it. If the value of a random variable tends to increase the value of safety margin, Z, further numerical combinations become pointless (Z may turn out to be negative with no impact on the value of p_f). The number of combinations is then reduced to an indispensable minimum.

Zonal and trend optimization can reduce the number of computational operations approximately by half from our experience up to now. Their effect is not as great as using grouping and interval optimization, but their significance is not negligible. All the optimization procedures can be combined.

Both optimization procedures have been published early in the development of the method briefly in (Janas *et al.* 2010).

3.4.3 Parallelizing

Some parts of the DOProC computations can be performed simultaneously in computers with two or more CPUs/cores. It is advisable to split the scope of operations into several parts. The number of those parts is the same as the number of available computational units. After parallel computations have been carried out, the partial results are compiled to create the final histogram of the quantity of interest, for instance, Z for probabilistic assessment. Parallelized computation can be combined particularly with interval optimization and with grouping.

The possibilities of using the described method for parallel computations were briefly described in conference papers (Krejsa *et al.* 2016d, Krejsa *et al.* 2016e).

4. Using DOProC

4.1 Basic DOProC applications

To demonstrate the probabilistic computation using the described algorithm, the ProbCalc software application

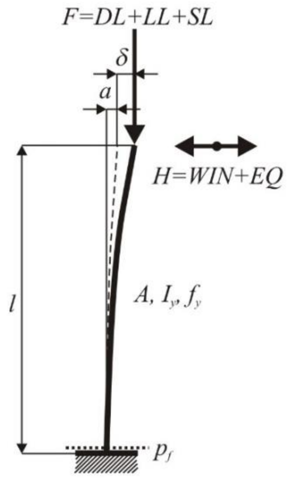

Fig. 15 Static scheme of the column loaded with a system of random variable forces

(Krejsa *et al.* 2014), developed for the practical use of DOProC, was used. If the goal is the reliability assessment of a structure, as well as the probability assessment in accordance with the Eurocodes (EC), the user can utilize the histogram of the analyzed reliability function, Z. The software, which is still under development, includes the optimization utilities (interval, zonal, trend optimization and parallelizing) discussed above, making it possible to reduce significantly the number of computational steps and, in turn, the computer time.

One shortcoming of the ProbCalc code is its somewhat demanding user interface. This is due to the requirement that the code be universally applicable; but this challenge may be eliminated by creating custom-made application software, e.g., for the probabilistic design and reliability assessment of the anchoring reinforcement of long mine workings and underground constructions (Krejsa *et al.* 2013) or for the probabilistic computation of fatigue cracks in steel structures and bridges subjected to cyclical loading (Krejsa *et al.* 2017, 2016c).

4.2 Example with analysis of optimization steps

The importance and rate of optimization steps are presented in the following example solved using ProbCalc code. The steel load-carrying component made from an HE300B profile is loaded by 2D bending and axial loads (simple compression) pursuant to the ultimate limit state and second-order theory. Attention was also paid to the impacts of initial imperfections. The static scheme of the structure is shown in Fig. 15.

The computational model is defined analytically and is based on the direct stiffness method. The maximum horizontal displacement at the top of the column is expressed analytically by (Marek *et al.* 1995)

$$\delta = \frac{WIN + EQ + \frac{a}{l}\cdot\left(DL + LL + SL\right)}{\frac{DL + LL + SL}{l \cdot K}}, \tag{33}$$

where WIN, EQ, DL, LL and SL represents the individual

Table 4 Description of input variables in the example.

Input variable	Minimum	Maximum	N_j	Histogram
Column height l	6 m	-	-	-
Yield stress f_y	200 MPa	435 MPa	217	FY235-01
Dead load DL	260 kN	320 kN	256	DEAD1*
Long-lasting load LL	0 kN	120 kN	256	LONG1*
Short-lasting load SL	0 kN	75 kN	256	SHORT1*
Wind WIN	-45 kN	45 kN	256	WIND1*
Earthquake EQ	-30 kN	30 kN	256	EARTH*
Geometric imperfections a	-30 mm	30 mm	16	IMP016
Variability of cross-section properties A, W_y and I_y	-	-	10^3	3DHE300B**
Cross-sectional area A	13076 mm^2	16048 mm^2	10	1DHE300BA
Cross-section modulus W_y	$1.44 \cdot 10^6$ mm^3	$1.77 \cdot 10^6$ mm^3	10	1DHE300BW
Moment of inertia I_y	$2.19 \cdot 10^8$ mm^4	$2.70 \cdot 10^8$ mm^4	10	1DHE300BI

* Histograms are taken from (Marek et al. 1995).
** 3D histogram was used for computation considering the statistical dependence of cross-section properties A, W_y and I_y. Histograms marked as 1DHE300BA, 1DHE300BW, and 1DHE300BI are based on this, as well.

Table 5 Results of analysis of the optimization of probabilistic reliability assessment considering statistical independence of input random variables, specifications of reliability classes RC, and consequence classes CC according to EN 1990

Optimization used	Computer time*	p_f	RC/CC
Without optimization	>> 24 hour	computation was not performed	
Grouping of output quantities	>> 24 hour	computation was not performed	
Grouping of input quantities	>> 24 hour	computation was not performed	
Grouping of input variables, zonal optimization	>> 24 hour	computation was not performed	
Grouping of input variables, interval optimization	2:33:22 hours	$5.6736 \cdot 10^{-5}$	RC2/CC2
Grouping of input variables, interval and zonal optimization	2:17:29 hours	$5.5559 \cdot 10^{-5}$	RC2/CC2
Grouping of input variables, interval, zonal and the trend optimization	1:20:43 hours	$5.5559 \cdot 10^{-5}$	RC2/CC2
Grouping of input and output variables	37:05 min.	$5.1330 \cdot 10^{-5}$	RC2/CC2
Grouping of input and output variables, zonal optimization	28:29 min.	$5.2469 \cdot 10^{-5}$	RC2/CC2
Grouping of input and output variables, parallelization (2~cores)	9:06 min.	$5.1330 \cdot 10^{-5}$	RC2/CC2
Grouping of input and output variables, interval optimization	4:30 min.	$5.0480 \cdot 10^{-5}$	RC2/CC2
Grouping of input and output variables, zonal and interval optimization	3:35 min.	$5.8711 \cdot 10^{-5}$	RC2/CC2
Grouping of input and output variables, parallelization (8~cores)	3:20 min.	$5.1330 \cdot 10^{-5}$	RC2/CC2

* The computation was performed on an Intel(R) Core(TM) processor i7-2600 CPU@3.40~GHz.

load components according to the scheme in Fig. 15, a is the initial imperfection in column, l is the height of the column and K is defined by

$$K = \frac{\tan(l) \cdot \sqrt{\dfrac{DL + LL + SL}{E \cdot I_y}}}{l \cdot \sqrt{\dfrac{DL + LL + SL}{E \cdot I_y}}} - 1, \tag{34}$$

where E is Young's modulus of steel and I_y is the moment of inertia in a relevant load plane.

Then, the bending moment M in the critical cross-section - in the fixed support of the column, and the normal stress in the outer fibers of this cross-section σ are subsequently equal to

$$M = \frac{\delta \cdot (1 + K)}{K} \cdot (DL + LL + SL) \tag{35}$$

and

$$|\sigma| = \frac{DL + LL + SL}{A} + \frac{|M|}{W_y} = (DL + LL + SL) \cdot \left[\frac{1}{A} + \frac{|\delta| \cdot (1 + K)}{K \cdot W_y} \right]. \tag{36}$$

The example includes 10 random input variables (vertical load: dead, long-lasting, and short-lasting; the horizontal load: wind and earthquake; variability of cross-section variables, yield stress, and geometric imperfections), seven of which are statistically independent

Table 6 Number of computing steps for probabilistic reliability assessment of individual types of optimization steps used considering statistical independence of input random variables

Optimization	Total number of computational steps
Without optimization	$4.13554 \cdot 10^{18}$
Grouping of output quantities	$1.75235 \cdot 10^{16}$
Grouping of input quantities	$2.27541 \cdot 10^{11}$
Grouping of input variables, zonal optimization	$1.83501 \cdot 10^{11}$
Grouping of input variables, interval optimization	$4.59571 \cdot 10^{9}$
Grouping of input variables, interval, zonal optimization	$3.38479 \cdot 10^{9}$
Grouping of input variables, interval, zonal and the trend optimization	$2.04303 \cdot 10^{9}$
Grouping of input and output variables	$1.04858 \cdot 10^{9}$
Grouping of input and output variables, zonal optimization	$8.22473 \cdot 10^{8}$
Grouping of input and output variables, interval optimization	$1.35032 \cdot 10^{8}$
Grouping of input and output variables, zonal and interval optimization	$1.06021 \cdot 10^{8}$

Table 7 Results of analysis of the optimization of probabilistic reliability assessment considering correlated cross-section properties, specifications of reliability classes RC, and consequence classes CC according to EN 1990

Optimization used	Computer time*	p_f	RC/CC
Without optimization	>> 24 hour	computation was not performed	
Grouping of output quantities	>> 24 hour	computation was not performed	
Grouping of input quantities	3:52:03 hours	$5.2442 \cdot 10^{-5}$	RC2/CC2
Grouping of input and output variables	1:09 min.	$5.2467 \cdot 10^{-5}$	RC2/CC2
Grouping of input and output variables, parallelization (2~cores)	19 sec.	$5.2467 \cdot 10^{-5}$	RC2/CC2
Grouping of input and output variables, parallelization (8~cores)	9 sec.	$5.2467 \cdot 10^{-5}$	RC2/CC2

* The computation was performed on an Intel(R) Core(TM) processor i7-2600 CPU@3.40~GHz.

and three are random variables which form a group of correlated random variables. For details, see Table 4.

The computation was performed for the statistically non-correlated random input variables as well as for the statistically correlated cross-section characteristics. The obtained p_f, computer time needed and the final classification into reliability classes, and consequences with statistically non-correlated variables are listed in Table 5. The number of operations, which depends on the number of optimization steps, is given in Table 6. A similar analysis was carried out with the statistically correlated cross-section

Table 8 Number of computation steps in a probabilistic reliability assessment of the individual types of optimization steps used considering correlated cross-section properties

Optimization	Total number of computational steps
Without optimization	$9.50648 \cdot 10^{16}$
Grouping of output quantities	$5.43227 \cdot 10^{14}$
Grouping of input quantities	$5.68852 \cdot 10^{9}$
Grouping of input and output variables	$3.25059 \cdot 10^{7}$

Table 9 Results of probabilistic reliability assessment by the Monte Carlo method for one million simulation steps, specifications of reliability classes RC, and consequence classes CC according to EN 1990

Number of computation	p_f	RC/CC
1	$5.1 \cdot 10^{-5}$	RC2/CC2
2	$5.7 \cdot 10^{-5}$	RC2/CC2
3	$5.8 \cdot 10^{-5}$	RC2/CC2
4	$6.8 \cdot 10^{-5}$	RC2/CC2
5	$5.9 \cdot 10^{-5}$	RC2/CC2
6	$6.3 \cdot 10^{-5}$	RC2/CC2
7	$4.7 \cdot 10^{-5}$	RC2/CC2
8	$4.9 \cdot 10^{-5}$	RC2/CC2
9	$6.5 \cdot 10^{-5}$	RC2/CC2
10	$5.2 \cdot 10^{-5}$	RC2/CC2

characteristics, which were expressed in a triple histogram (for results, see Tables 7 and 8).

It follows from the results that the optimization techniques (such as grouping, interval optimization, zonal and trend analysis, and parallelizing) may considerably reduce the computation time at the expense of minor impact on the result without any influence on the class/level of reliability or the class of consequences in accordance with EN 1990.

In order to verify the degree of accuracy of the DOProC, the probabilistic analysis was performed for the specified load-carrying element using the Monte Carlo method. In order to have objective comparison, results based on Monte Carlo simulations serve as an independent reference. One million simulations were repeated ten times; for partial results and comparison, see Table 9. The computer time needed for each computation was 24.5 seconds. The time of these computations is, of course, dependent on the way how the algorithm is implemented and on the computer performance. In this case, the computation was performed on an Intel(R) Core(TM) processor i7-2600 CPU@3.40 GHz using the Anthill code (Marek *et al.* 1995).

Results were slightly different when using DOProC method. If compared results from MC with DOProC (for instance, $p_f = 5.1330 \cdot 10^{-5}$ when grouping the input and output variables) the final p_f obtained by the Monte Carlo method is comparable (the average p_f obtained in 10 repetitions was $5.69 \cdot 10^{-5}$).

4.3 Analysis of results

Based on the results stored in Tables 5-9 one can conclude:

• Parallelized computations do not affect accuracy while significantly reduce the computer time. It is the most efficient strategy.

• The spread in p_f achieved utilizing different techniques in DOProc and stored in Table 5 is smaller than that obtained using MC (Table 9).

• Applying multivariate histograms to express the statistical dependence of input variables appears to be duly effective for it reduces the number of operations as well the computer time.

• It is worth noting that as oppose to MC-based approaches the application of DOProC does not come across any substantial complications when incorporating statistically dependent input variables.

5. Conclusions

The paper introduced the development of probabilistic methods and the use of such methods in the reliability assessment of structures. Particular attention was paid to a new method, DOProC, which is still under further development. DOProC appears to be a very efficient method whose solutions suffer only from numerical errors and errors resulting from discretization of input and output quantities. One shortcoming of DOProC is the considerable increase in the required computer time for probabilistic operations for models with many random variables. The maximum number of random variables depends on the complexity of this model and, importantly, whether it is possible to use any of the described optimization steps.

DOProC has proved to be suitable not only for reliability assessment, but also for other probabilistic computations. A simple example was deliberately used to illustrate the calculation procedure. Optimization and parallelization of the method predetermine DOProC for the solution of the real problems of engineering practice, tentatively in the range typical of the standard analytical methods. For these kinds of computations ProbCalc can also be used (Krejsa *et al.* 2014).

The possible applications for DOProC are far from exhausted. Other areas of investigation could include reliability assessment of structural systems, development of numerical approaches to make DOProC more efficient, and the use of parametric distributions for creating multi-dimensional histograms.

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