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# Vibration analysis of micro composite thin beam based on modified couple stress

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**Abstract.** In this article, analytical solution for free vibration of micro composite laminated beam on elastic medium based on modified couple stress are presented. The surrounding elastic medium is modeled as the Winkler elastic foundation. The governing equations and boundary conditions are obtained by using the principle of minimum potential energy for Euler-Bernoulli beam. For investigating the effect of different parameters including material length scale, beam thickness, some numerical results on different cross ply laminated beams such as (90,0,90), (0,90,0), (90,90,90) and (0,0,0) are presented on elastic medium. Free vibration analysis of a simply supported beam is considered utilizing the Fourier series. Also, the fundamental frequency is obtained using the principle of Hamilton for four types of cross ply laminations with hinged-hinged boundary conditions and different beam theories. The fundamental frequency for different thin beam theories are investigated by increasing the slenderness ratio and various foundation coefficients. The results prove that the modified couple stress theory increases the natural frequency under the various foundation for free vibration of composite laminated micro beams.

**Keywords:** composite laminated beam; modified couple stress theory; elastic foundation; generalized differential quadrature

## 1. Introduction

In recent years, researches on composite structures in the order of micron and sub-micron scales have been growing, rapidly. The experimental researches by Herakovich in 2012 show that the material strength and stiffness in microscale are higher than their bulky materials which can be explained by the size or scale effects. Since, theories of classical continuum mechanics overlook the material length scale parameter, so they are not appropriate for microscale applications, and the use of related nonclassical theories such as the couple stress theory is necessary many theories have been introduced such as the couple stress theory Toupin (1962), Mindlin and Tiersen (1962), Mindlin (1963, 1964), Yang *et al*. (2002), the modified couple stress theory By Park *et al*. (2006), Kocaturk *et al*. (2013), Jahangiri *et al*. (2015), strain gradiant in 1998 by Nix and Gao, and the nonlocal elasticity by Eringen in (1972, 1983). Gürses *et al* (2012) investigated the effects of nonlocal parameter, mode numbers, sector angle and radius ratio on the vibration frequencies in detail.

Modified strain gradient for non-classical sinusoidal plate model new non-classical microstructure-dependent sinusoidal plate model is developed based on the modified strain gradient by Akgöz and Civalek (2015).

Recent proposed modified couple stress theory studied the isotropic Euler-Bernoulli beam by Park and Gao (2006). Kapania and Raciti (1989) reviewed and compared classical laminated composite beams theories. Ghadiri *et al* (2016) investigated thermal stress of a simply supported micro laminated composite beam based on modified couple stress theory both analytically and nummerically. Mohammad-Abadi and Daneshmehr (2015) have studied free vibration analysis of Euler-Bernoulli, Timoshenko and Reddy beams based on the modified couple stress theory for several boundary conditions and they considered the free vibration analysis of a simply supported beam analytically as well as GDQ method.

Investigations on laminated microstructures with defects such as micro-cracks and impurities in laminated microstructures are of attention according to widely usage of composite materials in different scales. In this regard, Chen, and Li (2011) have suggested a new model for laminated composites. In 2011, they investigated bending for simply supported laminated composite beams with the first order shear deformation and solved the governing equations analytically. Bending of simply supported laminated composite Reddy beams were studied analytically by Chen and Sze (2012). Chen and Li (2013) studied free behavior of a simply supported laminated composite Timoshenko beam based on the new modified couple stress theory. Static bending and buckling behaviors of microbeams are investigated by Akgöz and Civalek (2015).

For considering the size effect many researchers have been concentrated on the beam theories in the recent years. Some papers have been published on attempts of developing the couple stress beam models and applying them to examine nanobeams and other small beam-like members/devices. Vibration analysis of functionally graded

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micro beams based on modified couple stress theory was done by Tounsi *et al.* (2015). The material length constants are predicted in the rotational equilibrium equations in the couple-stress theory. Miniature devices such as actuators or sensors (in small scale engineering applications) in micro-electromechanical systems (MEMS) and nano-electromechanical systems (NEMS) are often in the different forms with beams, plates and membranes which shows the importance of these models based on the couple-stress and strain-gradient. A zeroth-order shear deformation theory for free vibration analysis of functionally graded (FG) nanoscale plates resting on elastic foundation based on using the nonlocal differential constitutive expressions was investigated by Tounsi and *et al.* (2016). In recent years many researches have been accomplished around FGMs and composite materials and related analysis have been considered, Ait Yahia *et al*. (2015), Belabed *et al*. (2014), Bellifa *et al*. (2016), Bennoun *et al*. (2016), Bouderba *et al*. (2013), Bourada *et al*. (2015), Bousahla *et al*. (2014), Hamidi *et al*. (2015), Houari *et al*.(2016), Mahi *et al*. (2015), Tounsi *et al*. (2016), Zemri *et al*. (2015), Beldjelili *et al*. (2016), Attia *et al*. (2015), Bouderba *et al*. (2016), Bousahla *et al* (2016), Draiche *et al*.(2016), Chikh *et al*. (2017), Bessaim *et al*. (2013).

The size of elements in micro- and nano-electromechanical systems (MEMS and NEMS) is very small and as a result, non-classical continuum theories such as the modified couple stress theory are appropriate for modeling these material behaviors.

Beam structures are frequently found to be resting on the earth in various engineering applications. These include railway lines, geotechnical areas, highway pavement, building structures, offshore structures, transmission towers and transversely supported pipe lines. This motivated many researchers to analyze the behavior of beam structures on various elastic foundations.

Studies on homogeneous isotropic beams resting on variable Winkler foundation are found in various papers. Zhou (1993) studied vibration of a uniform single span beam resting on variable Winkler elastic foundation. Employing the finite element method, Thambiratnam and Zhuge (1996) studied the free vibration analysis of beams resting on elastic foundations. Au *et al*. Zheng (1999) considered an Euler-Lagrangian approach with C1 continuity functions for the vibration and stability analyses of non-uniform beams resting on elastic foundation. For two elastic foundation-parameters, Matsunaga (1999) studied the linear vibration of nonprismatic beams resting on two-parameter elastic foundations and non-homogenous microbeams embedded in an elastic medium is investigated based on modified strain gradient elasticity theory in conjunctions with various beam theories by Akgöz (2015). Ying *et al*. Lu and Chen (2008) presented solutions for bending and free vibration of FG beams resting on a Winkler-Pasternak elastic foundation based on the two-dimensional theory of elasticity. However, works related to sandwich beams or composite laminated beams on variable Winkler foundation is limited in the literature. In some biomechanical, biomedical and MEMS applications, a microbeam is found to be embedded in elastic matrix. In addition thermo-mechanical vibration analysis of functionally graded (FG) beams and functionally graded sandwich (FGSW) beams are presented by Pradhan in (2009).

In this study free vibration analyses of micro-sized composite laminated beams embedded in an elastic medium have been presented by using the modified couple stress theory.

## 2. The modified couple stress theory

The modified couple stress theory was presented by Yang *et al*. (2002), in which the strain energy $U$ of the isotropic linear elastic material occupying region $V$ can be written as

$$U = \frac{1}{2}\int_{v} (\sigma_{ij} : \varepsilon_{ij} + m_{ij} : \chi_{ij}) dV \quad (i, j = 1, 2, 3) \tag{1}$$

Where the strain tensor $\varepsilon_{ij}$, stress tensor $\sigma_{ij}$, curvature tensor $\chi_{ij}$ and deviatory part of the couple stress tensor $m_{ij}$ can be defined as

$$\varepsilon_{ij} = \frac{1}{2}\left(\nabla u + \left(\nabla u\right)^{T}\right) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) = \varepsilon_{ji} \tag{2}$$

$$\sigma_{ij} = \lambda tr(\varepsilon_{ij})\delta_{ij} + 2\mu\varepsilon_{ij} \tag{3}$$

$$\chi_{ij} = \frac{1}{2}\left(\nabla\theta + \left(\nabla\theta\right)^{T}\right) = \frac{1}{2}\left(\theta_{i,j} + \theta_{j,i}\right) = \chi_{ji} \tag{4}$$

$$m_{ij} = 2\zeta^{2}\mu\chi_{ij} \tag{5}$$

The components of rotation vector are given by

$$\theta = \tfrac{1}{2} curl\left(u\right) \tag{6}$$

The Lame's constants are $\mu$ and $\lambda$, where $\mu$ also is known as the shear modulus which is illustrated by $G$, and $u$ shows the components of the displacement vector. $E$, $\nu$ and $B$ denote as Young's modulus, Poisson's ratio and the modulus of curvature or bending, respectively. $\zeta$ is the material length scale parameter and is explained as the material property of the couple stress theory and has the dimension of length.

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \tag{7}$$

$$\mu = G = \frac{E}{2(1+\nu)} \tag{8}$$

$$\zeta^{2} = \frac{2(1+\nu)B}{E} = \frac{B}{G} \tag{9}$$

Constitutive relations of isotropic beams in Eq. (3) could not be written for composite materials. So, Cauchy stress-strain relation for $k$th ply of the laminated composite beam in local coordinate system ($x'$,$y'$,$z$) which $x'$ in local coordinate system shows the fiber's direction and can be rewritten as follows

$$\sigma'^k=\begin{Bmatrix}\sigma'^{\,k}_{x'}\\ \sigma'^{\,k}_{y'}\\ \tau^k_{x'y'}\\ \tau^k_{x'z}\\ \tau^k_{y'z}\end{Bmatrix}=C_1^k\varepsilon'^k=C_1^k\begin{Bmatrix}\varepsilon_{x'}\\ \varepsilon_{y'}\\ \gamma_{x'y'}\\ \gamma_{x'z}\\ \gamma_{y'z}\end{Bmatrix}=C_1^k\begin{Bmatrix}\partial u_1'/\partial x'\\ \partial u_2'/\partial y'\\ \partial u_1'/\partial y'+\partial u_2'/\partial x'\\ \partial u_1'/\partial z+\partial u_3/\partial x'\\ \partial u_2'/\partial z+\partial u_3/\partial y'\end{Bmatrix}\quad(10)$$

The stiffness matrix for $k$th ply of laminated composite beam in local coordinate system can be simplified as

$$C_1^k=\begin{bmatrix}C_{11}^k & C_{12}^k & 0 & 0 & 0\\ C_{21}^k & C_{22}^k & 0 & 0 & 0\\ 0 & 0 & C_{66}^k & 0 & 0\\ 0 & 0 & 0 & C_{44}^k & 0\\ 0 & 0 & 0 & 0 & C_{55}^k\end{bmatrix}\quad(11)$$

Where

$$C_{12}^k=C_{21}^k=\frac{\nu_{12}^kE_2'}{1-\nu_{12}^k\nu_{21}^k},\quad C_{22}^k=\frac{E_2'}{1-\nu_{12}^k\nu_{21}^k},$$
$$C_{11}^k=\frac{E_1'}{1-\nu_{12}^k\nu_{12}^k}$$
$$C_{66}^k=G_{12}^k,\quad C_{44}^k=G_{13}^k,\quad C_{55}^k=G_{23}^k,\quad(12)$$
$$\nu_{21}^k=\frac{\nu_{21}^kE_2^k}{E_1^k}$$

While $E_1^k$ and $E_2^k$ are elastic moduli, $G_{12}^k$, $G_{13}^k$ and $G_{23}^k$ are shear moduli and $\nu_{12}^k$ and $\nu_{21}^k$ are Poisson's ratios for the $k$th ply; the couple stress-curvature tensor can be written as

$$m'^k=\begin{Bmatrix}m^k_{x'}\\ m^k_{y'}\\ m^k_{x'y'}\\ m^k_{y'x'}\\ m^k_{y'z}\\ m^k_{zy'}\\ m^k_{x'z}\\ m^k_{zx'}\end{Bmatrix}=C_2^k\chi'^k=C_2^k\begin{Bmatrix}\chi_{x'}\\ \chi_{y'}\\ \chi_{x'y'}\\ \chi_{y'x'}\\ \chi_{y'z}\\ \chi_{zy'}\\ \chi_{x'z}\\ \chi_{zx'}\end{Bmatrix}=\begin{bmatrix}A' & 0\\ 0 & B'\end{bmatrix}\begin{Bmatrix}\partial\theta_{x'}/\partial x'\\ \partial\theta_{y'}/\partial y'\\ \partial\theta_{y'}/\partial x'\\ \partial\theta_{x'}/\partial y'\\ \partial\theta_{z}/\partial y'\\ \partial\theta_{y'}/\partial z\\ \partial\theta_{z}/\partial x'\\ \partial\theta_{x'}/\partial z\end{Bmatrix}\quad(13)$$

$\theta_{x'}$ Signifies the rotation about the $x'$-axis, $\theta_{y'}$ shows the rotation about the $y'$-axis $\theta_{z'}$ defines the rotation about the $z'$-axis and $A'$ and $B'$ are introduced as

$$A'=\begin{bmatrix}2C_{55}^k\zeta_{kb}^2 & 0 & 0 & 0\\ 0 & 2C_{44}^k\zeta_{km1}^2 & 0 & 0\\ 0 & 0 & 2C_{44}^k\zeta_{kb}^2 & 2C_{55}^k\zeta_{km1}^2\\ 0 & 0 & 2C_{66}^k\zeta_{kb}^2 & 2C_{55}^k\zeta_{km2}^2\end{bmatrix}\quad(14)$$

$$B'=\begin{bmatrix}2C_{66}^k\zeta_{km1}^2 & 2C_{44}^k\zeta_{km1}^2 & 0 & 0\\ 2C_{66}^k\zeta_{km1}^2 & 2C_{44}^k\zeta_{km1}^2 & 0 & 0\\ 0 & 0 & 2C_{66}^k\zeta_{kb}^2 & 2C_{55}^k\zeta_{km1}^2\\ 0 & 0 & 2C_{66}^k\zeta_{kb}^2 & 2C_{55}^k\zeta_{km2}^2\end{bmatrix}\quad(15)$$

Dissimilar to the isotropic beams, the laminated composite beams have three material length scale parameters. $\zeta_{km1}$ is the $y'$-direction material length scale parameter concerns with $\partial\theta_{y'}/\theta y'$, $\partial\theta_{x'}/\theta y'$ and $\partial\theta_{z'}/\theta y'$, $\zeta_{km2}$ is the $z$-direction material length scale parameter related to $\partial\theta_{y'}/\theta z$ and $\partial\theta_{x'}/\theta z$ and $\zeta_{kb}$ is the $x'$-direction material length scale parameter concerns with $\partial\theta_{x'}/\partial x'$, $\partial\theta_{y'}/\partial x'$ and $\partial\theta_{z'}/\partial x'$. This is obvious that these curvatures are not symmetric; however, the couple stress moments are symmetric. For isotropic materials, the couple stress moments and curvatures are symmetric.

Looking at relations introduced in the local coordinate system, these matrices can be defined as

$$T_1^k=\begin{bmatrix}m^2 & n^2 & 2mn & 0 & 0\\ n^2 & m^2 & -2mn & 0 & 0\\ -mn & mn & m^2-n^2 & 0 & 0\\ 0 & 0 & 0 & m & n\\ 0 & 0 & 0 & -n & m\end{bmatrix}\quad(16)$$

$$T_2^k=\begin{bmatrix}m^2 & n^2 & mn & mn & 0 & 0 & 0 & 0\\ n^2 & m^2 & -mn & -mn & 0 & 0 & 0 & 0\\ -mn & mn & m^2 & -n^2 & 0 & 0 & 0 & 0\\ -mn & mn & -n^2 & m^2 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & m & 0 & -n & 0\\ 0 & 0 & 0 & 0 & 0 & m & 0 & -n\\ 0 & 0 & 0 & 0 & n & 0 & m & 0\\ 0 & 0 & 0 & 0 & 0 & n & 0 & m\end{bmatrix}\quad(17)$$

Applying $m=\cos\psi^k$ and $n=\sin\psi^k$ while $\psi^k$ is fiber angel with respect to the $x$-axis, the couple stress-curvature tensor and the stress-strain relation for the $k$th ply of laminated composite beams in the global coordinate system are defined as

$$\sigma^k=\begin{Bmatrix}\sigma_x^{\,k}\\ \sigma_y^{\,k}\\ \tau^k_{xy}\\ \tau^k_{xz}\\ \tau^k_{yz}\end{Bmatrix}=Q_1^k\varepsilon^k=Q_1^k\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\\ \gamma_{xz}\\ \gamma_{yz}\end{Bmatrix}=Q_1^k\begin{Bmatrix}\partial u_1/\partial x\\ \partial u_2/\partial y\\ \partial u_1/\partial y+\partial u_2/\partial x\\ \partial u_1/\partial z+\partial u_3/\partial x\\ \partial u_2/\partial z+\partial u_3/\partial y\end{Bmatrix}\quad(18)$$

$$m^k=\begin{Bmatrix}m^k_x\\ m^k_y\\ m^k_{xy}\\ m^k_{yx}\\ m^k_{yz}\\ m^k_{zy}\\ m^k_{xz}\\ m^k_{zx}\end{Bmatrix}=Q_2^k\chi^k=Q_2^k\begin{Bmatrix}\chi_x\\ \chi_y\\ \chi_{xy}\\ \chi_{yx}\\ \chi_{yz}\\ \chi_{zy}\\ \chi_{xz}\\ \chi_{zx}\end{Bmatrix}=Q_2^k\begin{Bmatrix}\partial\theta_x/\partial x\\ \partial\theta_y/\partial y\\ \partial\theta_y/\partial x\\ \partial\theta_x/\partial y\\ \partial\theta_z/\partial y\\ \partial\theta_y/\partial z\\ \partial\theta_z/\partial x\\ \partial\theta_x/\partial z\end{Bmatrix}\quad(19)$$

Where

$$Q_1^k=T_1^{kT}C_1^kT_1^k\quad(20)$$

$$Q_2^k=T_2^{kT}C_2^kT_2^k\quad(21)$$

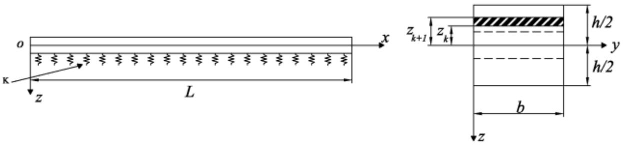


Fig. 1 Configuration of the beam and the coordinate system

## 3. The laminated composite beam model

Configuration of the coordinate system of the laminated composite beam is shown in Fig. 1 which length, width and thickness of the beam are $L$, $b$ and $h$, respectively; $u_1$, $u_2$ and $u_3$ are components of displacement vector in $x$, $y$ and $z$ direction, respectively and the displacement field for Reddy beam theory is described as

$$\begin{aligned} u_1 &= u(x) - z\phi(x) - c_1 z^3\left(-\phi(x) + \partial w(x)/\partial x\right) \\ u_2 &= 0 \\ u_3 &= w(x) \end{aligned} \tag{22}$$

Where $u(x)$ is the axial displacement of the mid-plane and $w(x)$ is the deflection of the microbeam along the thickness ($z$-direction). Also $\phi(x)$ is the rotation angle of cross section about the y-axis with respect to the thickness direction. $c_1$ is a constant introduced as

$$c_1 = 4\alpha \big/ 3h^2 \tag{23}$$

Setting $\alpha$=0 and $\phi(x)$=$\partial w(x)/\partial x$ in Eqs. (21) and (22) Euler-Bernoulli beam is achieved. In this case ,the cross section of the microbeam remains normal to the mid-plane and undistorted after deformation.

Also, the thick beam theory (Timoshenko) is achieved by setting $\alpha$=0; due to consideration of shear deformation, the cross section does not remain normal to the axial direction and it still remains plane and does not be distorted after deformation.

Furthermore, one can obtain the relations of Reddy beam theory by setting $\alpha$=1 and the shear stress vanishes on the upper and lower surfaces of the beam. So, there is no need to use shear correction factor in the Reddy beam theory unlike the Timoshenko beam theory. In addition, the cross section does not stay normal to the mid-plane and will be even undistorted after deformation in Reddy theory.

Considering all beam theories, the rotation vector is defined by using the displacement field in Eq. (21) and can be simplified as

$$\theta_y = \tfrac{1}{2}(\phi + \partial w/\partial x) - \tfrac{1}{2} c_1 z^2\left(-\phi + \partial w/\partial x\right) \tag{24}$$

The zero components of the strain and the curvature are defined by substituting Eq. (21) and (23) into Eq. (17) and can be written as

$$\varepsilon_y = \gamma_{yz} = \gamma_{xy} = \chi_{yx} = \chi_{yz} = \chi_{xz} = \chi_{zx} = \chi_x = \chi_y = 0 \tag{25}$$

So, the nonzero components are defined as

$$\varepsilon_x, \gamma_{xz}, \chi_{xy}, \chi_{zy} \neq 0 \tag{26}$$

The couple stress-curvature tensor and the stress-strain relation for the $k_{th}$ ply of laminated composite beams in the global coordinate system are simplified as

$$\sigma^k = \begin{Bmatrix} \sigma_x^{\ k} \\ \tau_{xz}^k \end{Bmatrix} = \begin{bmatrix} Q_{11}^k & 0 \\ 0 & Q_{44}^k \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \gamma_{xz} \end{Bmatrix} \tag{27}$$

$$m^k = \begin{Bmatrix} m_{xy}^k \\ m_{zy}^k \end{Bmatrix} = \begin{bmatrix} \hat{Q}_{44}^k & 0 \\ 0 & \hat{Q}_{66}^k \end{bmatrix} \begin{Bmatrix} \chi_{xy} \\ \chi_{zy} \end{Bmatrix} \tag{28}$$

In the present work, other components of stress and couple stress-curvature tensors are not zero but they are existed in the governing equations. The coefficients in Eqs. (26) and (27) are written as

$$\begin{aligned} Q_{11}^k &= m^2 C_{44}^k + n^2 C_{55}^k + 2m^2 n^2 (C_{12}^k + 2C_{66}^k) \\ Q_{44}^k &= m^2 C_{44}^k + n^2 C_{55}^k \\ \hat{Q}_{44}^k &= C_{44}^k \zeta_{kb}^2 m^2 (m^2 - n^2) + C_{55}^k \zeta_{kml}^2 C_{55}^k n^2 (m^2 - n^2) + \\ & \quad 2m^2 n^2 (C_{55}^k \zeta_{kb}^2 + C_{44}^k \zeta_{kml}^2) \\ \hat{Q}_{66}^k &= \zeta_{km2}^2 (m^2 C_{44}^k - n^2 C_{55}^k) \end{aligned} \tag{29}$$

A cross-ply laminated, $mn$=0 because $\psi$=$\pi$/2 or $\psi$=0, so coefficients in Eq. (27) are rewritten as the following

$$\begin{aligned} Q_{11}^k &= m^4 C_{11}^k + n^4 C_{22}^k \\ Q_{44}^k &= m^2 C_{44}^k + n^2 C_{55}^k \\ \hat{Q}_{44}^k &= C_{44}^k \zeta_{kb}^2 m^4 + C_{55}^k \zeta_{km1}^2 C_{55}^k n^4 \\ \hat{Q}_{66}^k &= \zeta_{km2}^2 (m^2 C_{44}^k - n^2 C_{55}^k) \end{aligned} \tag{30}$$

For isotropic materials, by neglecting the Poisson ratio and applying the coefficients of Eq. (29) coefficients are archived as

$$\begin{aligned} Q_{11}^k &= C_{11}^k = C_{22}^k = E \\ Q_{44}^k &= C_{44}^k = C_{55}^k = G \\ \hat{Q}_{44}^k &= Q_{66}^k = G\zeta^2 = \mu\zeta^2 \end{aligned} \tag{31}$$

Considering the material length scale parameters equal to zero in Eq. (29), the coefficients for classical laminated composite beams are achieved.

## 4. Principle of Hamilton for laminated composite for thin beam theory

The principle of Hamilton is used for achieving the equilibrium equations and the boundary conditions. The principle of virtual work for laminated composite beams of the couple stress theory can be defined by

$$\delta \int_0^T K - (U - W)dt = 0 \tag{32}$$

The first variation of the total strain energy in the beam is represented as

$$\delta U=\int_0^L b\left[\sum_{k=1}^{n}\int_{Z_k}^{Z_{k+1}}((\sigma^k)^T:\delta\varepsilon+(m^k)^T:\delta\chi)dz\right]dx \quad (33)$$

The first variation of virtual work done by external forces in the beam is expressed as

$$\begin{aligned}\delta W=&\int_0^L\left[f_u\delta u+\left(kw+f_w+\frac{\partial f_c}{\partial x}\right)\delta w_c\right]dx+\\&\left[\bar N\delta u+\left(\bar V-f_c\right)\delta w+\bar M\delta\phi+\bar Y\delta\left(\frac{\partial w}{\partial x}\right)\right]\Bigg|_{x=0}^{x=L}\end{aligned} \quad (34)$$

In which $f_u$ and $f_w$ are the $x$-and $z$-components of the body force per unit length of the beam, repectively. $kw$ is the lateral reaction force due to elastic medium and $f_c$ is the body moment about the $z$-axis per unit length of the beam, ( $f_c=\int_A CdA$ in which C is body couple per unit volume). $\bar N$ , $\bar V$ , $\bar M$ and $\bar Y$ are the axial force,-transverse force, the first-order and the third order bending moments applied at the ends of the beam, respectively. $f_c$ also can be as below

$$\begin{aligned}&\int_0^L f_c\delta w dx=-\frac{1}{2}\int_0^L f_c\delta\left(\theta+w_{,x}\right)dx=\\&-\frac{1}{2}\left(\int_0^L f_c\delta\theta dx+f_c\delta w\Big|_{x=0}^{x=L}-\int_0^L\left(\partial f_c/\partial x\right)\delta w dx\right)\end{aligned} \quad (35)$$

The first variation of the kinetic energy is

$$\begin{aligned}&\delta\int_0^L Kdt=-\int_0^T\int_0^L\left[m_0\frac{\partial^2 u}{\partial t^2}\delta u+\left(m_0\frac{\partial^2 w}{\partial t^2}-m_2\frac{\partial^4 w}{\partial x^2\partial t^2}\right)\delta w\right]dxdt\\&+\int_0^L-\left[m_2\frac{\partial^3 w}{\partial x\partial t^2}\delta w\right]\Bigg|_{x=0}^{x=L}dt\\&+\int_0^L\left[m_0\frac{\partial u}{\partial t}\delta u+m_0\frac{\partial w}{\partial t}\delta w+m_2\frac{\partial^2 w}{\partial x\partial t}\delta\left(\frac{\partial w}{\partial x}\right)\right]\Bigg|_{x=0}^{x=L}dx\end{aligned} \quad (36)$$

The composite laminated governing equations and boundary conditions of thin beam achieved as below

$$\begin{aligned}&\delta u\to -m_0\frac{\partial^2 u}{\partial t^2}+\bar Q_{11}\frac{\partial^2 u}{\partial x^2}+\left(-\bar J_{11}\right)\frac{\partial^2\phi}{\partial x^2}+f_u=0\\&\delta w\to m_2\frac{\partial^4 w}{\partial t^2\partial x^2}-m_0\frac{\partial^2 w}{\partial t^2}+\bar J_{11}\frac{\partial^3 u}{\partial x^3}-\\&\qquad\left(\bar I_{11}+\bar Q_{44}\right)\frac{\partial^4 w}{\partial x^4}+kw+f_w+\frac{1}{2}\frac{\partial f_c}{\partial x}=0\end{aligned} \quad (37)$$

Where the boundary conditions are in the following form

$$\begin{aligned}&\bar Q_{11}\frac{\partial u}{\partial x}+\left(-\bar J_{11}\right)\frac{\partial^2 w}{\partial x^2}=\bar N;\\&\quad or\; u=\bar u\; at\; x=0\; and\; x=L\\&m_2\frac{\partial^3 w}{\partial x\partial t^2}+\bar J_{11}\frac{\partial^2 u}{\partial x^2}-\left(\bar I_{11}+\bar Q_{44}\right)\frac{\partial^3 w}{\partial x^3}+f_c=\bar V;\\&\quad or\; w=\bar w\; at\; x=0\; and\; x=L\\&\bar J_{11}\frac{\partial u}{\partial x}-\left(\bar I_{11}+\bar Q_{44}\right)\frac{\partial^2 w}{\partial x^2}=-\bar Y;\\&\quad or\;\frac{\partial w}{\partial x}=\frac{\partial\bar w}{\partial x}\; at\; x=0\; and\; x=L\end{aligned} \quad (38)$$

Governing equations of Euler-Bernoulli beam can be obtained by substituting $\zeta_{kb}=\zeta_{km1}=\zeta_{km2}=0$ and isotropic material coefficients in Eq. (A.6) and $\bar{\bar Q}_{44}=0$ in Eq. (39) and Eq. (40) as the following form

$$\begin{aligned}&EA\frac{\partial^2 u}{\partial x^2}+f_u=m_0\frac{\partial^2 u}{\partial t^2}\\&\left[EA+\mu A\zeta^2\right]\frac{\partial^4 w}{\partial x^4}+kw-f_w-\frac{\partial f_c}{\partial x}=\\&m_0\frac{\partial^2 w}{\partial t^2}-m_2\frac{\partial^4\Omega}{\partial x^2\partial t^2}\end{aligned} \quad (39)$$

## 5. The analytical solution of free vibration of Euler-Bernoulli beam

For illustrating this model, the governing and the boundary conditions explained analytically. It must be mentioned that this model expressed three-layer laminated which the thickness of each layer is equal to each other. In this model, $f_u$, $f_w$ and $C$ components of body force per unit length in $x$ and $z$ direction and the body couple per unit volume respectively assumed zero in this study. The composite laminated governing equations and the boundry conditions solved by the Fourier series expantions that satisfy the boundry conditions of hinged-hinged and writen as following:

Hinged-hinged

$$u=w=\bar M=\bar Y=0 \text{ at } x=0 \text{ and } x=L \quad (40)$$

$$w(x,t)=\sum_{n=1}^{\infty}W_n\sin\left(\frac{n\pi x}{L}\right)e^{i\omega_n t} \quad (41)$$

Which $W_n$ and $\omega_n$ are the Fourier coefficient and natural frequency respectively and $i^2=-1$.

Substituting Eq. (40) in Eq. (38) governing equations of composite laminated Euler-Bernoulli beam written as

$$\begin{aligned}&\left[\bar I_{11}+\bar Q_{44}\right]\left(\frac{n\pi}{L}\right)^4 W_n+kW_n-\\&\omega_n^2\left[m_0+m_2\left(\frac{n\pi}{L}\right)^2\right]W_n=0\end{aligned} \quad (42)$$

Natural frequencies of composite laminated and isotropic beam written as

$$\omega_n^2=\frac{\left[\bar I_{11}+\bar{\bar Q}_{44}\right]\left(\frac{n\pi}{L}\right)^4+k}{\left[m_0+m_2\left(\frac{n\pi}{L}\right)^2\right]} \quad (43)$$

## 6. The numerical solution of Euler-Bernoulli beam

The results obtained in this paper focused on, effect of slenderness ratio ($h/\zeta$) with various foundation on fundamental frequency of both isotropic and composite laminated beams by considering material length scale

Table 1 The properties of materials

| | |
|---|---|
| *Isotropic beam* | $E$=$E_1$=$E_2$=1.44 GPa, $G$=$G_{12}$=$G_{13}$=$G_{23}$=$\dfrac{E}{2(1+\nu)}$, $\nu$=$\nu_{12}$=$\nu_{13}$=$\nu_{23}$=0.38, $\zeta$=$\zeta_{kb}$=$\zeta_{km1}$=$\zeta_{km2}$=17.6×10$^{-6}$ m, $b$=2$h$, $L$=20$h$ |
| *Micro composite laminated beam* | $E_1/E_2$=25, $E_2$=6.98 Gpa, $\nu_{12}$=$\nu_{13}$=$\nu_{23}$=0.25, $G_{12}$=$G_{13}$=0.5$E_2$, $G_{23}$=0.2$E_2$, $L$=200×10$^{-6}$ m, $b$=25×10$^{-6}$ m, $h$=25×10$^{-6}$ m |

Table 2 The fundamental frequency of Euler-Bernoulli isotropic beam (Hz)

| | | Elastic Foundation coefficient | | | | | Ref Mohammad-Abadi (2015) |
|---|---|---|---|---|---|---|---|
| | | $k$=10$^3$ | $k$=10$^4$ | $k$=10$^5$ | $k$=10$^6$ | $k$=10$^7$ | $k$=0 |
| EBT | $h$=$\xi$ | 1.1639e+6 | 1.1690e+6 | 1.2188e+6 | 1.6359e+6 | 3.8189e+6 | 1.1632e+6 |
| | $h$=5$\xi$ | 9.8191e+4 | 1.0059e+5 | 1.2199e+5 | 2.5002e+5 | 7.3404e+5 | 9.7921e+4 |
| | $h$=10$\xi$ | 4.5414e+4 | 4.6706e+4 | 5.8071e+4 | 1.2361e+5 | 3.6655e+5 | 4.5267e+4 |

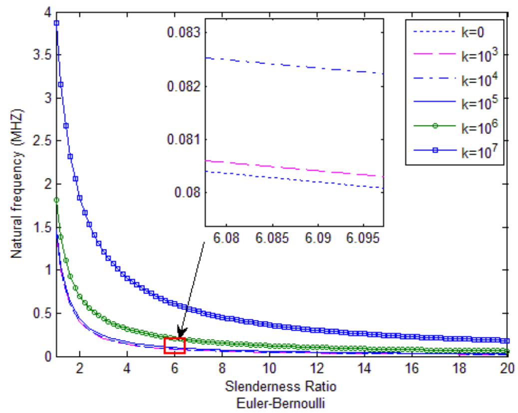


Fig. 2 The fundamental frequency of composite laminated Euler-Bernoulli beam (0, 90, 0) (MHz) on elastic foundation

parameter which defined in theory of modified couple stress. The material length scale parameter is concluded by experimental data while for micro composite laminated beam there is not experimental data, so for micro composite laminated beam, the material length scale parameter is assumed in order of material length scale parameter of epoxy that has been evaluated by Lam *et al*. (2003). Dimensions and properties of isotropic beam and micro composite laminated beam considered as Table 1.

For obtaining the fundamental frequency of isotropic Euler-Bernoulli beam on elastic medium some numerical results which are function of slenderness ratio (beam thickness to length scale parameter ratio) ($h/\zeta$) are depicted in Fig. 2 which achieved using material properties of isotropic beam in Table 1 considering various elastic foundation. For validating some numerical results achieved in Table 2, the elastic foundation coefficient is taken to be zero $k$=0 and the results show the accuracy of this attempt in comparision with Mohammad Abadi (2015).

Fig. 2 presents effects of slenderness ratio $h/\zeta$ increasing for composite laminated Euler-Bernoulli beam

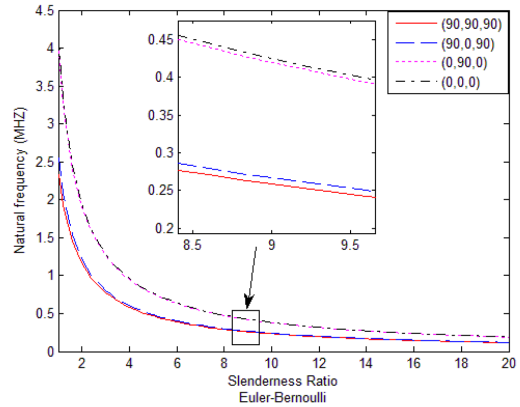


Fig. 3 The fundamental frequency of composite laminated Euler-Bernoulli beams for different cross ply laminated beams such as (90,0,90), (0,90,0), (90,90,90) and (0,0,0), (MHz) on elastic foundation

which achieved by increasing length of the beam for increasing slenderness ratio above the effect of increasing coefficient of elastic foundation in (0,90,0) composite laminated beam.

It is observed in Fig. 3 for different cross ply laminated beams such as (90,0,90), (0,90,0), (90,90,90) and (0,0,0), the fundamental frequency is increased by increasing length scale parameter and assumed elastic foundation coefficient is $k$=10$^7$ because of the effect of elastic medium on stiffness of chosen laminated beam.

In all prepared models by increasing the slenderness ratio the fundamental frequencies decreased and also by increasing the stiffness of elastic medium the fundamental frequencies increased but in the composite laminated models in all types of cross ply laminated beams such as (90,0,90), (0,90,0), (90,90,90) and (0,0,0) there is a sensitivity in changing the stiffness coefficient for example when in Fig. 3 the zero laminated beam (0,0,0), the stiffness coefficient increased from 10$^6$ to 10$^7$ there is a noticable increase. Fig. 2 shows the variation of the fundamental frequency for cross ply laminated (0, 90, 0) based on CLEBB versus beam thickness to length scale parameter ratio ($h/\zeta$) with $\zeta_b$=25×10$^{-6}$ m, $L$=20$h$ and $b$=2$h$. It is necessary to notice that the width and length of beam increased due to $b$=2$h$ and $L$=20$h$ by increasing the beam thickness. In Fig. 3 length scale parameter ($h/\zeta$) with $\zeta_b$=25×10$^{-6}$ m, $L$=20$h$ and $b$=2$h$ four types of lamination such as (90,0,90), (0,90,0), (90,90,90) and (0,0,0) are considered and effect of these cross ply laminated on the fundamental frequency of composite laminated beams are studied on elastic medium.

As it can\` be seen, the highest values of fundamental frequency are predicted by cross ply (0, 0, 0) laminated and the lowest values are obtained by cross ply (90, 90, 90) laminated and predicted values by cross ply (0, 90, 0) and (90, 0, 90) laminated lie between them.

## 7. Conclusions

In this study, free vibration of a anisotropic microstructure-dependent model for a thin laminated composite beams, based on modified couple-stress theory, and for hinged-hinged boundary conditions on different types of cross ply laminations such as (90,0,90), (0,90,0), (90,90,90) and (0,0,0) were considered. The fundamental frequency has been defined by analytical. Also, the classical theory was achieved by considering the material length scale parameter, $\zeta_b$=0 and it was compared with the modified couple-stress theory for different beam theories and the results illustrate that by considering the size effect, the stiffness of an anisotropic microstructure-dependent model for the thin laminated composite beams has been increased. For investigating different parameters including material length scale parameter, beam thickness, some numerical results on different cross ply laminated beams are presented in addition, the Fundamental frequency of different thin beam theory is investigated by increasing slenderness ratio and various foundations. The results prove that the modified couple stress theory increases the natural frequency under the various foundation for free vibration of composite laminated micro beams.

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## Appendix

The coefficients strain energy formula can be expanded as

$$\left(N_x, M_x, P, Q, R, M_{xy}, M_{2xy}, M_{1zy}\right)= b\sum_{k=1}^{n}\int_{z_k}^{z_{k+1}}\left(\sigma_x^k, z\sigma_x^k, z^3\sigma_x^k, \tau_{xz}^k, z^2\tau_{xz}^k, m_{xy}^k, z^2 m_{xy}^k, z m_{zy}^k\right) \quad \text{A.1}$$

$N_x$, $M_x$ and $M_{xy}$ in Bernoulli-Euler composite laminated beam in terms of displacement parameters can be written as

$$\begin{aligned} N_x &= \bar{Q}_{11}\frac{\partial u}{\partial x} - \bar{J}_{11}\frac{\partial^2 w}{\partial x^2} \\ M_x &= \bar{J}_{11}\frac{\partial u}{\partial x} - \bar{I}_{11}\frac{\partial^2 w}{\partial x^2} \\ M_{xy} &= -\bar{\bar{Q}}_{11}\frac{\partial^2 w}{\partial x^2} \end{aligned} \quad \text{A.2}$$

And the coefficients in above equation ($\bar{Q}_{11}, \bar{J}_{11}, \bar{I}_{11}, \overline{\overline{Q_{11}}}$) are introduced in Eq. (A.4). By using Equation of stress-strain relation and couple stress-curvature relation and (A.1), the state coefficients in Eq. (A.1) for composite laminated Timoshenko and Reddy beams beam in terms of displacement parameters can be given as

$$\begin{aligned} N_x &= \bar{Q}_{11}\frac{\partial u}{\partial x} - \bar{J}_{11}\frac{\partial^2 w}{\partial x^2} - c_1\bar{R}_{11}\left(-\frac{\partial \phi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) \\ M_x &= \bar{J}_{11}\frac{\partial u}{\partial x} - \bar{I}_{11}\frac{\partial^2 w}{\partial x^2} - c_1\bar{S}_{11}\left(-\frac{\partial \phi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) \\ P &= \bar{R}_{11}\frac{\partial u}{\partial x} - \bar{S}_{11}\frac{\partial \phi}{\partial x} - c_1\bar{T}_{11}\left(-\frac{\partial \phi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) \\ Q &= k_s\bar{Q}_{44}\left(-\phi + \frac{\partial w}{\partial x}\right) - 3c_1\bar{I}_{44}\left(-\phi + \frac{\partial w}{\partial x}\right) \\ R &= \bar{I}_{44}\left(-\phi + \frac{\partial w}{\partial x}\right) - 3c_1\bar{S}_{44}\left(-\phi + \frac{\partial w}{\partial x}\right) \\ M_{xy} &= -\frac{1}{2}\bar{\bar{Q}}_{44}\left(-\frac{\partial \phi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) - \frac{3}{2}c_1\bar{I}_{44}\left(-\frac{\partial \phi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) \\ M_{2xy} &= -\frac{1}{2}\bar{\bar{I}}_{11}\left(-\frac{\partial \phi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) - \frac{3}{2}c_1\bar{\bar{S}}_{44}\left(-\frac{\partial \phi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right) \\ M_{1zy} &= -\frac{3}{2}c_1\bar{\bar{I}}_{66}\left(-\phi + \frac{\partial w}{\partial x}\right) \end{aligned} \quad \text{A.3}$$

Where

$$\begin{aligned} \left\{\bar{Q}_{jj}, \bar{I}_{jj}, \bar{S}_{jj}\right\} &= b\sum_{k=1}^{n}\left\{\left(z_{k+1}-z_k\right), \left(\frac{z_{k+1}^3 - z_k^3}{3}\right), \left(\frac{z_{k+1}^5 - z_k^5}{5}\right)\right\}Q_{jj}^k \\ & j = 1, 4 \\ \left\{\bar{J}_{11}, \bar{R}_{11}, \bar{T}_{11}\right\} &= b\sum_{k=1}^{n}\left\{\left(\frac{z_{k+1}^2 - z_k^2}{2}\right), \left(\frac{z_{k+1}^4 - z_k^4}{4}\right), \left(\frac{z_{k+1}^7 - z_k^7}{7}\right)\right\}Q_{11}^k \\ \left\{\bar{\bar{Q}}_{44}, \bar{\bar{J}}_{jj}, \bar{\bar{S}}_{44}\right\} &= b\sum_{k=1}^{n}\left\{\left(z_{k+1}-z_k\right), \left(\frac{z_{k+1}^3 - z_k^3}{3}\right), \left(\frac{z_{k+1}^5 - z_k^5}{5}\right)\right\}Q_{jj}^k \\ & j = 4, 6 \end{aligned} \quad \text{A.4}$$

By substituting Eq. (28) in Eq, (A.3), these coefficients for isotropic material simplified as

$$\begin{aligned} &\left\{\bar{Q}_{11}, \bar{I}_{11}, \bar{S}_{11}\right\} = \left\{bhQ_{11}^k, \frac{1}{12}bh^3Q_{11}^k, \frac{1}{80}bh^5Q_{11}^k\right\} = \left\{EA, EI_2, EI_4\right\} \\ &\left\{\bar{Q}_{44}, \bar{I}_{44}, \bar{S}_{44}\right\} = \left\{bhQ_{44}^k, \frac{1}{12}bh^3Q_{44}^k, \frac{1}{80}bh^5Q_{44}^k\right\} = \left\{\mu A, \mu I_2, \mu I_4\right\} \\ &\left\{\bar{J}_{11}, \bar{R}_{11}, \bar{T}_{11}\right\} = \left\{0, 0, \frac{1}{480}bh^7Q_{11}^k\right\} = \left\{0, 0, \mu I_6\right\} \\ &\left\{\overline{\overline{Q}}_{44}, \bar{\bar{I}}_{44}, \bar{\bar{S}}_{44}, \bar{\bar{I}}_{66}\right\} = \\ &\quad \left\{bhQ_{44}^k, \frac{1}{12}bh^3Q_{44}^k, \frac{1}{80}bh^5Q_{44}^k, \frac{1}{12}bh^3Q_{66}^k\right\} = \\ &\quad \left\{\mu\zeta^2 A, \mu\zeta^2 I_2, \mu\zeta^2 I_4, \mu\zeta^2 I_2\right\} \end{aligned} \quad \text{A.5}$$

Inertia coefficients that used in Eq. (38), where $\rho$ is the mass density of the beam material and independent of time, given as

$$\left(m_0, m_2\right) = \int_A \rho\left(1, z^2\right)dA = \left(\rho bh, \frac{1}{12}\rho bh^3\right) \quad \text{A.6}$$