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Consistent couple-stress theory for free vibration analysis of Euler-Bernoulli nano-beams made of arbitrary bi-directional functionally graded materials

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Abstract. In this paper, using consistent couple stress theory and Hamilton's principle, the free vibration analysis of Euler-Bernoulli nano-beams made of bi-directional functionally graded materials (BDFGMs) with small scale effects are investigated. To the best of the researchers' knowledge, in the literature, there is no study carried out into consistent couple-stress theory for free vibration analysis of BDFGM nanostructures with arbitrary functions. In addition, in order to obtain small scale effects, the consistent couple-stress theory is also applied. These models can degenerate into the classical models if the material length scale parameter is taken to be zero. In this theory, the couple-tensor is skew-symmetric by adopting the skew-symmetric part of the rotation gradients as the curvature tensor. The material properties except Poisson's ratio are assumed to be graded in both axial and thickness directions, which it can vary according to an arbitrary function. The governing equations are obtained using the concept of Hamilton principle. Generalized differential quadrature method (GDQM) is used to solve the governing equations for various boundary conditions to obtain the natural frequencies of BDFG nano-beam. At the end, some numerical results are presented to study the effects of material length scale parameter, and inhomogeneity constant on natural frequency.

Keywords: Euler-Bernoulli nano-beams; free vibration; consistent couple-stress theory; bi-directional functionally graded materials (BDFGMs); size effect; generalized differential quadrature method (GDQM)

1. Introduction

Recently, nano and micro structural elements such as beams, membranes and plates have attracted worldwide attention from the researches community for their superior properties and extensive applications in nano and micro electromechanical (NEMS and MEMS) devices (Gopalakrishnan and Narendar 2013, Nejad *et al*. 2017b). At nano and micro meter scales, size effects often become important. Both experimental and molecular dynamics simulation results have shown that the small-scale effects in the analysis of mechanical properties of nano and micro structures cannot be neglected and classical continuum theories is not usable. Molecular dynamics simulation is convenient method to simulate the mechanical behavior of small size structures but it is computationally expensive for structures with large number of atoms (Gopalakrishnan and Narendar 2013, Keivani *et al*. 2016). Thus researchers stimulated to develop several higher-order continuum theories such as nonlocal theory (Eringen 1972a, b, 1983, 2002), strain gradient theory (Lam *et al*. 2003) and etc. which could predict size effect by considering material length scale parameters. In 1960s, the couple-stress theory, introduced by Toupin (1962), Mindlin and Tiersten (1962), and Koiter (1964). This theory is an appropriate non-classical theory for analyzing micro and nano scale structures. By this theory the stiffness of these type of structures is predicted more than what is done by the classical theory. The modified version of the couple-stress theory has been proposed by Yang *et al*. (2002) by considering the couple-stress tensor to be symmetric. In this theory two higher order material length scale parameters are introduced in addition to the two Lame constants. One of the good aspects of this theory was that the four additional parameters in the micropolar theory and five additional parameters in the strain gradient theory were reduced to two additional parameters. This property has attracted some researchers in recent years to derive formulations for the mechanical analyzing of the micro-beams and micro-plates and investigate their mechanical behavior based on this theory. The formulations and mechanical behavior investigations for homogeneous linear micro-beams (Ma *et al*. 2008, Park and Gao 2006), homogenous nonlinear micro-beams (Asghari *et al*. 2010, Şimşek 2014, Xia *et al*. 2010), functionally graded linear micro-beams (Asghari *et al*. 2011), functionally graded nonlinear micro-beams (Ke *et al*. 2012), linear micro-plates (He *et al*. 2015, Jomehzadeh *et al*. 2011, Yin *et al*. 2010), nonlinear micro-plates (Asghari 2012, Lou and He 2015), composite laminated beams (Mohammad-Abadi and Daneshmehr 2015), micro beam under the effect of an impact force (Kocaturk and Akbas 2013) have been presented in recent years based on the modified couple-stress theory. Recently, by considering true continuum kinematical displacement and rotation, Hajesfandiari and Dargush (Hadjesfandiari and Dargush 2011) demonstrate the couple-tensor is skew-symmetric. Thus, they present the consistent couple-stress theory by

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adopting the skew-symmetric part of the rotation gradients as the curvature tensor.

Functionally graded materials (FGMs) are heterogeneous composite materials whose properties change smoothly and continuously along desired dimension(s). This continuously varying composition eliminates interface problems, and thus, the stress distributions are smooth (Zenkour 2013). A number of papers considering various aspects of FGM have been published in recent years (Nejad and Rahimi 2009, 2010, Ghannad *et al*. 2012, Hadi *et al*. 2013, Şimşek and Reddy 2013, Xue and Pan 2013, Nejad *et al*. 2014a, b, Ziegler and Kraft 2014, Gan *et al*. 2015, Jabbari *et al*. 2015, Kolahchi *et al*. 2015, Nejad and Fatehi 2015, Nejad *et al*. 2015a, b, Akbarov 2016, Hadji *et al*. 2016, Hosseini *et al*. 2016, Mazarei *et al*. 2016, Dehghan *et al*. 2016, Jabbari *et al*. 2016, Li and Hu 2016, 2017, Li *et al*. 2017, Nejad *et al*. 2017). Thanks to the advances in technology, FGMs have started to find their ways into micro-nano-electro-mechanical systems (MEMS/NEMS), for example in the form of shape memory alloy thin films with a global thickness in micro- or nano-scale (Lü *et al*. 2009b), electrically actuated MEMS devices (Zhang and Fu 2012), and atomic force microscopes (AFMs) (Kahrobaiyan *et al*. 2010). It should be noted that most of the above-mentioned analyses are related to FGMs with material properties varying in one direction only. However, there are practical occasions which require tailored grading of macroscopic properties in two or even three directions. As reported by Steinberg (1986), the fuselage of an aerospace craft undergoes an extremely high temperature field with excessive temperature gradient on the surface and through the thickness, when the plane sustains flight at a speed of Mach 8 and at an altitude of 29 km. In this circumstance, the conventional unidirectional FGMs may not be so appropriate to resist multi-directional severe variations of temperature. Therefore, it is of great significance to develop novel FGMs with macroscopic properties varying in two or three directions (2D or 3D FGMs) to withstand a more general temperature field. The number of studies on beams and plates made of two-directional functionally graded material (2D-FGM) is still very limited. Lü *et al*. (2009a) proposed the state-space based differential quadrature method for the thermo-elastic analysis of bi-directional FGM plates. In addition, dynamic behavior of multi-directional FGM annular plates was investigated by Nie and Zhong (2010). Zhao *et al*. (2012) suggested a symplectic framework for the analysis of plane problems of BDFGMs in which the elastic modulus varies exponentially both along the longitudinal and transverse coordinates. The fully coupled thermo-mechanical behavior of BDFGMs beam structures, using isogeometric finite element model, was studied by Lezgy-Nazargah (2015). Şimşek (2015) investigated free and forced vibration of BDFG Timoshenko beam under the action of a moving load. The material properties of the beam varied exponentially in both axial and thickness directions. Wang *et al*. (2016) studied the free vibration of a two-directional functionally graded beam which had variable material properties along the beam length and thickness. It is assumed that material properties vary through the length according to a simple power law distribution with an arbitrary power index and have an exponential gradation along the beam thickness. Li *et al*. (2016) derived a size-dependent Timoshenko beam model which accounts for through-thickness power-law variation of a two-constituent functionally graded (FG) material, in the framework of the non-local strain gradient theory. Fakhrabadi and Yang (2015) presented the nonlinear electromechanical behavior of nano-beams under electrostatic actuation based on the consistent couple-stress theory. Nejad *et al*. (2016) presented buckling analysis of arbitrary two-directional functionally graded Euler-Bernoulli nano-beams based on non-local elasticity theory. In other studies, Nejad and Hadi (2016a) presented bending analysis and free vibration analysis (Nejad and Hadi 2016b) of arbitrary bi-directional functionally graded Euler-Bernoulli nano-beams based on non-local elasticity theory.

In this article, to the best of the researchers' knowledge, for the first time, using consistent couple-stress theory, free vibration analysis of BDFGMs Euler-Bernoulli nano-beams is presented. The effects of changes of some important parameters such as material length scale, FG index on the values of frequencies, and Frequency in different modes are studied. The results of this study can be a reference for designing the elastic types bi-directional FGM Euler-Bernoulli nano-beams.

2. Analysis

Consider a nano-beam of length L, width b, and thickness h made of bi-directional functionally graded materials (Fig. 1). Cartesian coordinates (x, y, z) are considered.

The investigation on 2D-FGMs has shown that it is more capable of reducing thermal and residual stresses than one-directional FGMs (Nemat-Alla 2003). So, the modulus of elasticity E and density ρ are assumed to vary as arbitrary functions in both axial and thickness directions, as indicated below

$$E = f_1(x)g_1(z) \tag{1}$$

$$\rho = f_2(x)g_2(z) \tag{2}$$

where $f_1(x)$, $f_2(x)$, $g_1(z)$, and $g_2(z)$ are arbitrary functions.

In the consistent couple-stress theory, the equations of equilibrium of the linear isotropic materials are formulated as (Hadjesfandiari and Dargush 2011)

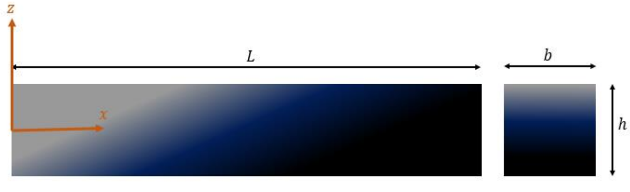

Fig. 1 Geometry of the bi-directional functionally graded Euler-Bernoulli nano-beam

$$\sigma_{ji,j} + f_i = 0 \tag{3}$$

$$m_{ji,j} + \varepsilon_{ijk}\sigma_{jk} = 0 \tag{4}$$

where σ_{ji} and m_{ji} represent the force-(classical) and couple-stress tensors, respectively. In addition, f_i and ε_{ijk} denote the body force per unit volume and permutation or Levi-Civita symbol, respectively. As mentioned before, Hadjesfandiari and Dargush (2011) proved that in the couple-stress theory, the body force and body couple are not distinguishable from each other and the body couple transform to the equivalent body force (Hadjesfandiari and Dargush 2011). Moreover, in the couple-stress theory, unlike the classical elasticity, the stress tensor is generally non-symmetric. Thus, it can be decomposed to the symmetric and skew-symmetric components as following

$$\sigma_{ji} = \sigma_{(ji)} + \sigma_{[ji]} \tag{5}$$

where $\sigma_{(ji)}$ is the symmetric part and $\sigma_{[ji]}$ is the skew-symmetric part of the force-stress tensor. In order to define the elements of Eqs. (3)-(5) required in the couple-stress theory, the kinematic parameters should be utilized. The displacement gradient can be decomposed into two distinct parts

$$u_{ij} = e_{ij} + \omega_{ij} \tag{6}$$

where

$$e_{ij} = u_{(i,j)} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{7}$$

$$\omega_{ij} = u_{[i,j]} = \frac{1}{2}\left(u_{i,j} - u_{j,i}\right) \tag{8}$$

In the above relations, e_{ij} and ω_{ij} are strain and rotations tensors, respectively. Similar to the couple-stress tensor, the rotation tensor is skew-symmetrical and a vector can be defined dual to it as

$$\omega_i = \frac{1}{2}\varepsilon_{ijk}\omega_{kj} \tag{9}$$

The gradient of rotation tensor can be decomposed into two sub-tensors as

$$\omega_{i,j} = \chi_{ij} + \kappa_{ij} \tag{10}$$

where

$$\chi_{ij} = \omega_{(i,j)} = \frac{1}{2}\left(\omega_{i,j} + \omega_{j,i}\right) \tag{11}$$

$$\kappa_{ij} = \omega_{[i,j]} = \frac{1}{2}\left(\omega_{i,j} - \omega_{j,i}\right) \tag{12}$$

The diagonal arrays of the former known as the torsion tensor show the pure torsion of the element about the coordinate axis and the off-diagonal terms are deviations from sphericity. It does not contribute as a fundamental measure of deformation and will not be included in the strain energy. On the other hand, in the couple-stress theory, the curvature tensor (κ_{ij}) plays a crucial role in the strain energy. The corresponding dual vector of the skew-symmetric curvature tensor can be formulated as

$$\kappa_i = \frac{1}{2}\varepsilon_{ijk}\kappa_{kj} \tag{13}$$

It is now the time of formulating the force and couple-stresses corresponding to the above kinematic parameters. The symmetrical part of the force-stress tensor in Eq. (5) is same as the force-stress tensor in classical elasticity and can be obtained from Eq. (14)

$$\sigma_{(ji)} = \lambda e_{kk}\delta_{ij} + 2\mu e_{ij} \tag{14}$$

where λ and μ are the Lame's constants. The couple-stress tensor is skew-symmetrical (m_{ij}=-m_{ji}) and a vector m_i can be introduced dual to the tensor

$$m_i = \frac{1}{2}\varepsilon_{ijk}m_{kj} \tag{15}$$

For the isotropic linear materials, Hadjesfandiari and Dargush (2011) proved that the couple-stress can be computed from Eq. (16)

$$m_i = -8\eta\kappa_i \tag{16}$$

The above relation shows that the couple-stress theory for the isotropic linear materials has only one extra size-dependent parameter. The ratio $\eta=\mu l^2$ is the constant makes difference between the classical and consistent couple-stress theories. The size-dependen parameter, l, varies from one material to another or from one scale to another scale. For the zero value of this parameter, the latter reduces to the former.

In addition, Hadjesfandiari and Dargush (2011) showed that the skew-symmetric component of the stress tensor can be obtained from Eq. (17)

$$\sigma_{[ji]} = -m_{[i,j]} \tag{17}$$

According to the consistent couple-stress developed by Hadjesfandiari and Dargush (2011), the strain energy density of an isotropic linear elastic material with volume Ω experiencing an infinitesimal displacement is defined as

$$U = \frac{1}{2}\int_{\Omega}\left(\sigma_{(ji)}e_{ij} + m_{ji}\kappa_{ij}\right)dv \tag{18}$$

Components of displacement vector (u_1, u_2, and u_3) for Nano-beams based on Euler-Bernoulli beam theories can be expressed as

$$\begin{cases} u_1 = -z\left(dw/dx\right) \\ u_2 = 0 \\ u_3 = w\left(x,t\right) \end{cases} \tag{19}$$

Substitution of Eq. (19) into the Eq. (12), the skew-symmetric curvature tensor is expressed as

$$\kappa = \frac{1}{2}\frac{\partial^2 w}{\partial x^2}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{20}$$

From Eq. (16), the couple-stress tensor is defined as follows

$$m = 4\mu l^2 \frac{\partial^2 w}{\partial x^2}\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{21}$$

From the displacement field, the strain components can be calculated by substituting Eq. (19) into Eq. (7)

$$e = -z\frac{\partial^2 w}{\partial x^2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{22}$$

For a slender beam with a large aspect ratio, the Poisson effect is secondary and can be disregarded to simplify the formulation of the beam theory. Hence, the stress component is presented as

$$\sigma = -Ez\frac{\partial^2 w}{\partial x^2}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{23}$$

Substituting Eq. (1) and Eqs. (20)-(23) into Eq. (18), the variation of strain energy is simplified to

$$\int_0^t \delta U dt = \int_0^t\int_0^L \left(I_2 + SI_0 l^2\right)\left(f_1''\frac{\partial^2 w}{\partial x^2} + +2 f_1'\frac{\partial^3 w}{\partial x^3} + f_1\frac{\partial^4 w}{\partial x^4}\right)\delta w dx dt \tag{24}$$

where

$$\begin{Bmatrix} I_0 \\ I_2 \end{Bmatrix} = \int_A g_1(z)\begin{Bmatrix} 1 \\ z^2 \end{Bmatrix} dA \tag{25}$$

$$S = \frac{2}{1+\upsilon} \tag{26}$$

The variation of the kinetic energy can be written in the following form

$$\int_0^t \delta K dt = \int_0^t\int_0^L \left(f_2' m_2 \frac{\partial^3 w}{\partial x \partial t^2} + + f_2 m_2 \frac{\partial^4 w}{\partial x^2 \partial t^2} - m_0\frac{\partial^2 w}{\partial t^2}\right)\delta w dx dt \tag{27}$$

where

$$\begin{Bmatrix} m_0 \\ m_2 \end{Bmatrix} = \int_A g_2(z)\begin{Bmatrix} 1 \\ z^2 \end{Bmatrix} dA \tag{28}$$

The governing equations of the BDFGM Euler-Bernoulli nano beam can be obtained, using the concept of Hamilton principle, that is

$$\int_{t_1}^{t_2} \left(\delta U - \delta K\right) dt = 0 \tag{29}$$

Substituting Eqs. (24) and (26) into Eq. (29), the Navier equation is expressed as

$$\left(I_2 + SI_0 l^2\right)\left(f_1''\frac{\partial^2 w}{\partial x^2} + 2 f_1'\frac{\partial^3 w}{\partial x^3} + f_1\frac{\partial^4 w}{\partial x^4}\right) = = f_2' m_2 \frac{\partial^3 w}{\partial x \partial t^2} + f_2 m_2 \frac{\partial^4 w}{\partial x^2 \partial t^2} - m_0\frac{\partial^2 w}{\partial t^2} \tag{30}$$

For free vibration analysis, it is assumed that w varies harmonically with respect to the time variable t, as follows

$$w(x,t) = W(x)e^{i\omega t} \tag{31}$$

Substituting Eq. (31) into Eq. (30) and by assuming $f_1(x) = f_2(x) = e^{\frac{\beta}{L}x}$, the following equation is obtained

$$\left(I_2 + SI_0 l^2\right)\left[\left(\frac{\beta}{L}\right)^2\frac{d^2W}{dx^2} + 2\left(\frac{\beta}{L}\right)\frac{d^3W}{dx^3} + \frac{d^4W}{dx^4}\right] = -\omega^2\left[\left(\frac{\beta}{L}\right)m_2\frac{dw}{dx} + m_2\frac{d^2W}{dx^2} - m_0 W\right] \tag{32}$$

3. Generalized differential quadrature method

In the case of the general boundary conditions, the analytical solution of Eq. (32) is difficult to obtain, so a GDQ approach is adopted for this equations The GDQ approach may be an easy and useful tool for the purpose of analyzing more complex problems. In addition, GDQM is an efficient numerical method for the solution of differential equations. It is assumed that the grid points are located on the zeros of the Chebyshev polynomials (Shu and Chew 1998) and to discretize the solution domain, one can assume a set of *N* grid points in the x-direction as

$$X_i = \frac{L}{2}\left\{1 - cos\left(\frac{i-1}{N-1}\right)\pi\right\}, \qquad i = 1,\ldots,N-1 \tag{33}$$

In this method, the derivatives of a function *f*(*x*), at a point x_i, are expressed as

$$f_x^{(n)}(x_i) = \sum_{j=1}^{N} C_{ij}^{(n)} f(x_j), \qquad n = 1,\ldots,N-1 \tag{34}$$

where N is the number of the grid points over the x direction. $C_{ij}^{(n)}$ is the respective weighting coefficients through the x direction obtained through the following equations.

If n=1, i.e., for the first order derivative, then

$$C_{ij}^{(1)} = \frac{M(X_i)}{(X_i - X_j)M(X_j)}, \quad i,j = 1,\dots,N \;\; j \neq i \tag{35}$$

where

$$M(X_i) = \prod_{\substack{j=1 \\ j \neq i}}^{N_x} (X_i - X_j) \tag{36}$$

To obtain the weighting coefficients for the second-order or higher-order derivatives, the matrix multiplication procedure is implemented

$$C_{ij}^{(n)} = n\left(C_{ii}^{(n-1)} C_{ij}^{(1)} - \frac{C_{ij}^{(n-1)}}{X_i - X_j}\right), \; i,j = 1,\dots,N \;\; j \neq i \tag{37}$$

$$C_{ii}^{(n)} = -\sum_{\substack{j=1 \\ j \neq i}}^{N_x} C_{ij}^{(n)}, \qquad \begin{cases} i = 1,\dots,N \\ n-1,2,\dots,N-1 \end{cases} \tag{38}$$

Substituting Eq. (34) into the first governing equation (Eq. (32)), the following equation is obtained

$$\begin{aligned}&\left(I_2 + SI_0 l^2\right)\left[\sum_{j=1}^{N} C_{ij}^{(4)} W(x_j) + 2\frac{\beta}{L}\sum_{j=1}^{N} C_{ij}^{(3)} W(x_j) + \right. \\ &\left. + \left(\frac{\beta}{L}\right)^2 \sum_{j=1}^{N} C_{ij}^{(2)} W(x_j)\right] = -\omega^2 \left(m_2 \sum_{j=1}^{N} C_{ij}^{(2)} W(x_j) + \right. \\ &\left. + \left(\frac{\beta}{L}\right) m_2 \sum_{j=1}^{N} C_{ij}^{(1)} W(x_j) - m_0 W(x_j)\right)\end{aligned} \tag{39}$$

Then arranging the displacement variable and corresponding coefficient, the governing equations in the following form could be obtained

$$\begin{bmatrix} A_{bb} & A_{bd} \\ A_{db} & A_{dd} \end{bmatrix}\begin{bmatrix} X_b \\ X_d \end{bmatrix} = [\varpi]\begin{bmatrix} 0 & 0 \\ B_{db} & B_{dd} \end{bmatrix}\begin{bmatrix} X_b \\ X_d \end{bmatrix} \tag{40}$$

where subscripts b and d denote boundary and domain sample points, respectively. In addition, coefficients A and B are matrices and their dimensions depend on the number of domain and boundary sample points. After eliminating boundary nodes X_b including (W) in Eq. (40) by using the boundary conditions, the dimension of the coefficient matrices reduces. Finally, Eq. (40) can be rewritten to give an eigenvalue problem as

$$[K][X_d] = [\varpi][I][X_d] \tag{41}$$

Solving the obtained eigenvalue problem gives the natural frequency (ϖ) of the BDFG Euler-Bernoulli nano-beam based on consistent couple-stress theory.

3. Results and discussion

In this section, the free vibration of BDFG Euler-Bernoulli nano-beams based on consistent couple-stress theory is investigated by numerical results.

In order to verify the validity and reliability of the present work, when β and l are neglected, a comparison of the dimensionless frequency in mode 1 of beams with various boundary conditions (S-S: simply supported-simply supported, C-C: clamped-clamped and C-S: clamped-simply supported) at two ends is made with Nejad and Hadi (2016b) and Eltaher *et al.* (2013), as shown in Table 1. Here, the beam's geometric and material properties are given as follows: Young's modulus is 30 MPa, density is 1 Kg/m^3, width of the beam is 1 nm and the length of the beam is 10 nm. It can be seen that there is an excellent agreement between the results obtained in this paper and those reported in Nejad and Hadi (2016b) and Eltaher *et al.* (2013). Dimensionless frequency and frequency ratio are defined as follows

$$\bar{\omega} = \omega L^2 \sqrt{\frac{m_0}{I_2}} \tag{42}$$

$$\mathrm{Fr} = \text{Frequency ratio} = \frac{\omega_{NL}}{\omega_L} \tag{43}$$

In the above relation, ω_L is the frequency when the size scale parameter is taken to be zero. Also it should be noted that when the frequency ratio approaches 1, size effects are negligible. The material properties considered in the current study are tabulated in Table 2.

It is proposed that the modulus of elasticity and density of the nano-beam material vary in the x and z directions, as follows

$$E(x,z) = e^{\frac{\beta}{L}x}\left[E_c\left(\frac{2z+h}{2h}\right)^{n_1} + E_m\left(1 - \left(\frac{2z+h}{2h}\right)^{n_1}\right)\right] \tag{44}$$

$$\rho(x,z) = e^{\frac{\beta}{L}x}\left[\rho_c\left(\frac{2z+h}{2h}\right)^{n_2} + \rho_m\left(1 - \left(\frac{2z+h}{2h}\right)^{n_2}\right)\right] \tag{45}$$

Figs. 2 and 3 illustrate the variation of the dimensionless modulus of elasticity through the length and thickness of the beam for various values of n_1=n_2=n and β. The value of n equal to zero, represents a fully ceramic beam, whereas infinite n indicates a fully metallic beam. The variation of the combination of ceramic and metal is linear for n=1. According to Fig. 3, in the same position (0<x/L<1), it is observed that for higher values of β, the stiffness increases.

Table 1 Comparison of non-dimensional frequency in mode 1 for a clamped-clamped nano-beam with (Eltaher *et al.* 2013; Nejad and Hadi 2016b)

L/h	$\bar{\omega}_1$ (Present work)	$\bar{\omega}_1$ (Nejad and Hadi 2016b)	$\bar{\omega}_1$ (Eltaher *et al.* 2013)
10	22.2594	22.2594	22.4926
20	22.3446	22.3446	22.4022
100	22.3721	22.3721	22.3744

Table 2 Material properties used in the numerical study

Materials	Properties		
	E (GPa)	ρ (kg/m^3)	υ
C	69	2700	0.3
M	339	3800	0.3

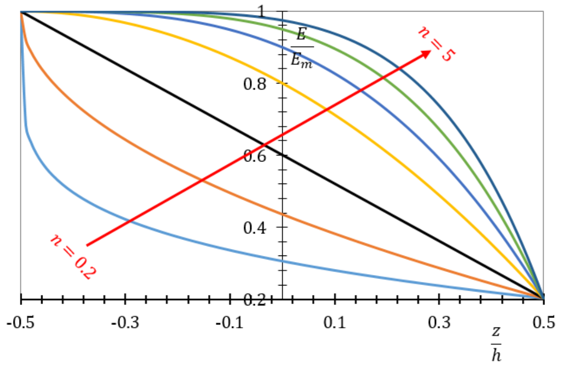

Fig. 2 Distribution of dimensionless modulus of elasticity versus *z/h* at *x*=0

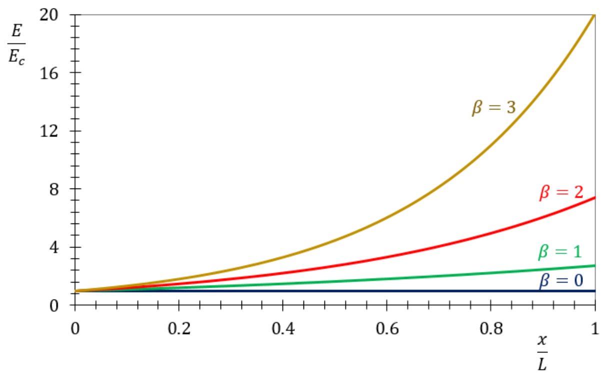

Fig. 3 Distribution of dimensionless modulus of elasticity versus *x/L* at *z*=-*h*/2

Fig. 4 illustrates the convergence of the GDQM in obtaining the non-dimensional frequency. It is observed that considering more than 9 sample points does not affect the accuracy of the results significantly. In this figure, the non-dimensional frequency error is defined as

$$e = \left|\frac{\bar{\omega}_{N+1} - \bar{\omega}_N}{\bar{\omega}_N}\right| \times 100 \tag{46}$$

where e is a small value number and in this analysis, it is taken to be 10^{-2}.

Fig. 5 shows the ratio of frequency in the case of considering couple-stress effect to the classic case in terms of dimensionless thickness, *h/l*. It can be seen, with increasing the dimensionless thickness, the frequency ratio tend to 1 which shows that with increasing the thickness against size scale parameter, couple-stress effect decreases. For the dimensionless thickness equal to 1, relative frequency ratio is equal to 4.5717 which shows the difference between classic and couple-stress theory in small sizes.

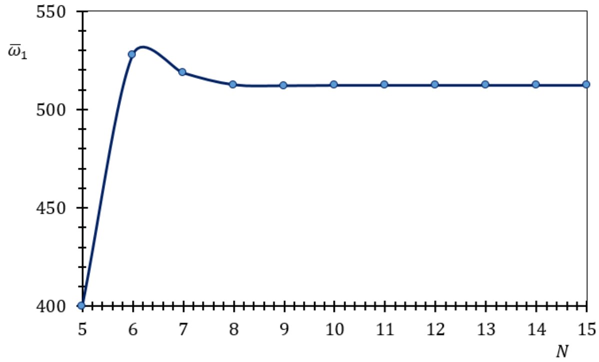

Fig. 4 Convergence of dimensionless frequency in mode 1 clamped-clamped BDFG (β=2, n=2, b=h=1 nm, L=100 nm, and l=5 nm)

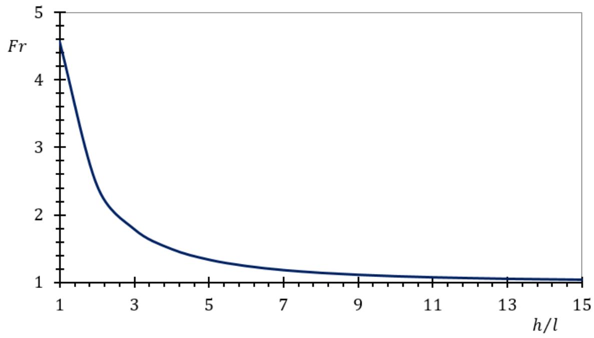

Fig. 5 Frequency ratio of clamped-clamped BDFG nano beam versus to dimensionless thickness (L=100 nm, b=h, n=2, β=2, and l=1 nm)

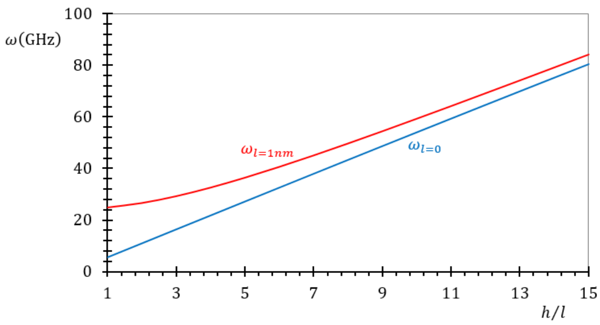

Fig. 6 Frequency of clamped-clamped BDFG nano beam by considering the effects of couple-stress theory and the classic theory versus to dimensionless thickness (L=100 nm, b=h, n=2, β=2, and l=1 nm)

The natural frequency is depicted in Fig. 6 by considering the effects of couple-stress and the classic theory in terms of dimensionless thickness. The more the dimensionless thickness, the more the natural frequency would be in both theories. Also this figure shows that in a constant dimensionless thickness, the natural frequency is greater in the case of considering couple-stress effects than the classic one. This shows the effects of size. With increasing the dimensionless thickness in both cases, both cases tend to reach each other.

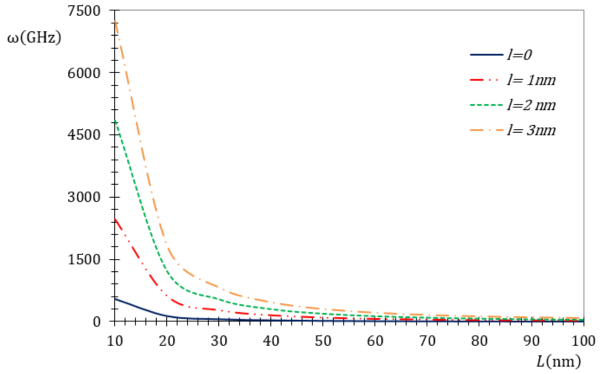

Fig. 7 Frequency in mode 1 of clamped-clamped BDFG nano beam versus to length in various values of l (b=h=1 nm, n=2, β=2, and l=5 nm)

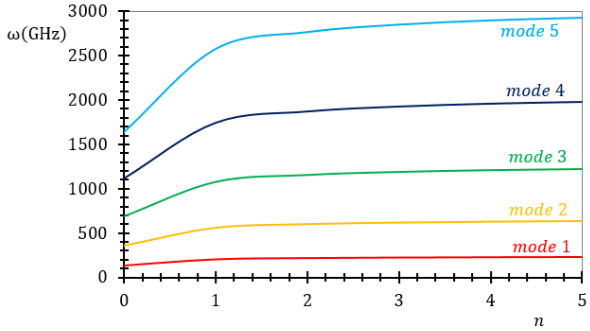

Fig. 8 Frequency of clamped-clamped BDFG nano-beam versus n_1=n_2=n in different modes (b=h=5 nm, L=75 nm, β=2, and l=5 nm)

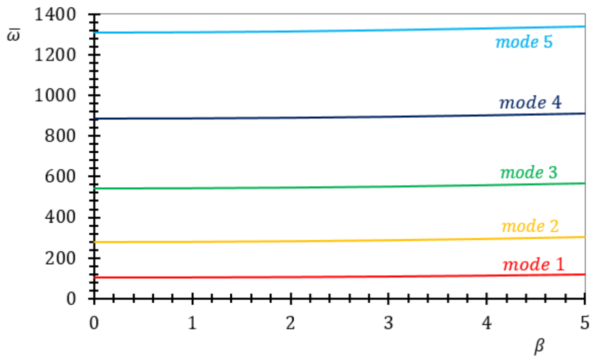

Fig. 9 Dimensionless frequency of clamped-clamped FG nano-beam versus β in different modes (b=h=5 nm, L=75 nm, β=2, and l=5 nm)

Fig. 7 shows the change in frequency in terms of the nano beam length in various values of l. This figure shows that the increase in nano beam length results in frequency reduction. Also, with increasing the nano-beam length, all cases tend to reach each other. In other words, with increasing the nano beam length, the effects of size disappear.

Given that n_1=n_2=n, natural frequency for the first five modes against n is shown in Fig. 8. It is apparent from the curve that there is an increase in the value of natural frequency as n increases.

Fig. 9 illustrates the dimensionless natural frequency against β for the first five modes. This figure shows that as β increases, the dimensionless natural frequency increases too. In other words, this figure shows that for higher values of β, stiffness increases.

5. Conclusions

The present paper has discussed the applicability of a non-classical continuum theories by consistent couple-stress theory to obtain the size dependent on vibration behavior of Euler-Bernoulli nano beams made of bi-directional functionally graded materials. The BDFG Euler-Bernoulli nano-beam is assumed to be graded through thickness and length directions, following the arbitrary material distribution. The governing equations and the boundary conditions are derived, using the Hamilton principle. Afterwards, GDQM is applied to solve the governing equations to obtain the natural frequencies of FG nano-beam. Results of this paper show small scale effects significantly contribute to the mechanical behavior of BDFG nano-beam, a significant fact which cannot be neglected. Further, frequency ratio decreases with the increase in the size scale parameter value. The results presented in this work may provide useful guidelines for designing and developing BDFG nano-beams based on nano devices that make use of the vibration properties of BDFG nano-beam. To show the effect of inhomogeneity on the vibration properties of FG nano-beam, different values were considered for material inhomogeneity parameters n and β. The presented results show that the material inhomogeneity has a significant influence on the mechanical behaviors of the BDFG Euler-Bernoulli nano-beams. Finally, the comparison between the results obtained from the classical and consistent couple-stress theory reveals that application of the latter leads to a model of the nano-beam with higher stiffness and larger natural frequency.

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