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# A nonlocal zeroth-order shear deformation theory for nonlinear postbuckling of nanobeams

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**Abstract.** In this work, a nonlocal zeroth-order shear deformation theory is developed for the nonlinear postbuckling behavior of nanoscale beams. The beauty of this formulation is that, in addition to including the nonlocal effect according to the nonlocal elasticity theory of Eringen, the shear deformation effect is considered in the axial displacement within the use of shear forces instead of rotational displacement like in existing shear deformation theories. The principle of virtual work together of the nonlocal differential constitutive relations of Eringen, are considered to obtain the equations of equilibrium. Closed-form solutions for the critical buckling load and the amplitude of the static nonlinear response in the postbuckling state for simply supported and clamped clamped nanoscale beams are determined.

**Keywords:** nanobeams; postbuckling; nonlocal elasticity

## 1. Introduction

In recent years, the scientific community gives a great attention to nanostructured elements because of their important properties. Establish experiments with nanoscale size elements are found to be difficult and expensive. Thus, elaborating appropriate theoretical models for nanostructures is an important topic for nanoengineering applications. In the scientific literature, we can found three approaches to model structures at nanoscale, namely: (a) atomistic (Ball 2001, Baughman *et al*. 2002), (b) hybrid atomistic-continuum mechanics (Bodily and Sun 2003, Li and Chou 2003a, b, Pradhan and Phadikar 2008) and (c) continuum mechanics. Both atomistic and hybrid atomistic-continuum mechanics are often computationally expensive and are not simple and practical for studying systems at large scale. As against, continuum mechanics method is less computationally expensive than the former two approaches. In addition, it has been remarked that this approach provides almost accurate results compared to those of atomistic and hybrid approaches.

Bending, buckling and vibration of nanoscale structures are of great importance in nanotechnology. Understanding mechanical behavior of nanoscale structures is the main step for many NEMS devices such as oscillators, clocks and sensor devices. There are already exploratory works on the continuum models for mechanical response of carbon nanotubes (CNTs) (Wang *et al*. 2006, Wang and Varadan 2006, Lu *et al*. 2007, Heireche *et al*. 2008, Tounsi *et al*. 2008, Benzair *et al*. 2008, Amara *et al*. 2010, Mustapha and Zhong 2010, Song *et al*. 2010, Roque *et al*. 2011, Naceri *et al*. 2011, Tounsi *et al*. 2013a, Benguediab *et al*. 2014) or nanobeam (Reddy 2007, Murmu and Adhikari 2012, Eltaher *et al*. 2012, Emam 2013, Berrabah *et al*. 2013). More reports on the behavior of nanostructures may be also found in the open literature (see, e.g., Bouafia *et al*. 2017, Ebrahimi and Salari 2015, Ebrahimi *et al*. 2015, Zemri *et al*. 2015, Larbi Chaht *et al*. 2015, Belkorissat *et al*. 2015, Adda Bedia *et al*. 2015, Bounouara *et al*. 2016, Ahouel *et al*. 2016, Ebrahimi and Barati 2016a, b, Ahouel *et al*. 2016, Besseghier *et al*. 2017). In these above-mentioned works it has been suggested that nonlocal elasticity theory developed by Eringen (1983, 2002) should be employed in the continuum models for accurate prediction of mechanical behaviors of nanostructures. Contrary to the local theories which assume that the stress at a point is a function of strain at that point, the nonlocal elasticity theory assumes that the stress at a point is a function of strains at all points in the continuum.

In the present work, attempt is made to study the static nonlinear postbuckling behavior of nanoscale beams according to the nonlocal zeroth-order shear deformation theory (ZSDT). The ZSDT was first developed by Shimpi (1999) for isotropic plates and thus it seems to be important to extend this theory to nanostructures by using the nonlocal elasticity theory of Eringen. The ZSDT accounts for the transverse shear deformation effect through the use of shear forces instead of rotational displacements as in existing shear deformation theories. The ZSDT contains the same

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unknowns as in the Timoshenko beam theory (TBT), but satisfies the traction-free boundary conditions on the top and bottom surfaces of the beam without requiring any shear correction factor. By employing the principle of virtual work together with the Von-Karman theory for large deflections, the equilibrium equations are obtained. The stress resultants are reformulated to consider the small scale effect according to the nonlocal elasticity theory of Eringen. A beam with simply supported and clamped-clamped boundary conditions is considered in this work and closed-form solutions for the critical buckling load and the static postbuckling response are predicted for the two cases. The effects of different parameters like the nonlocal parameter and the length-to-height ratio on the critical buckling load and the postbuckling response is analyzed. The obtained results are compared with the existing solutions to verify the accuracy of present theory in predicting the critical buckling load and the postbuckling response of nanobeams.

## 2. Theoretical formulations

### 2.1 Kinematics

The displacement field of the ZSDT is adopted based on the assumption that the transverse shear stress vary parabolically within the depth of the beam in such a way that it vanishes on the top and bottom surfaces. Consequently, there is no need to use shear correction factor. Based on this assumption, the following displacement field can be obtained (Hadji *et al*. 2015)

$$
\begin{aligned}
u(x,z) &= u_0(x) - z\frac{\partial w_0}{\partial x} + \frac{1}{\lambda_x}\left[\frac{3}{2}\left(\frac{z}{h}\right) - 2\left(\frac{z}{h}\right)^3\right]Q_x(x) \\
v(x,z) &= 0 \\
w(x,z) &= w_0(x)
\end{aligned}
\tag{1}
$$

where $u_0$ and $w_0$ are the displacements of a point on the mid-plane of the beam in the $x$ and $z$ directions, respectively; $h$ is the beam thickness; $Q_x$ is the transverse shear force; and $\lambda_x$ is unknown constant determined based on the definition of the transverse shear force as

$$Q_x = \int_{-h/2}^{h/2} \tau_{xz}\, dz \tag{2}$$

The Von-Karman-type of geometric non-linearity is taken into consideration in the strain-displacement relations which are as follows

$$\varepsilon_x = \varepsilon_x^0 + z\,k_x + f\,\frac{dQ_x}{\lambda_x dx} \quad \text{and} \quad \gamma_{xz} = g\,\frac{Q_x}{\lambda_x} \tag{3}$$

where $\varepsilon_x^0$ and $k_x$ are, respectively, the nonlinear longitudinal strain and curvature defined as

$$\varepsilon_x^0 = \frac{du_0}{dx} + \frac{1}{2}\left(\frac{d\,w_0}{dx}\right)^2 \quad \text{and} \quad k_x = -\frac{d^2 w_0}{dx^2} \tag{4a}$$

and

$$f = \frac{3}{2}\left(\frac{z}{h}\right) - 2\left(\frac{z}{h}\right)^3 \quad \text{and} \quad g = \frac{df}{dz} = \frac{3}{2h}\left[1 - 4\left(\frac{z}{h}\right)^2\right] \tag{4b}$$

### 2.2 Constitutive relations

Nonlocal elasticity theory was first proposed by Eringen (1983), and he assumed that the stress at a reference point is a functional of the strain field at every point of the continuum. The differential form of the nonlocal constitutive relation proposed by Eringen (1983) for the normal stress $\sigma_x$ and shear stress $\tau_{xz}$ are given by

$$\sigma_x - \mu\frac{d^2\sigma_x}{dx^2} = E\varepsilon_x \tag{5a}$$

$$\tau_{xz} - \mu\frac{d^2\tau_{xz}}{dx^2} = G\gamma_{xz} \tag{5b}$$

where $E$ and $G$ are the elastic modulus and shear modulus of the nanobeam, respectively; $\mu=(e_0 a)^2$ is the nonlocal parameter, $e_0$ is a constant appropriate to each material and $a$ is an internal characteristic length. When the nonlocal parameter is taken as $\mu=0$, the constitutive relation of the local theory is obtained.

### 2.3 Equilibrium equations

The principle of virtual displacements is employed herein to derive the equilibrium equations. The principle can be stated in analytical form as

$$\int_0^L \int_A \left(\delta U + \delta V\right) dA\,dx = 0 \tag{6}$$

where $\delta U$ is the virtual variation of the strain energy; $\delta V$ is the virtual variation of the work done by the external applied loads. The variation of the strain energy of the beam can be stated as

$$
\begin{aligned}
\delta U &= \int_0^L \int_A \left(\sigma_x \delta\,\varepsilon_x + \tau_{xz}\delta\,\gamma_{xz}\right) dA\,dx \\
&= \int_0^L \left(N_x \delta\,\varepsilon_x^0 + M_x \delta\,k_x + P_x\frac{d\delta\,Q_x}{\lambda_x dx} + R_x\frac{\delta\,Q_x}{\lambda_x}\right)dx
\end{aligned}
\tag{7}
$$

where $N_x$, $M_x$, $P_x$ and $R_x$ are the stress resultants defined as

$$
\begin{aligned}
(N_x, M_x, P_x) &= \int_A (1, z, f)\,\sigma_x dA \quad \text{and} \\
R_x &= \int_A g\,\tau_{xz} dA
\end{aligned}
\tag{8}
$$

The variation of the work done by the external applied loads can be written as

$$\delta V = -\int_0^L N_0 \frac{d\, w_0}{dx}\frac{d\delta\, w_0}{dx}dx \tag{9}$$

where $N_0$ is an external compressive load applied at the beam's ends.

Substituting the expressions for $\delta U$ and $\delta V$ from Eqs. (7) and (9) into Eq. (6) and integrating by parts, and collecting the coefficients of $\delta w_0$, and $\delta Q_x$, the following equilibrium equations of the proposed beam theory are obtained

$$\frac{dN_x}{dx} = 0 \tag{10a}$$

$$\frac{d^2 M_x}{dx^2} + \frac{d}{dx}\left(N_x \frac{d\, w_0}{dx}\right) - N_0 \frac{d^2\, w_0}{dx^2} = 0 \tag{10b}$$

$$\frac{dP_x}{dx} - R_x = 0 \tag{10c}$$

By substituting Eq. (3) into Eq. (5) and the subsequent results into Eq. (8), the stress resultants are obtained as

$$N_x - \mu \frac{d^2 N_x}{dx^2} = A_{11}\left[\frac{du_0}{dx} + \frac{1}{2}\left(\frac{d\, w_0}{dx}\right)^2\right] \tag{11a}$$

$$M_x - \mu \frac{d^2 M_x}{dx^2} = -D_{11}\frac{d^2 w_0}{dx^2} + D_{11}^s \frac{dQ_x}{\lambda_x dx} \tag{11b}$$

$$P_x - \mu \frac{d^2 P_x}{dx^2} = -D_{11}^s\frac{d^2 w_0}{dx^2} + H_{11}^s \frac{dQ_x}{\lambda_x dx} \tag{11c}$$

$$R_x - \mu \frac{d^2 R_x}{dx^2} = A^s \frac{Q_x}{\lambda_x} \tag{11d}$$

where

$$\left(A_{11}, D_{11}, D_{11}^s, H_{11}^s\right) = \int_A \left(1, z^2, z\, f, f^2\right) E dA, \qquad A^s = \int_A g^2 G dA \tag{12}$$

In light of Eqs. (4a) and (10a), Eq. (11a) can be expressed in local form as follows

$$N_x = A_{11}\left(\frac{du_0}{dx} + \frac{1}{2}\left(\frac{d\, w_0}{dx}\right)^2\right) \tag{13}$$

Solving Eq. (10b) for $d^2M_x/dx$, substituting the outcome into Eq. (11b), and noting that $dN_x/dx=0$, we obtain the nonlocal moment stress resultant as

$$M_x = -D_{11}\frac{d^2 w_0}{dx^2} + \mu\left(N_0 - N_x\right)\frac{d^2 w_0}{dx^2} + D_{11}^s \frac{dQ_x}{\lambda_x dx} \tag{14}$$

To develop the equilibrium equations in terms of the displacement components, we can used both Eq. (14) and Eq. (10b), and noting that $dN_x/dx=0$, we obtain the moment equilibrium equation as

$$D_{11}\frac{d^4 w_0}{dx^4} + \left(N_0 - N_x\right)\frac{d^2}{dx^2}\left(w_0 - \mu\frac{d^2 w_0}{dx^2}\right) - D_{11}^s \frac{d^3 Q_x}{\lambda_x dx^3} = 0 \tag{15}$$

By integrating Eq. (13) once with respect to $x$ and by setting $u(0)=u(L)=0$ (i.e., the beam ends do not move), we obtain

$$N_x = \frac{EA}{2L}\int_0^L \left(\frac{dw_0}{dx}\right)^2 dx \tag{16}$$

Substituting Eq. (16) into Eq. (15), we obtain

$$D_{11}\frac{d^4 w_0}{dx^4} + \left(N_0 - \frac{EA}{2L}\int_0^L \left(\frac{dw_0}{dx}\right)^2 dx\right)\frac{d^2}{dx^2}\left(w_0 - \mu\frac{d^2 w_0}{dx^2}\right) - D_{11}^s \frac{d^3 Q_x}{\lambda_x dx^3} = 0 \tag{17}$$

To express the third equilibrium equation in terms of the displacements, we solve Eq. (10c) for $P_x$ and obtain

$$\frac{dP_x}{dx} = R_x \tag{18}$$

Differentiating Eq. (18) once with respect to $x$ and substituting the outcome into Eq. (11c), one obtains

$$P_x - \mu\frac{dR_x}{dx} = -D_{11}^s \frac{d^2 w_0}{dx^2} + H_{11}^s \frac{dQ_x}{\lambda_x dx} \tag{19}$$

Solve the above equation for $P_x$, differentiate the outcome once with respect to $x$, and substitute into Eq. (18), we obtain

$$R_x - \mu\frac{d^2 R_x}{dx^2} + D_{11}^s \frac{d^3 w_0}{dx^3} - H_{11}^s \frac{d^2 Q_x}{\lambda_x dx^2} = 0 \tag{20}$$

Finally, the above equation, in light of Eq. (11d), can be expressed as follows

$$A^s \frac{Q_x}{\lambda_x} - H_{11}^s \frac{d^2 Q_x}{\lambda_x dx^2} + D_{11}^s \frac{d^3 w_0}{dx^3} = 0 \tag{21}$$

Finally, we outline that Eqs. (17) and (21) govern the nonlocal nonlinear response of beams subjected to an external compressive load.

## 3. Analytical solutions

The closed-form solution of Eqs. (17) and (21) for the critical buckling load and the static postbuckling response of the beam with simply-supported and clamped–clamped end conditions can be constructed.

### *3.1 Analytical solutions for simply supported boundary conditions*

The boundary conditions for the simply-supported beam are

$$w_0(x) = a \sin \frac{\pi x}{L} \tag{22a}$$

$$Q_x(x) = b \cos \frac{\pi x}{L} \tag{22b}$$

where $a$ and $b$ are unknowns to be determined. It should be noted that $a$ is the maximum static deflection at the midspan of the beam. Substituting Eqs. (22) into Eqs. (17) and (21) and solving for $a$ and $b$. Three solutions for the static postbuckling amplitude $a$ are obtained. The first is the trivial solution, $a$=0, that corresponds to the unstable equilibrium position of the prebuckling state. The other two solutions are given by

$$a = \pm \frac{2}{\pi} \sqrt{\frac{N_0 L^2}{A_{11}} - \frac{\pi^2 L^2}{\left(\pi^2 \mu + L^2\right)} \left( \frac{D_{11}}{A_{11}} - \frac{\pi^2 (D_{11}^s)^2}{A_{11}\left(A^s L^2 + \pi^2 H_{11}^s\right)} \right)} \tag{23}$$

These two solutions correspond to the stable equilibrium positions in the postbuckling state.

On the other hand, the critical buckling load, $N_{cr}$, can be obtained by solving the linear counterpart of Eq. (17). The result is

$$N_{cr} = \frac{\pi^2 \left[ D_{11} A^s L^2 + \pi^2 \left( D_{11} H_{11}^s - (D_{11}^s)^2 \right) \right]}{\left(\pi^2 \mu + L^2\right)\left(A^s L^2 + \pi^2 H_{11}^s\right)} \tag{24}$$

The nondimensional critical buckling load, $\overline{N}_{cr}$, is defined as follows

$$\overline{N}_{cr} = N_{cr} \frac{L^2}{D_{11}} \tag{25}$$

### 3.2 Analytical solutions for clamped - clamped beam

For clamped-clamped beams, the following displacement field is assumed

$$w_0(x) = \frac{1}{2} a \left( 1 - \cos \frac{2\pi x}{L} \right) \tag{26a}$$

$$Q_x(x) = b \sin \frac{2\pi x}{L} \tag{26b}$$

Substituting Eqs. (26) into Eqs. (17) and (21), one obtains the static postbuckling amplitude as

$$a = \pm \frac{2}{\pi} \sqrt{\frac{N_0 L^2}{A_{11}} - \frac{4\pi^2 L^2}{\mu\left(4\pi^2 + L^2\right)} \left( \frac{D_{11}}{A_{11}} - \frac{4\pi^2 (D_{11}^s)^2}{A_{11}\left(A^s L^2 + 4\pi^2 H_{11}^s\right)} \right)} \tag{27}$$

$$N_{cr} = \frac{4\pi^2 \left[ D_{11} A^s L^2 + 4\pi^2 \left( D_{11} H_{11}^s - (D_{11}^s)^2 \right) \right]}{\left(4\pi^2 \mu + L^2\right)\left(A^s L^2 + 4\pi^2 H_{11}^s\right)} \tag{28}$$

The nondimensional critical buckling load, $\overline{N}_{cr}$, is defined as follows

$$\overline{N}_{cr} = N_{cr} \frac{L^2}{D_{11}} \tag{29}$$

## 4. Results and discussion

In this section, various numerical results are established and discussed to check the accuracy of present theory in predicting the buckling and the post-buckling responses of nanoscale beams. The side of nanoscale beam $L$ is assumed to be 10 nm, the modulus of elasticity $E$ and the Poisson's ratio $v$, are supposed to be 30 MPa and 0.3, respectively. The obtained results using the present nonlocal ZSDT for different values of the small scale parameter and length-to-depth ratio $L/h$ are compared with those reported by Emam (2013) based on Euler-Bernoulli beam theory (EBT), the Timoshenko beam theory (TBT) and Reddy beam theory (RBT). It can be seen that the results of present theory are in excellent agreement with those of RBT used by Emam (2013) for all values of small scale parameter and length-to-depth ratio even for clamped-clamped beams.

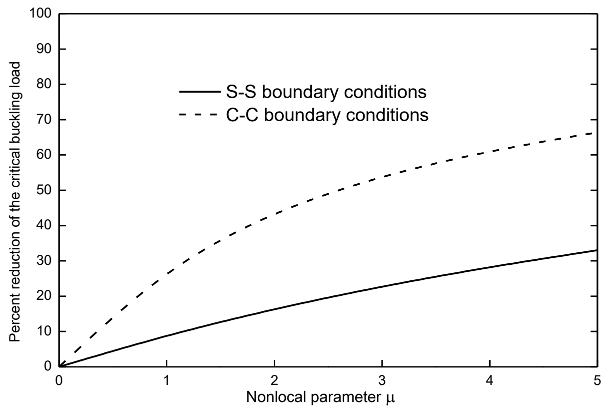


Fig. 1 Variation of the percent reduction of the critical buckling load with the nonlocal parameter

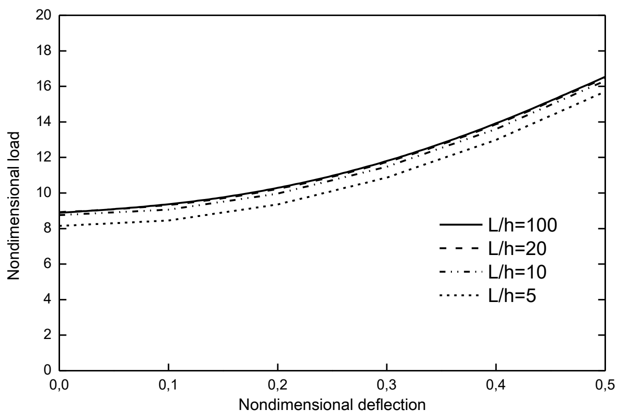


Fig. 2 Variation of the maximum buckling with the applied axial load for simply supported beams with $\mu$=1

Fig. 1 plotted the percent reduction of the critical buckling load versus the small scale parameter ($\mu$) for both simply supported and clamped-clamped nanoscale beams. It is seen that the critical buckling load is very affected by the small scale parameter and the results of the simply supported beam are less than that of the clamped-clamped beam.

Fig. 2 gives mechanical postbuckling load-deflection curves for different values of the thickness parameter ($L/h$) according to the present formulation. It can be seen that the effect of the thickness parameter ($L/h$) on the arising amplitude of buckling is insignificant in the case of simply supported beam. However, in the case of the clamped-clamped beam, this effect is highly significant as is shown in Fig. 3.

Figs. 4 and 5 illustrate the mechanical postbuckling load-deflection curves for different values of the small scale parameter ($\mu$) for the simply supported and the clamped-clamped beams, respectively. It can be seen from these figures that the small scale parameter not only reduces the critical buckling load, but it also magnifies the arising amplitude of buckling. Thus, it can be concluded that the nonlocal parameter softens the beam and understanding this behavior is crucial in investigating and designing nanoscale beams as sensors. For clamped-clamped beams, the contribution of the small scale parameter on the postbuckling behavior is more significant as is shown in Fig. 5.

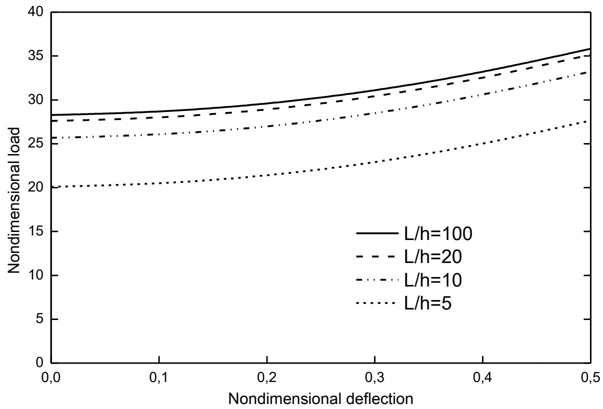


Fig. 3 Variation of the maximum buckling with the applied axial load for clamped-clamped beams with $\mu$=1

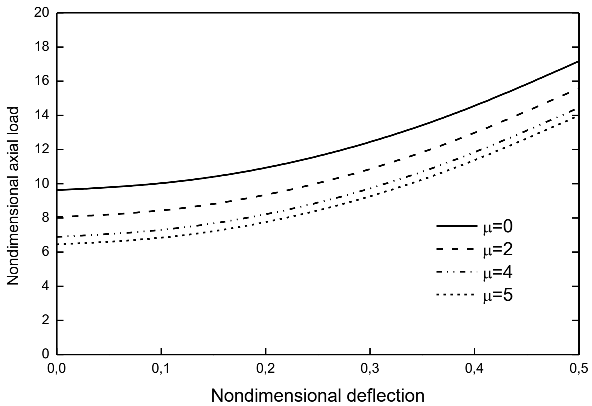


Fig. 4 Variation of the maximum buckling with the applied axial load with for simply supported beams with $L/h$=10

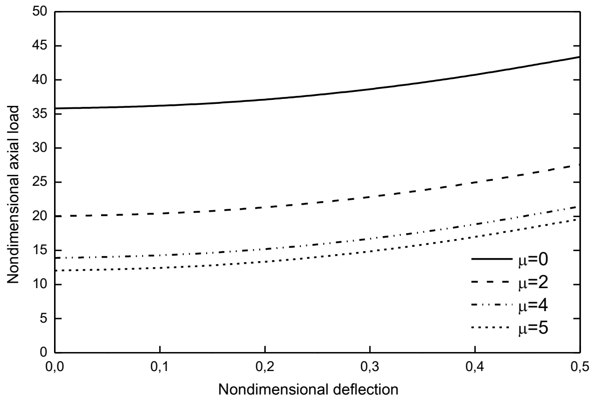


Fig. 5 Variation of the maximum buckling with the applied axial load with for clamped-clamped beams with $L/h$=10

Table 1 Nondimensional first critical buckling load for simply supported nanoscale beams

| *L/h* | $\mu$ | EBT | TBT | RBT | Present |
|---|---|---|---|---|---|
| 100 | 0 | 9.8696 | 9.8671 | 9.8671 | 9.8671 |
| | 1 | 8.9830 | 8.9807 | 8.9807 | 8.9807 |
| | 2 | 8.2426 | 8.2405 | 8.2405 | 8.2405 |
| | 3 | 7.6149 | 7.6130 | 7.6130 | 7.6130 |
| | 4 | 7.0761 | 7.0743 | 7.0743 | 7.0743 |
| | 5 | 6.6085 | 6.6068 | 6.6068 | 6.6068 |
| 20 | 0 | 9.8696 | 9.8067 | 9.8067 | 9.8067 |
| | 1 | 8.9830 | 8.9258 | 8.9258 | 8.9258 |
| | 2 | 8.2426 | 8.1900 | 8.1900 | 8.1900 |
| | 3 | 7.6149 | 7.5664 | 7.5664 | 7.5664 |
| | 4 | 7.0761 | 7.0310 | 7.0310 | 7.0310 |
| | 5 | 6.6085 | 6.5663 | 6.5663 | 6.5663 |
| 10 | 0 | 9.8696 | 9.6227 | 9.6228 | 9.6228 |
| | 1 | 8.9830 | 8.7583 | 8.7583 | 8.7583 |
| | 2 | 8.2426 | 8.0364 | 8.0364 | 8.0364 |
| | 3 | 7.6149 | 7.4244 | 7.4245 | 7.4245 |
| | 4 | 7.0761 | 6.8990 | 6.8991 | 6.8991 |
| | 5 | 6.6085 | 6.4431 | 6.4432 | 6.4432 |

Table 2 Nondimensional first critical buckling load for clamped-clamped nanoscale beams

| *L/h* | $\mu$ | EBT | TBT | RBT | Present |
|---|---|---|---|---|---|
| 100 | 0 | 39.4784 | 39.4379 | 39.4379 | 39.4379 |
| | 1 | 28.3043 | 28.2753 | 28.2753 | 28.2753 |
| | 2 | 22.0603 | 22.0377 | 22.0377 | 22.0377 |
| | 3 | 18.0733 | 18.0548 | 18.0548 | 18.0548 |
| | 4 | 15.3068 | 15.2911 | 15.2911 | 15.2911 |
| | 5 | 13.2749 | 13.2613 | 13.2613 | 13.2613 |

Table 2 Continued

| *L/h* | $\mu$ | EBT | TBT | RBT | Present |
|---|---|---|---|---|---|
| 20 | 0 | 39.4784 | 38.4907 | 38.4910 | 38.4910 |
| | 1 | 28.3043 | 27.5962 | 27.5964 | 27.5964 |
| | 2 | 22.0603 | 21.5084 | 21.5085 | 21.5085 |
| | 3 | 18.0733 | 17.6211 | 17.6212 | 17.6212 |
| | 4 | 15.3068 | 14.9239 | 14.9240 | 14.9240 |
| | 5 | 13.2749 | 12.9428 | 12.9429 | 12.9429 |
| 10 | 0 | 39.4784 | 35.8034 | 35.8075 | 35.8075 |
| | 1 | 28.3043 | 25.6695 | 25.6724 | 25.6724 |
| | 2 | 22.0603 | 20.0067 | 20.0090 | 20.0090 |
| | 3 | 18.0733 | 16.3909 | 16.3927 | 16.3927 |
| | 4 | 15.3068 | 13.8819 | 13.8835 | 13.8835 |
| | 5 | 13.2749 | 12.0391 | 12.0405 | 12.0405 |

## 5. Conclusions

This work presents analytical solutions for the nonlinear postbuckling of nanoscale beams subjected to axial compression within the context of the zeroth-order shear deformation theory and the nonlocal differential constitutive relations of Eringen. The present formulation is able of capturing both shear deformation and nonlocal effects of nanoscale beams, and does not require shear correction factors. Closed-form solutions for the critical buckling load and the nonlinear static amplitude in the postbuckling state for simply supported and clamped clamped beams are given. It is observed that buckling and post buckling response of nanoscale beams is very susceptible to the nonlocal parameter. An improvement of present formulation will be considered in the future work dealing with composite and functionally graded materials (Tounsi *et al*. 2013b, Bouderba *et al*. 2013, Ait Amar Meziane *et al*. 2014, Zidi *et al*. 2014, Ait Atmane *et al*. 2015, Ait Yahia *et al*. 2015, Mahi *et al*. 2015, Taibi *et al*. 2015, Attia *et al*. 2015, Tounsi *et al*. 2016, Bousahla *et al*. 2016, Bouderba *et al*. 2016, Bellifa *et al*. 2016, Beldjelili *et al*. 2016, Boukhari *et al*. 2016, Houari *et al*. 2016, Fahsi *et al*. 2017, Chikh *et al*. 2017, Meksi *et al*. 2017, and to account for the thickness stretching effect by using quasi-3D shear deformation models (Bessaim *et al.* 2013, Bousahla *et al.* 2014, Swaminathan and Naveenkumar 2014, Sayyad and Ghugal 2014; Belabed *et al.* 2014, Fekrar *et al.* 2014, Hebali *et al.* 2014, Hamidi et al. 2015, Meradjah *et al.* 2015, Bourada *et al.* 2015, Bennoun *et al*. 2016, Draiche *et al*. 2016).

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