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A novel evidence theory model and combination rule for reliability estimation of structures

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Abstract. Due to the discontinuous nature of uncertainty quantification in conventional evidence theory(ET), the computational cost of reliability analysis based on ET model is very high. A novel ET model based on fuzzy distribution and the corresponding combination rule to synthesize the judgments of experts are put forward in this paper. The intersection and union of membership functions are defined as belief and plausible membership function respectively, and the Murfhy's average combination rule is adopted to combine the basic probability assignment for focal elements. Then the combined membership functions are transformed to the equivalent probability density function by a normalizing factor. Finally, a reliability analysis procedure for structures with the mixture of epistemic and aleatory uncertainties is presented, in which the equivalent normalization method is adopted to solve the upper and lower bound of reliability. The effectiveness of the procedure is demonstrated by a numerical example and an engineering example. The results also show that the reliability interval calculated by the suggested method is almost identical to that solved by conventional method. Moreover, the results indicate that the computational cost of the suggested procedure is much less than that of conventional method. The suggested ET model provides a new way to flexibly represent epistemic uncertainty, and provides an efficiency method to estimate the reliability of structures with the mixture of epistemic and aleatory uncertainties.

Keywords: evidence theory; fuzzy set; combination rule; hybrid reliability

1. Introduction

Uncertainties always exist in engineering systems such as material properties, load, and so on. The traditional reliability analysis methods based on classical probability theory has been widely adopted in engineering fields (Acar *et al*. 2005). This methods include the Monte Carlo Simulation method (MCS), important sampling method (Melchers 1989), the Monte Carlo Markov Chain (Lu 2009), First Order Second Moment (Cornell 1969) and so on. However, due to the lack of data, incomplete information and other factors, the precise probability distribution of uncertain parameters is not entirely obtained. Under this situation, the uncertain variables may be modeled by fuzzy theory (Chakraborty *et al*. 2009, Fang *et al*. 2015, Zhang 2015), ET (Suo 2013), convex set theory (Bi 2014) or other non-probability theories (Li 2011). ET, for instance, an imprecise probability theory based on belief function and plausibility functions has been widely applied in in engineering practices (Oberkampf *et al*. 2001, Bai *et al*. 2012). However, due to the discontinuous nature of uncertainty quantification in ET, the computational cost of reliability analysis based on ET is high (Du 2006), which is still a challenging problem for reliability analysis. Therefore, some scholars have focused on this problem. For example, Jiang *et al*. (2013) presented a novel method to deal with the epistemic uncertainty. In this method, a uniformity approach is used to deal with the evidence variables, through which the original reliability problem can be transformed to a traditional reliability problem with only random uncertainty. Du *et al*. (2006) developed a unified uncertainty analysis framework using probability and ET, and a first order reliability method to conduct the hybrid uncertainty problems. However, the parameters processing is much computationally expensive. Xiao *et al*. (2012) proposed an unified uncertainty analysis method based on the maximum entropy approach and simulation. The method avoids the most probability point (MPP) searching and non-normal to normal transformation by directly sampling in random and interval variables. Tao (2014) proposed a novel method for safety analysis of structures based on probability and ET. In this method, the probability density function of the epistemic variables is assumed piecewise uniform distribution. Although the methods discussed above have many advantages in reliability analysis, the uniformity approach or piecewise uniform distribution may not fully represent the judgments of experts on the epistemic uncertainty. Therefore, it is urgent to investigate a novel ET model to flexibly represent epistemic uncertainty. For example, the fuzzy distribution may be adopted in focal elements to represent epistemic uncertainty.

Fuzzy set theory is suited to the situations where sufficient information is not available for defining a probability distribution and represents the available data

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with fuzzy variables. Zadeh (1965) introduced the concept of fuzzy subset $\tilde{A}$ of X by extending the characteristic function. The core concept in fuzzy theory is fuzzy subset $\tilde{A}$ of X which is defined by its membership function u_A $:X \rightarrow [0:1]$. The value $u_A(a)$ represents the membership degree of a point a in the set "A". Conventional fuzzy structural analysis is oriented to estimate the membership function of an output structural variable given the membership functions of the input ones, which are discretized in so-called α-cut levels. Li *et al*. (2015) explained the traditional failure possibility of the structures with fuzzy variables from the probability perspective and proposed to analyze the reliability of the structure with fuzzy variables based on the Kriging surrogate model method. Aghili *et al*. (2014) addressed the issues of both symmetric and asymmetric uncertainty distributions in fuzzy failure rates and that how a fuzzy result can profoundly come under the influence of them. Purba *et al*. (2014) proposed a fuzzy-based reliability approach to evaluate basic events of system fault trees whose failure precise probability distributions of lifetime are not available. Kumar *et al*. (2012) addressed the fuzzy system reliability analysis using different types of intuitionistic fuzzy numbers, by which membership functions and non-membership functions of fuzzy reliability of both series and parallel system could be constructed. In fuzzy structural analysis, Hurtado *et al*. (2012) suggested that the reliability analysis with conventional α-cut levels could be replaced by the first-order reliability analysis. Although fuzzy set is widely applied in reliability analysis, how to synthesize the judgments of experts on fuzzy variables is still a problem. For example, two experts independently give different membership functions on the same fuzzy variable, there is no appropriate ways to synthesize the two membership functions. Recent years, some researchers have investigated hybrid reliability method, such as Fuzzy Set and Monte Carlo Simulation (Canizes *et al*. 2012), randomness, fuzziness and non-probabilistic method (Ni *et al*. 2010), fuzzy and random (Li *et al*. 2014), intervals and fuzzy (Fuh *et al*. 2014), probability and fuzzy set theory (Chakraborty *et al*. 2007, Xiao *et al*. 2012), probability and ET (Bae 2004), interval and probability (Jiang *et al*. 2015), fuzzy probability (Purba *et al*. 2015).

Although researchers have proposed many methods to deal with hybrid uncertainty, how to represent the epistemic uncertainty flexibly and improve the computational efficiency and precise is still a challenging for reliability analysis. In this paper, a novel ET model based on fuzzy distribution and the corresponding combination rules are suggested, then, this model is adopted in reliability analysis. The authors provide a brief overview of ET and fuzzy set theory in Section 1. The ET model based on fuzzy distribution is developed in Section 2. Then, the combination rules for the novel ET model are provided in Section 3, and the reliability analysis procedure is given in Section 4. Numerical examples are provided in Section 5 to support the proposed method. Finally, the conclusion is presented in Section 6.

2. Basic concept of ET and fuzzy set

2.1 Basic concept of ET

ET is developed by Dempster and Shafer (Shafer 1976). It can be also regarded as an extension of the classical probability theory. Instead of one measure in traditional probability theory, ET employs belief $\boldsymbol{Bel}$ and plausibility $\boldsymbol{pl}$ to characterize the confidence interval of real reliability probability. Under different circumstances, ET can be equivalent to the classical probability theory, possibility theory, etc. It contains the following important concepts.

(1) Θ is defined to denote discernment frame. If set function $m: 2^{\Theta} \rightarrow [0,1] (2^{\Theta}$ is the power set of $\Theta)$ should satisfy the following two axioms

$$m(\phi) = 0$$
$$\sum_{A \subset \Theta} m(A) = 1 \tag{1}$$

where m is expressed as the Basic Probability Assignment (BPA), and $\forall A \subset \Theta$, $\boldsymbol{m(A)}$ is the basic probability of $\boldsymbol{A}$.

(2) The belief $\boldsymbol{Bel(A)}$, plausibility $\boldsymbol{Pl(A)}$ and a subset $\boldsymbol{A}$ of $\boldsymbol{C}$ are defined by $\boldsymbol{Bel(A)} = \sum_{\boldsymbol{C \cap A}} \boldsymbol{m(C)}$

$$Pl(A) = \sum_{C \cap A \neq \emptyset} m(C) \tag{2}$$

(3) Combination rules: while the information comes from multiple sources, such as two condition monitoring schemes or multiple experts, the multiple BPA structures can be aggregated by so-called rules of combination. The joint BPA is calculated by

$$m(A) = \begin{cases} 0 & , \quad A = \emptyset \\ \dfrac{\sum_{\substack{A_1 \cap \cdots \cap A_n = A \\ A_1, \cdots A_n \subset \Theta}} m_1(B_i) m_2(C_j)}{K} & , A \neq \emptyset \end{cases} \tag{3}$$

Where $K = \sum_{\substack{A_1 \cap \cdots \cap A_n \neq \emptyset \\ A_1, \cdots A_n \subset \Theta}} m_1(A_1) \cdots m_n(A_n)$ is the conflict factor.

The above rule is called Dempster's rule, which has been applied widely in combining belief functions. However, some researchers have provided examples of counter-intuitive results produced by Dempster's rule for combining belief functions. For example, Murphy (2000) proposed a combination rule to balance multiple evidences. The core of this rule includes the following steps.

(1) Allow mass in the null set, which means to eliminate the need for normalization $\boldsymbol{\Theta}$. This can eliminate division by $\boldsymbol{1 - K}$ from Eq. (2), Dempster's rule;

(2) Assign the mass in the null set to the base set $\boldsymbol{\Theta}$. Since the correct destination of the conflicting evidence is unknown, it should be distributed among all the elements, rather than just the elements which happen to be intersections of the combining masses;

(3) Average the masses assigned to a subset $\boldsymbol{Z}$ to determine its belief function $\boldsymbol{Bel}$. Eq. (4) covers the case where two rules are combined.

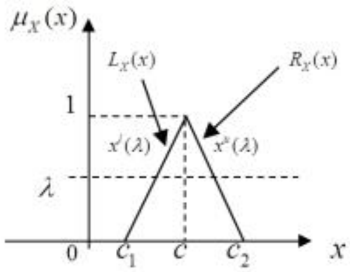


Fig. 1 Membership function of fuzzy set [33]

$$Bel = \frac{1}{2}[\sum_{X \subset Z} m_1(X) + \sum_{Y \subset Z} m_2(Y)] \tag{4}$$

where $\boldsymbol{X}$ and $\boldsymbol{Y}$ represent uncertain variables.

2.2 Basic concept of fuzzy set

A fuzzy input variable $\boldsymbol{X}$can be characterized by its normalized membership function $\boldsymbol{u_x(x)}$ as Eq. (5) (Li 2005)

$$0 \le \mu_x(x) \le 1 \tag{5}$$

Considering a triangle fuzzy variable, it's membership function is shown in Fig. 1, $\boldsymbol{L_X(x)}$ and $\boldsymbol{R_X(x)}$ are the left branch and right branch of the membership function $\boldsymbol{u_x(x)}$, which are linear increasing and decreasing monotonic functions with respect to the variable $\boldsymbol{X}$ respectively. The membership function of the fuzzy variable $\boldsymbol{X}$ can be expressed as

$$\mu_x(x) = \begin{cases} L_X(x) = \dfrac{x - c_1}{c - c_1} & c_1 \le x \le c \\ 1 & x = c \\ R_X(x) = \dfrac{c_2 - x}{c_2 - c} & c \le x \le c_2 \end{cases} \tag{6}$$

According to the $\boldsymbol{\lambda}$ level sets of the fuzzy variables, the convex fuzzy variable $\boldsymbol{X}$ can be characterized by a family of $\boldsymbol{\lambda}$ level sets $\boldsymbol{X(\lambda)}$, which is shown in Eq. (7) [34]

$$X(\lambda)=\{[x^l(\lambda), x^u(\lambda)], \lambda \in [0,1]\} \tag{7}$$

Based on the treatment of different level cut, the uncertainty under each cut level can be regarded as an interval. This cut subset method is always applied in conventional reliability analysis based on fuzzy set.

2.3 Basic concept of entropy

Entropy is always used to represent the uncertainty of variables. The larger entropy ,the greater uncertainty is. The computational method of entropy of fuzzy variable can be found in reference [33]. The entropy of continuous fuzzy variable can be solved by Eq. (8)

$$H(x) = \int_{-\infty}^{+\infty} S(c(x))dx \tag{8}$$

In Eq. (8), $\boldsymbol{c(x) = 0.5u(x)}$ and $\boldsymbol{u(x)}$ is the membership function of fuzzy variable. The integrand $\boldsymbol{S}$ is defined as Eq. (9)

$$S(t) = -t \ln t - (1 - t) \ln(1 - t) \tag{9}$$

3. The novel ET model based on fuzzy distribution

Based on the discussion of ET and fuzzy set, it may be a good idea to combine ET with fuzzy set to represent epistemic uncertainty. Although both fuzzy set and ET are adopted to represent epistemic uncertainty, the emphasis of the two models is different. In the conventional fuzzy sets model, shown in Fig. 1, the membership functions are defined in focal elements without BPA. Conversely, in the conventional ET model, the BPA are assigned to focal elements, and there is no distribution function defined in focal elements. Intuitively, the combination of the two models may represent epistemic uncertainty flexibly. Based on this idea, the authors suggest a novel model named as ET model based on fuzzy distribution. In this model, both the BPAs and the membership functions are assigned on all focal elements. Moreover, the height of the membership function is proportional to the BPA. An example of this novel model is shown in Fig. 2, in which all focal elements are assigned BPA and the sum of BPAs is equal to 1. In this example, the BPA in focal element $\mathbf{[c_1, c_2]}$, $[\mathbf{c_2}, \mathbf{c_3}]$ and $[\mathbf{c_3}, \mathbf{c_4}]$ is 0.2, 0.6 and 0.2 respectively. Moreover, the membership functions are assigned on the focal elements, such as triangle fuzzy number in interval $\mathbf{[c_1, c_2]}$, trapezoid fuzzy number in interval and $\mathbf{[c_2, c_3]}$ linear fuzzy number in interval $\mathbf{[c_3, c_4]}$. The normalized method presented in Ref (Rao *et al*. 2009) is adopted to normalize the membership function. In this method, the height of membership function is normalized by the maximum BPA. In this example, the maximum BPA is 0.6, which is in. $\mathbf{[c_2, c_3]}$ Therefore, the height of the membership function in the interval $\mathbf{[c_1, c_2]}$, $\mathbf{[c_2, c_3]}$ and $\mathbf{[c_3, c_4]}$ is 0.2/0.6, 0.6/0.6 and 0.2/0.6 respectively, i.e., 1/3, 1 and 1/3, which are show in Eq. (10).

Compared with the conventional fuzzy model, the BPA in this novel model may express expert's judgments on uncertain variables more effectively. Certainly, this model can be degenerated to conventional fuzzy model when the BPAs in all focal elements are identical. Compared with the conventional ET, the membership functions in the novel model are better for expressing expert's judgments on uncertain variables. In a word, the novel model possesses the advantages of conventional ET model and fuzzy model in representing epistemic uncertainty. Therefore, the computational efficiency of reliability analysis may be improved by the novel model

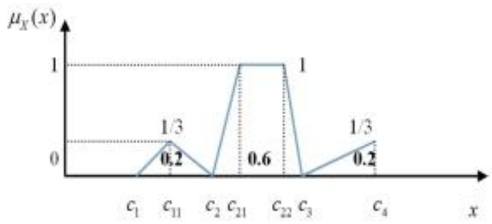


Fig. 2 The novel ET model based on fuzzy distribution

$$\mu_x(x)=\begin{cases}\dfrac{x-c_1}{c_{11}-c_1}\times\dfrac{1}{3} & c_1\le x\le c_{11}\\ \dfrac{c_2-x}{c_2-c_{11}}\times\dfrac{1}{3} & c_{11}\le x\le c_2\\ \dfrac{x-c_2}{c_{21}-c_2} & c_2\le x\le c_{21}\\ 1 & c_{21}\le x\le c_{22}\\ \dfrac{c_3-x}{c_3-c_{22}} & c_{22}\le x\le c_3\\ \dfrac{x-c_3}{c_4-c_3}\times\dfrac{1}{3} & c_3\le x\le c_4\end{cases}\qquad(10)$$

4. Combination rules for the novel ET model

In engineering practice, sometimes more than one expert give their judgments on the same epistemic uncertain variable, and the decider must take their judgments into account synthetically. Therefore, the uncertain information need be combined. With no doubt, a novel combination rules must be developed for this novel model. In other words, both the BPAs and the membership function of the novel ET model must be combined. To illustrate the suggested combination rules, an epistemic variable represented by the novel ET model is listed in Table 1. Two experts gave their judgments on $\boldsymbol{X}$, including the intervals and BPAs. The membership functions of $\boldsymbol{X}$ are assumed as triangle fuzzy number. The novel combination rules include three steps.

Step (1), combining BPA by the Murphy's combination rule presented in Section1.1. For this example, the combined BPAs are shown in Table.1, which indicates there are three focal elements in the combined evidence body, i.e., focal element [2,6], [6,7] and [7,9] with the BPA 0.5455, 0.3636 and 0.0909 respectively.

Step (2), normalizing fuzzy membership functions by the maximum BPA (Rao *et al*. 2009). In this example, the normalized fuzzy membership functions are shown in Fig. 2. The membership function 1 and 2 are depicted by red lines and black lines respectively.

Step (3), defining $\boldsymbol{Bel}$ and $\boldsymbol{pl}$ membership functions. Fig. 2 shows that the two membership functions are intersected and the overlapped area is filled by yellow dotted lines. Intuitively, the overlapped area is believed by the two experts. Inspired by the concepts of $\boldsymbol{Bel}$ in ET, the overlapped area is named as $\boldsymbol{Bel}$ area and the enveloping lines of this area are defined as $\boldsymbol{Bel}$ membership functions. The area filled by light blue lines is believed by only one of the experts. Similarly, this area is named as $\boldsymbol{pl}$ area and the enveloping lines of this area are defined as $\boldsymbol{pl}$ membership functions. Moreover, the area exceeding the $\boldsymbol{pl}$ area is named as against area, where no expert gives his confidence. For example, when $\boldsymbol{X}$ is equal to 3, the three intervals, namely, the $\boldsymbol{Bel}$ interval, $\boldsymbol{pl}$ interval and against interval are shown in Fig. 2.

Unlike the probability theory, ET measures uncertainty with two measures - $\boldsymbol{Bel}$ and $\boldsymbol{pl}$, which are regarded as lower and upper bounds of probability respectively. Similarly, for example, the $\boldsymbol{Bel}$ and $\boldsymbol{pl}$ membership functions of $\boldsymbol{X}$ shown in Fig. 3 and Fig. 4 may be viewed as the lower and upper bounds of the original membership function of $\boldsymbol{X}$ respectively. Therefore, the $\boldsymbol{Bel}$ and $\boldsymbol{pl}$ membership functions define a confidence interval for the original membership function, which can be verified by entropy. Solved by Eq. (6), the entropy of $\boldsymbol{X}$ before and after combination is listed in 2. The entropy of membership function 1 and membership function 2 are 0.6594 and 0.7483 respectively. The entropy of $\boldsymbol{Bel}$ membership function and $\boldsymbol{pl}$ membership function are 0.4116 and 0.7499 respectively. It is evident that the entropy of original membership functions (both membership function 1 and 2) is less than that of $\boldsymbol{pl}$ membership function and is more than that of $\boldsymbol{Bel}$ membership function. According the definition of entropy, the larger entropy, the greater uncertainty is. Therefore, it can be concluded that the uncertainty represented by $\boldsymbol{Bel}$ membership function is less than that represented by original membership functions, and it is rational that the $\boldsymbol{Bel}$ membership function is viewed as lower bounds of membership functions. Similarly, the $\boldsymbol{pl}$ membership function may be viewed as upper bounds of membership functions. It is noted, shown in le.1, Fig. 3 and Fig. 4, the number and the interval of focal elements and the BPAs are same in the $\boldsymbol{Bel}$ membership function and $\boldsymbol{pl}$ membership function.

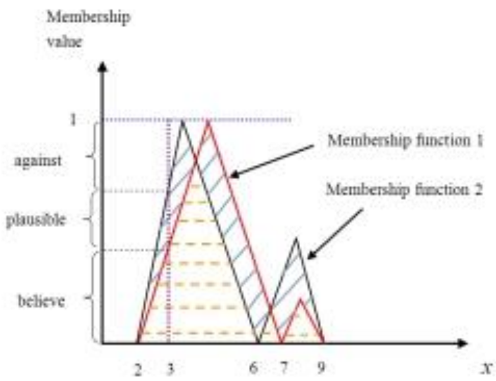


Fig. 3 The $\boldsymbol{Bel}$ and $\boldsymbol{pl}$ area in the novel model

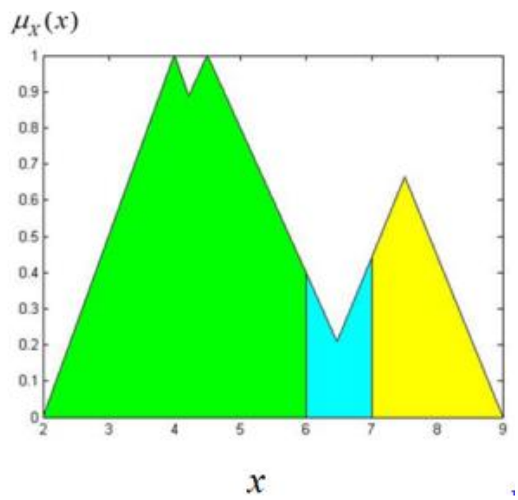


Fig. 4 the $\boldsymbol{pl}$ membership function of $\boldsymbol{x}$

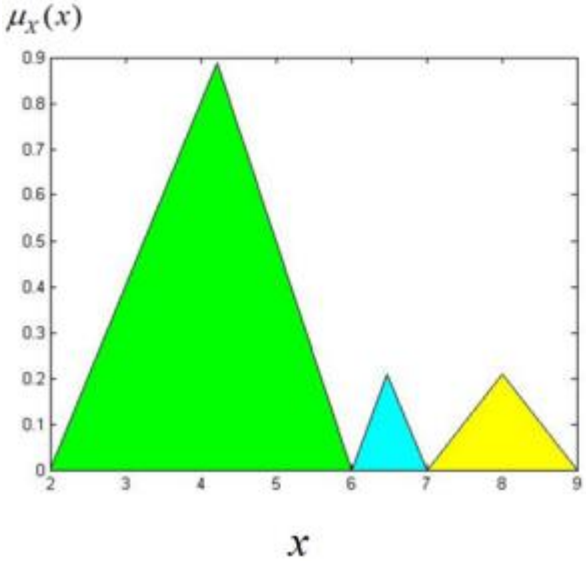


Fig. 5 the $\boldsymbol{Bel}$ membership function of $\boldsymbol{x}$

Table 1 Epistemic variable $\boldsymbol{x}$

Expert 1		Expert 2		The combined BPA	
Interval	BPA	Interval	BPA	Interval	BPA
[2,7]	0.8	[2,6]	0.6	[2,6]	0.5455
[7,9]	0.2	[6,9]	0.4	[6,7]	0.3636
				[7,9]	0.0909

Table 2 The entropy of $\boldsymbol{x}$ before and after combination

The entropy before combination		The entropy after combination	
membership function1	membership function 2	$\boldsymbol{pl}$ membership function	$\boldsymbol{Bel}$ membership function
0.6594	0.7483	0.7499	0.4116

5. The procedure for reliability analysis

The suggested novel ET model may be applied to represent epistemic uncertain variables flexiblyand provides the basis for reliability analysis. However, it is difficult to directly apply the novel ET model in conventional reliability analysis method based on probability theory such as the first order second moment (FOSM). The cause lies in that only the $\boldsymbol{Bel}$ membership function and $\boldsymbol{pl}$ membership function are defined in the novel ET model, not the probability density function ($\boldsymbol{pdf}$) which is the basis for conventional reliability analysis. Therefore, the $\boldsymbol{Bel}$ membership function and $\boldsymbol{pl}$ membership function need be transformed to the $\boldsymbol{pdf}$. Lu *et al*. (2004) advocated that the membership function of fuzzy set may be transformed to $\boldsymbol{pdf}$ with a normalization factor, which is shown in Eq. (11)

$$f_i^{(e)}(x_i) = \frac{f_i(x_i)}{\int f_i(x_i)dx_i} \tag{11}$$

Where $\boldsymbol{f_i^{(a)}(x_i)}$ is equivalent probability density function, $\boldsymbol{f_i(x_i)}$ represents $\boldsymbol{i-th}$ membership function.

In this paper, the transformation method shown in Eq. (11) is adopted to transform $\boldsymbol{pl}$ and $\boldsymbol{Bel}$ membership function to $\boldsymbol{pdf}$. It is noted that the $\boldsymbol{pl}$ and $\boldsymbol{Bel}$ membership function are piecewise continuous functions, which means the $\boldsymbol{pl}$ and $\boldsymbol{Bel}$ membership function differs in focal elements. Based on the definition of cumulative distribution function ($\boldsymbol{cfd}$) (Lu *et al*. 2009), the $\boldsymbol{cfd}$ $\boldsymbol{F(u)}$ of the novel model is calculated by Eq. (12)

$$F(u) = \sum_{i=1}^{p-1} m_i + \int_{x_p}^{x} f_p^{(e)} dx \tag{12}$$

Where the MPP is located in the $\boldsymbol{p-th}$ focal element, $\boldsymbol{\sum_{i=1}^{p-1} m_i}$ represents the sum of BPA from the first focal element to the $\boldsymbol{p-1-th}$ focal element, $\boldsymbol{\int_{x_p}^{x} f_p^{(e)} dx}$ represents the BPA of the $\boldsymbol{p-th}$ focal element.

The conventional equivalent normalization method (Lu *et al*. 2009) (belongs to FOSM) always be applied in reliability analysis for structures with aleatory uncertainties. Moreover, the $\boldsymbol{pdf}$ and $\boldsymbol{cdf}$ of epistemic uncertain variables represented by the novel ET model can be calculated by Eq. (11) and Eq. (12). Therefore, the FOSM may be adopted in reliability analysis for structures with the mixture of epistemic and aleatory uncertainties. Based on the conventional equivalent normalization method, the suggested reliability analysis procedure is shown as follows and the corresponding flow chart is shown in Fig. 6.

(1) Calculate the mean value and variance of uncertain variable modeled by the novel ET model through Eq. (13)~Eq. (15) (Tao *et al*. 2014)

$$\mu(x) = \int_{-\infty}^{+\infty} x f(x) dx \tag{13}$$

$$\mu(x^2) = \int_{-\infty}^{+\infty} x^2 f(x) dx \tag{14}$$

$$\sigma = \mu(x^2) - (\mu(x))^2 \tag{15}$$

Where $\boldsymbol{f(x)}$ represents the combined membership function (the $\boldsymbol{pl}$ or $\boldsymbol{Bel}$ membership function), $\boldsymbol{\mu}$ is mean value and $\boldsymbol{\sigma}$ is variance of uncertain variables.

(2) Initialize MPP $\boldsymbol{x^*}$ (for both the epistemic and aleatory uncertain variables), typically, the initial checking point may be assumed to be at the mean value of uncertain variables,

(3) Solve $\boldsymbol{cfd}$ of uncertain variable by Eq. (12) and select $\boldsymbol{pdf}$ (denoted by Eq. (11))according to the focal element where the uncertain variable located (this step is only for uncertain variables which denoted by novel ET model),

(4) Normalize non-normal variable by Eq. (16) and Eq. (17), $\boldsymbol{\mu_{x_i}}$ and $\boldsymbol{\sigma_{x_i}}$ is replaced by $\boldsymbol{\mu_{x_i'}}$ and $\boldsymbol{\sigma_{x_i'}}$

$$\mu_{X_i'} = x_i^* - \Phi^{-1}[F_{x_i}(x_2^*)]\sigma_{x_i'} \tag{16}$$

$$\alpha_{x_i}{'}=\frac{\varphi\{\Phi^{-1}[F_{x_I}(X_i^*)]\}}{f_{x_i}(x_i^*)} \tag{17}$$

where $\boldsymbol{x_i}$ is non-normal random variable, $\boldsymbol{x_i'}$ is equivalent normal random variable, $\boldsymbol{x^*}$ is checking point or MPP. $\boldsymbol{F}$ and $\boldsymbol{f}$ are $\boldsymbol{cdf}$ and $\boldsymbol{pdf}$ of no-normal variables respectively. $\boldsymbol{\Phi}$ is $\boldsymbol{cdf}$ and $\boldsymbol{\varphi}$ is $\boldsymbol{pdf}$ of standard normal variables, Φ^{-1} denotes the inverse function of $\boldsymbol{\Phi}$. $\boldsymbol{\Phi}$ is solved by Eq. (18)

$$\Phi(y)=\int_{-\infty}^{y}\varphi(y)dy \tag{18}$$

(5) Calculate sensitivity coefficient $\boldsymbol{\alpha_{x_i}}$

$$\alpha_{x_i}=\cos\theta_{x_i}=-\frac{\frac{\partial g_x(x^*)}{\partial X_i}\sigma_{x_i}}{\sqrt{\sum_{i=1}^{n}[\frac{\partial g_x(x^*)}{\partial X_i}]^2\sigma_{x_i}^2}} \tag{19}$$

where g is performance function.

(6) Solve reliability index β

$$\beta=\frac{\mu_{z_z}}{\sigma_{z_z}}=\frac{g_x(x^*)+\sum_{i=1}^{n}\frac{\partial g_x(x^*)}{\partial X_i}(\mu_{x_i}-x^*)}{\sqrt{\sum_{i=1}^{n}[\frac{\partial g_x(x^*)}{\partial X_i}]^2\sigma_{x_i}^2}} \tag{20}$$

(7)Compute the new MPP x^* by Eq. (21)

$$x^*=\mu_{x_i}+\beta\sigma_{x_i}cos\vartheta_{x_i} \tag{21}$$

(8)If the difference of adjacent $\|x^*\|$ is less than ε ($\boldsymbol{\varepsilon}$ is convergence criteria defined by researchers), end, else go to step (3).

The procedure is the same as the conventional equivalent normalization method except the solution of $\boldsymbol{\mu}$, $\boldsymbol{\sigma}$, $\boldsymbol{pdf}$ and $\boldsymbol{cdf}$ of epistemic uncertain variables represented by the novel ET model. More details of the conventional equivalent normalization method can be found in Ref (Lu *et al*. 2009).

5. Examples

5.1 Numerical example

A numerical example is selected to validate the effectiveness of the presented procedure. The performance function is shown as Eq. (22), which is strong non-linear function. In this performance function, $\boldsymbol{x_1}, \boldsymbol{x_3}, \boldsymbol{x_4}, \boldsymbol{x_5}$ and $\boldsymbol{x_6}$ are random variables which obey normal distribution. Their mean value and variance are shown in Table 3. $\boldsymbol{x_2}$ and $\boldsymbol{x_7}$ are represented by the novel ET model and their BPAs are shown in Table 4. The membership functions of $\boldsymbol{x_2}$ and $\boldsymbol{x_7}$ is assumed as triangle fuzzy number.

Combined the BPA of $\boldsymbol{x_2}$ and $\boldsymbol{x_7}$ through Eq. (4), the results are shown in Table 2, in which there are 5 focal elements and 4 focal elements for $\boldsymbol{x_2}$ and $\boldsymbol{x_7}$ respectively. Normalized by the normalization method (Rao *et al*. 2009), the membership functions of $\boldsymbol{x_2}$ and $\boldsymbol{x_7}$ are shown in Fig. 7(a) and Fig. 7(b) respectively, in which the membership

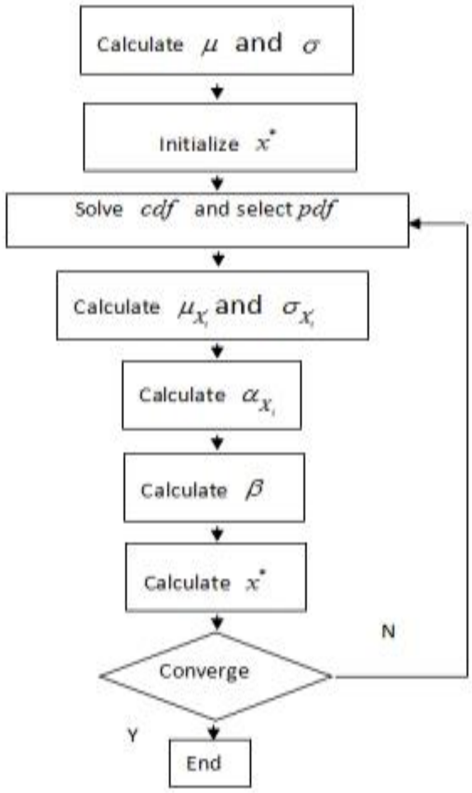


Fig. 6 Flow chart of reliability analysis

Table 3 Mean value and variance of uncertain variables

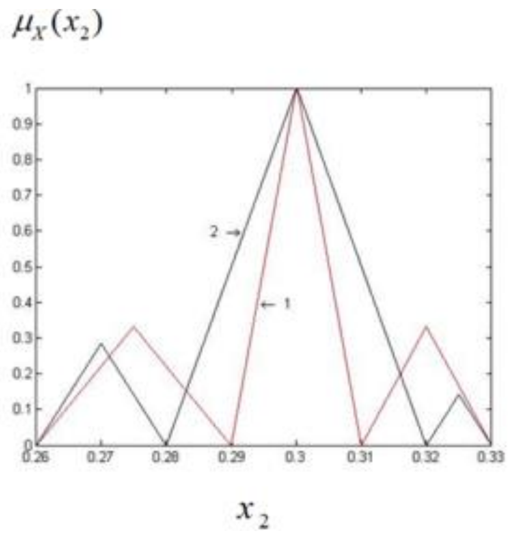
Uncertain variable	x_1	x_3	x_4	x_5	x_6
Mean value	0.01	360	226e-6	0.5	0.12
Variance	0.003	54	1.13e-005	0.05	0.006

Table 4 BPA of $\boldsymbol{x_2}$ and $\boldsymbol{x_7}$

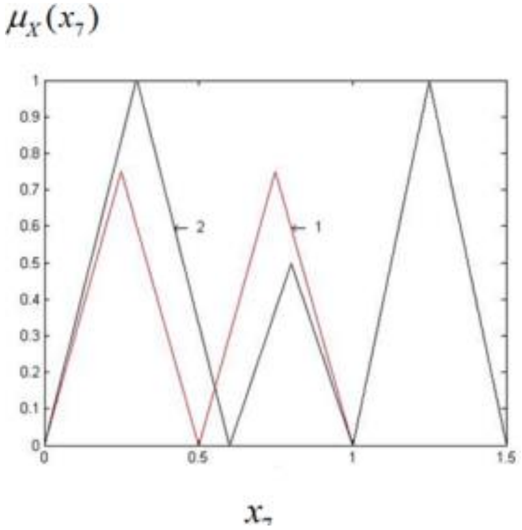
Uncertain variable	Expert 1		Expert 2		Combination results	
	Interval	BPA	Interval	BPA	Interval	Combined BPA
$\boldsymbol{x_2}$	[0.26,0.29]	0.2	[0.26,0.28]	0.2	[0.26,0.28]	31/180
	[0.29,0.31]	0.6	[0.28,0.32]	0.7	[0.28,0.29]	515/7200
	[0.31,0.33]	0.2	[0.32,0.33]	0.1	[0.29,0.31]	45/80
					[031,0.32]	75/800
					[0.32,0.33]	0.1
$\boldsymbol{x_7}$	[0,0.5]	0.3	[0 0.6]	0.4	[0,0.5]	61/180
	[0.5,1]	0.3	[0.6 1]	0.2	[0.5 0.6]	158/250/18
	[1,1.5]	0.4	[1 1.5]	0.4	[0.6 1]	113/500

functions given by expert 1 and expert 2 are denoted by black and red line respectively. The combination method for membership functions presented in Section 3 is adopted to solve the $\boldsymbol{Bel}$ and $\boldsymbol{pl}$ membership functions, which are shown in Fig. 8 and Fig. 9 respectively

$$g(x)=50-\frac{(x_1x_2x_3-x_3^2x_4^2x_5)}{x_6x_7}-x_1 \tag{22}$$

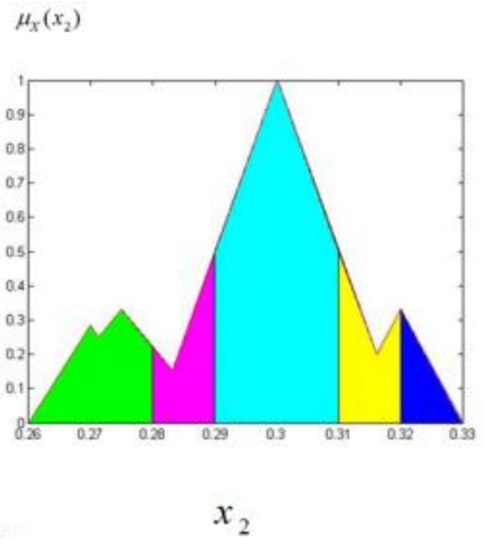


(a) The normalized membership function of $\boldsymbol{x_2}$



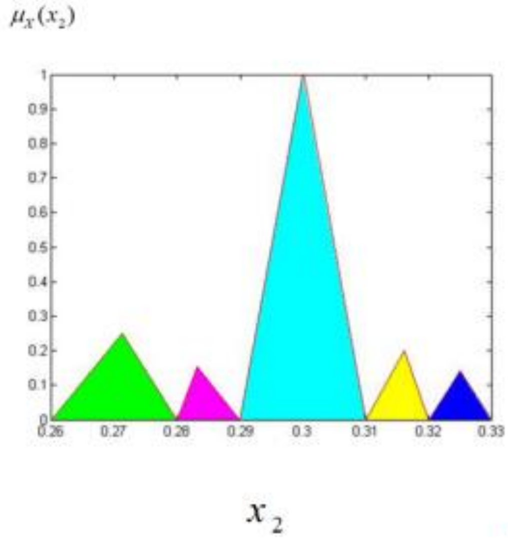
(b) The normalized membership functi on of $\boldsymbol{x_7}$

Fig. 7 The normalized membership function of $\boldsymbol{x_2}$ and $\boldsymbol{x_7}$



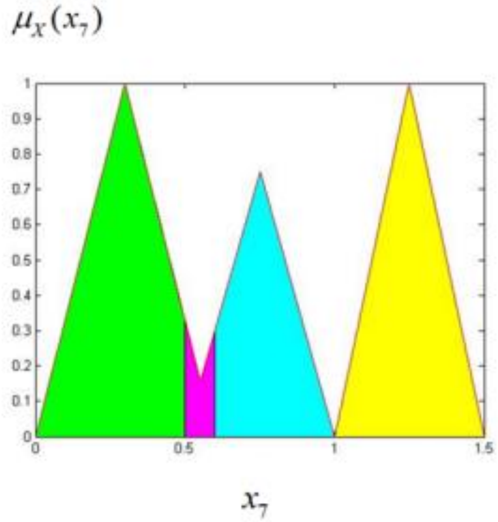
(a) $\boldsymbol{pl}$ membership function

Fig. 8 $\boldsymbol{Bel}$ and $\boldsymbol{pl}$ membership function of $\boldsymbol{x_2}$

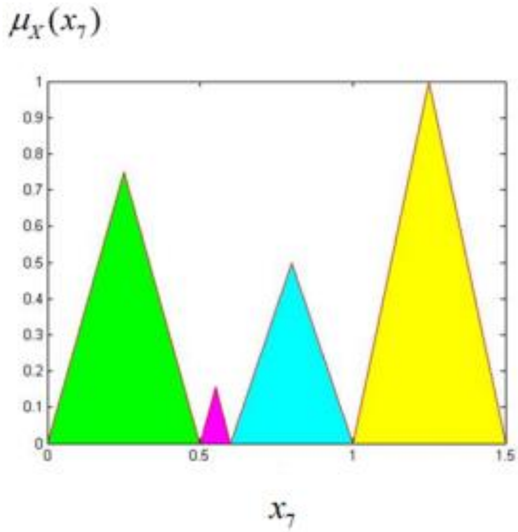


(b) $\boldsymbol{Bel}$ membership function

Fig. 8 Continued



(a) $\boldsymbol{pl}$ membership function



(b) $\boldsymbol{Bel}$ membership function

Fig. 9 $\boldsymbol{pl}$ and $\boldsymbol{Bel}$ membership function of $\boldsymbol{x_7}$

Solved by Eq. (8), the entropy of x_2 and x_7 is shown in Table 5. The results indicate that the entropy of x_2 and x_7 before combination is less than that of pl membership function and is more than that of Bel membership function. For example, the entropy of x_2 before combination is 0.4838 and 0.5425 for membership function 1 and membership function 2 respectively. This entropy is less than that of pl membership function 0.6279 and is more than the entropy of Bel membership function 0.3983. Compared with the original membership function, the pl membership function expands the uncertainty and the Bel membership function shrinks the uncertainty. Therefore, it is rational to view the pl and Bel membership function as upper and lower bounds of original membership functions respectively.

Solved this example by the presented reliability analysis procedure, the results are shown in Table 6. The lower band of reliability, solved from the pl membership function, is 0.8986, which is named as pl reliability. Similarly, the reliability 0.9352 calculated form the Bel membership function is viewed as the upper band of reliability and is named as Bel reliability. Therefore, the interval of reliability of this example is [0.8986, 0.9352], which is the reliability estimation of the numerical example with the mixture of epistemic and aleatory uncertainties.

To validate this results, the conventional cut subset method denoted by Eq. (7) and the MCS are adopted to calculate the reliability, which may be viewed as real reliability. The levels in cut subset method are 30 and the sample numbers in MCS is 1e6. The interval of reliability solved by the conventional method is [0.9172, 0.9269], which is almost identical to the interval [0.8986, 0.9352] solved by the suggested procedure. The maximum difference between the two reliability intervals is only 2.02%. Therefore, the suggested procedure poses high computational accuracy. The other advantage of this suggested procedure is the low computational cost. Due to the suggested method is based on FORM, there is only 15 cycles to calculate the reliability for this example. Compared with the sample numbers 1e6 in MCS, the computational cost in the suggested procedure is much less than that in conventional method.

Table 5 The entropy of x_2 and x_7

Uncertain variable	The entropy before combination		The entropy after combination	
	membership function 1	membership function 2	pl membership function	Bel membership function
x_2	0.4838	0.5425	0.6279	0.3983
x_7	0.6628	0.6635	0.7249	0.6014

Table 6 Reliability of example 1

Membership function	Reliability solved by the presented method	Reliability solved by MCS	Difference
pl membership function	0.8986	0.9172	2.02%
Bel Membership function	0.9352	0.9269	0.89%

5.2 Example 2-crank-slider mechanism

Engineering example 2 is derived from Du's paper (Du 2006). This example is a crank-slider mechanism, and its schematic diagram is shown in Fig. 8. Eq. (22) is the performance function. It is the safety margins for strength requirements of the coupler, which are defined by the difference between the material strength and the maximum stress

$$G_1 = g_1(X,Y) = S - \frac{4P(l+m)}{\pi\left(\sqrt{(l+m)^2 - e^2} - ke\right)(d_2^2 - d_1^2)} \tag{23}$$

The failure events are defined by Eq. (24)

$$s_1 = \{X,Y|g_1(X,Y) < 0\} \tag{24}$$

In this mechanism, the parameters $d = [d_1, d_2] = [15.5, 26.5]$.

Random variables include the length of the crank l, the length of the coupler m, the external force p, and the yield strength of the coupler s. The distributions of this random variables are given in Table 7. To validate the reliability analysis procedure suggested in this paper, the coefficient of friction k and the offset e are modeled by the novel ET model. Their intervals and BPAs are shown in Table 8, and their membership function is assumed as triangle fuzzy number. Combined by Murphy's average combination rule, the combined BPAs are also shown in Table 8.

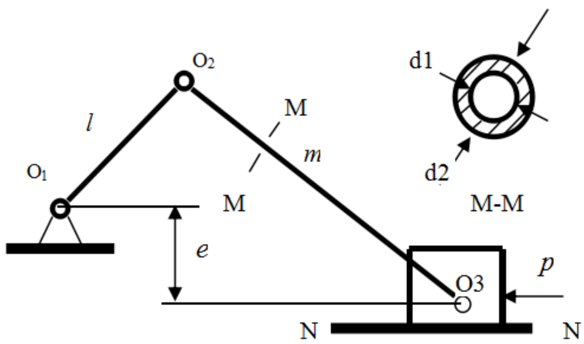


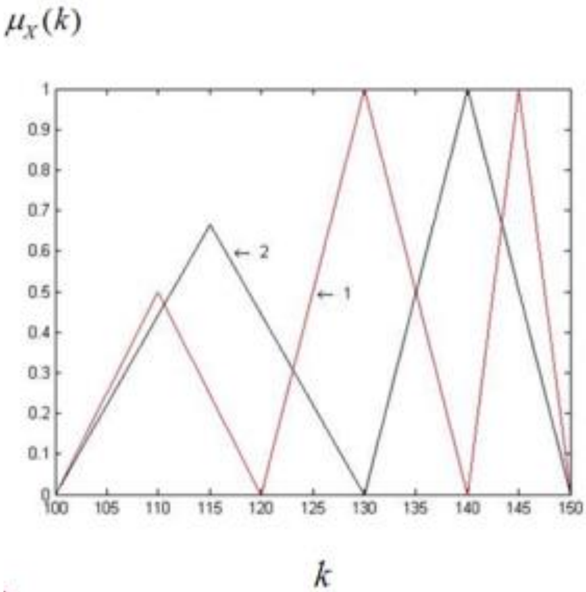
Fig. 9 A crank-slider mechanism

Table 7 Random variables x

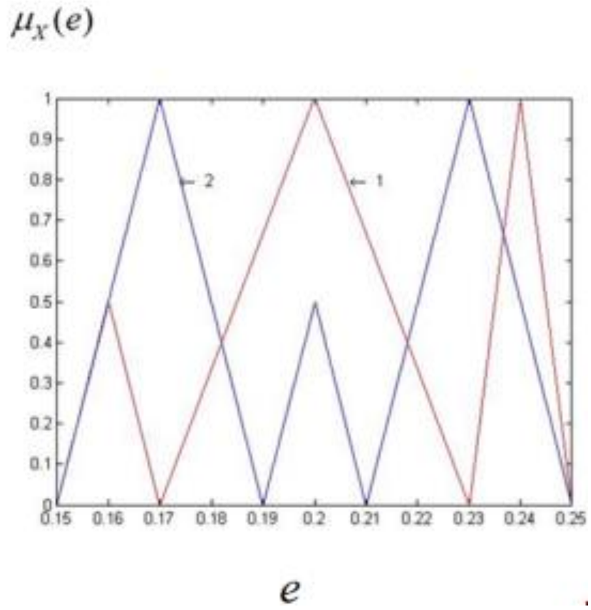
Variable	Symbols in Fig. 8	Mean value	Standard deviation	Distribution
x_1	l	100 mm	0.01 mm	normal
x_2	m	300 mm	0.01 mm	normal
x_3	p	250 KN	25 KN	normal
x_4	s	990 MPa	39 MPa	normal

Table 8 Epistemic variable $\boldsymbol{e}$ and $\boldsymbol{\mu}$

Uncertain variable	Expert 1		Expert 2		Combination results	
	interval	BPA	interval	BPA	interval	Combined BPA
$\boldsymbol{e}$	[100,120]	0.2	[100,130]	0.4	[100,120]	0.2
	[120,140]	0.4	[130,150]	0.6	[120,130]	0.2
	[140,150]	0.4			[130,140]	0.25
					[140,150]	0.35
	[0.15,0.17]	0.2	[0.15,0.19]	0.4	[0.15,0.17]	0.2
	[0.17,0.23]	0.4	[0.19,0.21]	0.2	[0.17,0.19]	0.2
$\boldsymbol{k}$	[0.23,0.25]	0.4	[0.21,0.25]	0.4	[0.19,0.21]	0.15
					[0.21,0.23]	0.15
					[0.23,0.25]	0.3

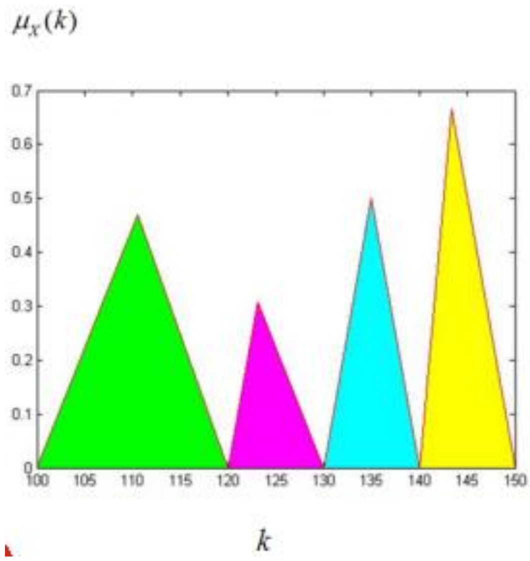


(a)The normalized membership functions of $\boldsymbol{k}$

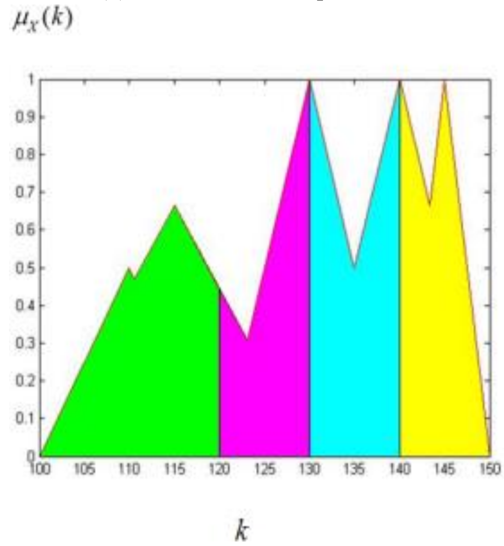


(b) The normalized membership functions of $\boldsymbol{e}$

Fig. 10 The normalized membership function of $\boldsymbol{k}$ and $\boldsymbol{e}$

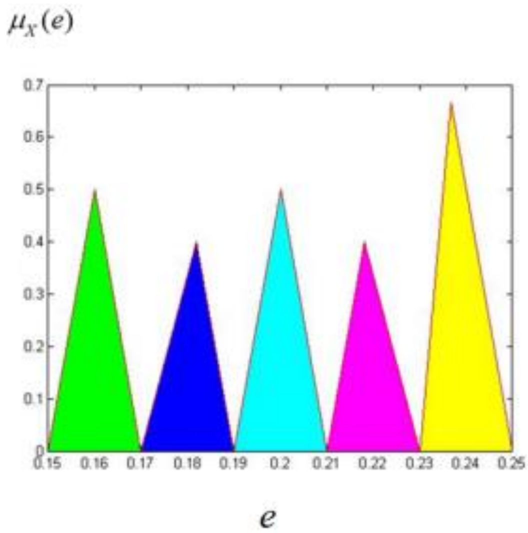


(a) $\boldsymbol{Bel}$ membership function



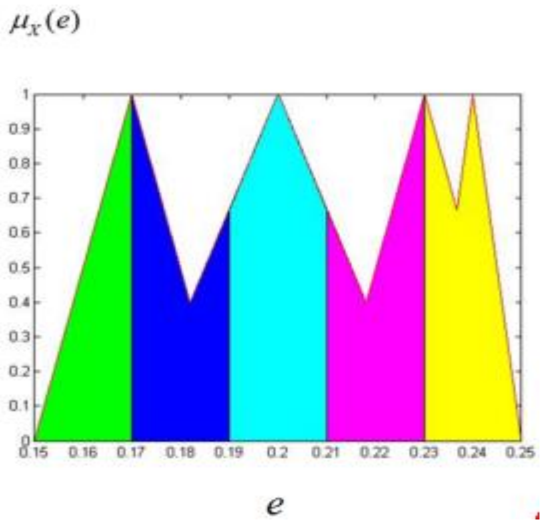
(b) $\boldsymbol{pl}$ membership function

Fig.11 The combined membership function of $\boldsymbol{k}$



(a) $\boldsymbol{Bel}$ membership function

Fig. 12 The combined membership function of $\boldsymbol{e}$



(b) $\boldsymbol{pl}$ membership function

Fig. 12 Continued

Table 8 The entropy of $\boldsymbol{e}$ and $\boldsymbol{k}$

Uncertain variable	The entropy before combination		The entropy after combination	
	Membership function1	Membership function 2	$\boldsymbol{p}$ membership function	$\boldsymbol{Bel}$ membership function
$\boldsymbol{e}$	0.6346	0.6459	0.7924	0.4880
$\boldsymbol{k}$	0.6780	0.7213	0.98597	0.4963

Table 9 Reliability of example

Membership function	Reliability solved by the presented method	Reliability solved by MCS	Difference
$\boldsymbol{pl}$ membership function	0.9842	0.9869	0.27%
$\boldsymbol{Bel}$ membership function	0.9919	0.9908	0.09%

Fig. 10(a) and Fig. 10(b) show the normalized membership functions of $\boldsymbol{\mu}$ and $\boldsymbol{e}$ respectively. The combined membership functions of $\boldsymbol{k}$ and $\boldsymbol{e}$ are exhibited in Fig. 11, including the $\boldsymbol{Bel}$ and $\boldsymbol{pl}$ membership function. The entropy of $\boldsymbol{e}$ and $\boldsymbol{k}$ is listed in Table 8, which indicates that the entropy of original membership functions is less than that of $\boldsymbol{pl}$ membership function and more than that of $\boldsymbol{Bel}$ membership function. Therefore, it is rational to the $\boldsymbol{pl}$ and $\boldsymbol{Bel}$ membership function are viewed as upper and lower bounds of original membership functions respectively.

Solved by the suggested procedure and conventional method, the results of example 2 are shown in Table 9. The $\boldsymbol{pl}$ reliability is 0.9842 and the $\boldsymbol{Bel}$ reliability is 0.9919, namely, the reliability interval is [0.9842, 0.9919], which is almost the same as the interval [0.9869, 0.9908] solved by the suggested procedure. Therefore, the reliability interval calculated by the suggested method may be considered as the estimation of the real reliability. Moreover, there is only 12 cycles to calculate the reliability of this example. Compared with the sample numbers 1e6 in MCS, the computational cost in the suggested procedure is much less than that in conventional method. Therefore, the suggested procedure can be adopted to calculate the reliability interval efficiently.

7. Conclusions

To solve the hybrid reliability for engineering structures, a novel ET model based on fuzzy distribution is suggested. Inspired by the concept of belief and plausible in ET, the intersection and union of membership functions are defined as the belief and plausible membership functions respectively. The Murfhy's average combination rule is adopted to combine the basic probability assignment for focal elements. Then the combined membership function is transformed to the equivalent probability density function by means of a normalizing factor, and the equivalent normalization method is used to solve the reliability of structure with the evidence variables and random variables. A numerical example and an engineering example are given to demonstrate the advantages of the proposed method. The results show that the reliability interval calculated by the suggested method is almost identical to that solved by MCS and cut subset method. Moreover, the results also indicate that the computational cost of the suggested procedure is much less than that of conventional method. The suggested ET model provides a new way to flexibly represent epistemic uncertainty, and the suggested procedure provides an efficiency method to estimate the reliability of structures with the mixture of epistemic and aleatory uncertainties.

Acknowledgments

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