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Stress analysis of an infinite rectangular plate perforated by two unequal circular holes under bi-axial uniform stresses

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Abstract. Exact solutions for stresses for an infinite rectangular plate perforated by two circular holes of different radii subjected to uni-axial or bi-axial uniform loads are investigated using the Airy stress function. The hoop stresses occurring at the edge of the circular hole are computed and plotted. Comparisons are made for the stress concentration factors for several types of loading conditions.

Keywords: perforated plate; circular hole; multi-holes; biaxial tension; stress concentration factor; airy stress function; exact solution

1. Introduction

Numerous researchers have investigated the mechanical behaviors of perforated plates, with main concerns being classified into four categories; stress concentration (Savin 1961, Muskhelishvili 1963, Miyata 1970, Peterson 1974, Iwaki and Miyao 1980, Yang and He 2002, Zhang *et al.* 2002, She and Guo 2007, Li *et al.* 2008, Yang *et al.* 2008, Yu *et al.* 2008, Kang 2014, Woo *et al.* 2014), vibration (Boay 1996, Sabir and Davies 1997, Sivakumar *et al.* 1999a, 1999b, Pan 1997, Chau and Wang 1999, Liu and Liew 1999a, 1999b, Yang and Zhou 1996, Yang and Park 1999, Geannakakes 1990, Laura *et al.* 1997, Liew *et al.* 1993a, 1993b, Liew *et al.* 1995, Liew and Sum 1998, Liew *et al.* 2001, Aksu and Ali 1976, Lam and Hung 1989, Liew *et al.* 2003), buckling (Rockey *et al.* 1967, Narayanan and Rockey 1981, Shanmugam and Narayanan 1982, Azizian and Roberts 1983, Narayanan and Chow 1984, Narayanan and Avanessian 1984, Roberts and Azizian 1985, Narayanan and Darwish 1985, Brown and Yettram 1986, Brown *et al.* 1987, Brown 1990, Nemeth 1996, Shakerley and Brown 1996, Shanmugam *et al.* 1999, Cheng and Fan 2001, El-Sawy and Nazmy 2001, Shanmugam *et al.* 2002, El-Sawy *et al.* 2004, Azhari *et al.* 2005, El-Sawy and Martini 2007, Paik 2007, 2008, Maiorana *et al.* 2008, Koomur and Sonmez 2008, Maiorana *et al.* 2009, Moen and Schafer 2009, Cheng and Zhao 2010, Komur 2011), and fatigue (Dalessandro 1976, Lacarac *et al.* 2000, Tokiyoshi *et al.* 2001, Chakherlou and Abazadeh 2011, Chan *et al.* 2013, Liu *et al.* 2013, Saleem *et al.* 2013). The various discrete methods have been used to study them. The finite element method is the most widely used. Diverse methods other than the finite element method have been used like the complex variable method, three-dimensional stress analysis, the Ritz method, the boundary element method, the differential quadrature element method, semi-analytical solution method, experimental method, conjugate load/displacement method, and Galerkin averaging method. Most of the shapes of perforated holes have three types of circular, elliptical, and rectangular cutout. Exact solutions for perforated plates by two circular holes loaded by uni-axial or bi-axial tensions have not been reported.

In the present study, exact solutions for stresses of an infinite rectangular plate perforated by two circular holes subjected to uni-axial or bi-axial tensions are investigated by two-dimensional theory of elasticity using the Airy stress function. The hoop stresses occurring at the edge of the circular hole are computed and plotted. Comparisons are made for the stress concentration factors for several types of loading conditions.

2. Stresses for a rectangular plate perforated by a circular hole under uni-axial tension

Fig. 1 shows a rectangular plate with a circular hole of radius of R under uni-axial tension σ_1 in the x-direction, and the rectangular (x,y) and polar (r,θ) coordinate systems. The plate is assumed to be very large compared with the hole.

First of all, considering a rectangular plate with no hole under uni-axial uniform tension σ_1 in the x-direction, the stress components are as below

$$\sigma_{xx}^0 = \frac{\partial^2 \phi^0}{\partial y^2} = \sigma_1, \quad \sigma_{xy}^0 = -\frac{\partial^2 \phi^0}{\partial x \partial y} = 0, \quad \sigma_{yy}^0 = \frac{\partial^2 \phi^0}{\partial x^2} = 0 \quad (1)$$

where ϕ^0 is a fundamental Airy stress function and satisfies

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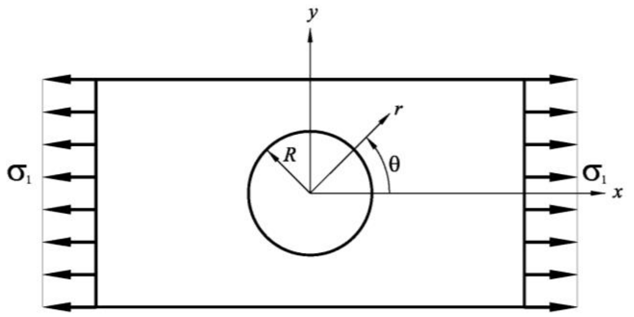

Fig. 1 A rectangular plate perforated by a circular hole subjected to uni-axial tension σ_1 in x-direction, and the rectangular (x, y) and the polar coordinates (r,θ)

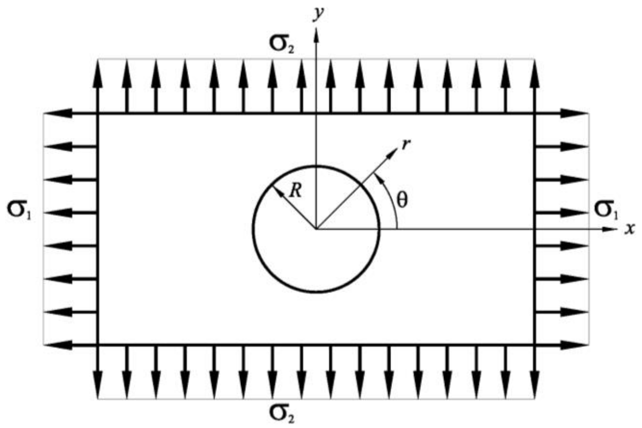

Fig. 2 A rectangular plate perforated by a circular hole subjected to biaxial tensions σ_1 and σ_2

the governing equation $\nabla^4\phi=\nabla^2(\nabla^2\phi)=0$ with no body forces in 2-D plane problems in elasticity, where the Laplacian differential operator ∇^2 is expressed as

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2} \tag{2}$$

and ∇^4 is the bi-harmonic differential operator defined by

$$\nabla^4=\nabla^2(\nabla^2)=\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4} \tag{3}$$

in the rectangular coordinates. From Eqs. (1), the function ϕ^0 can be assumed as

$$\phi^0=\frac{\sigma_1}{2}y^2+Ax+By+C \tag{4}$$

where A, B, and C are arbitrary integration constants. A linear function of x or y and a constant in the Airy stress function are trivial terms which do not give rise to any stresses and strains. Dropping the trivial terms in Eq. (4), the function ϕ^0 becomes

$$\phi^0=\frac{\sigma_1}{2}y^2 \tag{5}$$

Using the relation of

$$y=r\sin\theta \tag{6}$$

and the multiple angle formula

$$\sin^2\theta=\frac{1-\cos 2\theta}{2} \tag{7}$$

Eq. (5) can be transformed into the bi-harmonic functions as

$$\phi^0=\frac{\sigma_1}{4}(r^2-r^2\cos 2\theta) \tag{8}$$

which satisfies the governing equation $\nabla^4\phi^0=\nabla^2(\nabla^2\phi^0)=0$ $=0$, where ∇^2 is

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{9}$$

and ∇^4 is expressed as

$$\nabla^4=\nabla^2(\nabla^2)=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right) \tag{10}$$

in the polar coordinates. From the relation between stress components and the Airy stress function in the polar coordinates, the stresses in the plate with no hole subjected to remote uni-axial tension σ_1 in the x-direction can be calculated as below

$$\sigma_{rr}^0=\frac{1}{r}\frac{\partial\phi^0}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi^0}{\partial\theta^2}=\frac{\sigma_1}{2}(1+\cos 2\theta) \tag{11}$$

$$\sigma_{r\theta}^0=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi^0}{\partial\theta}\right)=-\frac{\sigma_1}{2}\sin 2\theta \tag{12}$$

$$\sigma_{\theta\theta}^0=\frac{\partial^2\phi^0}{\partial r^2}=\frac{\sigma_1}{2}(1-\cos 2\theta) \tag{13}$$

Let us return to the problem of a perforated rectangular plate by a circular hole under uni-axial tension σ_1 in the x-direction. The Airy function ϕ_x becomes

$$\phi_x=\phi^0+\phi^* \tag{14}$$

where ϕ^* is an Airy stress function to cancel unwanted traction due to ϕ^0 on $r=R$. The normal σ_{rr} and shear stress $\sigma_{r\theta}$ on $r=R$ must be free as

$$\sigma_{rr}\Big|_{r=R}=[\sigma_{rr}^0+\sigma_{rr}^*]_{r=R}=0 \tag{15}$$

$$\sigma_{r\theta}\Big|_{r=R}=[\sigma_{r\theta}^0+\sigma_{r\theta}^*]_{r=R}=0 \tag{16}$$

Therefore, σ_{rr}^* must have terms of $\cos 2\theta$ or a constant and $\sigma_{r\theta}^*$ must have $\sin 2\theta$ on $r=R$ in order to eliminate the stresses on $r=R$ due to ϕ^0 in Eqs. (11) and (12). Tables 1 and 2 show the potential candidates of the bi-harmonic functions identified as r^2, $\ln r$, $r^2\ln r$, $r^2\cos 2\theta$, $r^4\cos 2\theta$, $\cos 2\theta/r^2$, $\cos 2\theta$ from the tables by Dundurs (Fu 1996), which contain stresses and displacements corresponding to certain bi-harmonic functions in the polar coordinates. The symbols in Table 2 μ and κ are shear modulus and a secondary elastic constant, respectively. The constant κ is defined by $(3-\nu)/(1+\nu)$ for plane stress problems and $3-4\nu$ for plane strain ones where ν is Poisson's ratio. In order not to (1) disturb the uniform traction at infinity, (2) have multi-valued displacement and (3) have infinite stress at infinity we delete the terms r^2, $r^2\ln r$, $r^2\cos 2\theta$, and $r^4\cos 2\theta$. We thus arrive at the Airy stress function ϕ_x

$$\phi_x = \frac{\sigma_1}{4}\left(r^2 - r^2\cos 2\theta + C_1 R^2 \ln r + C_2 R^4 \frac{\cos 2\theta}{r^2} + C_3 R^2 \cos 2\theta\right) \tag{17}$$

where C_1~C_3 are arbitrary integration constants to be determined by the traction boundary conditions. In order to make the constants C_1~C_3 dimensionless, they are multiplied by R^2 or R^4. Applying the stress free boundary conditions on the edge of the circular hole at $r=R$

$$\sigma_{rr}\Big|_{r=R} = \left[\frac{1}{r}\frac{\partial \phi_x}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi_x}{\partial \theta^2}\right]_{r=R} = 0 \tag{18}$$

$$\sigma_{r\theta}\Big|_{r=R} = \left[-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi_x}{\partial \theta}\right)\right]_{r=R} = 0 \tag{19}$$

the unknown constants are computed as $C_1=-2$, $C_2=-1$, and $C_3=2$. Thus the function ϕ_x becomes

$$\phi_x = \frac{\sigma_1}{4}\left(r^2 - r^2\cos 2\theta - 2R^2 \ln r - R^4 \frac{\cos 2\theta}{r^2} + 2R^2 \cos 2\theta\right) \tag{20}$$

From the relations between the stresses and the Airy stress function in polar coordinates in Eqs. (11)-(13), the stress components in the plate with a circular hole under uniform tension σ_1 in x-direction can be expressed as

$$\sigma_{rr} = \frac{\sigma_1}{2}\left\{\left(1 - 4\frac{R^2}{r^2} + 3\frac{R^4}{r^4}\right)\cos 2\theta + 1 - \frac{R^2}{r^2}\right\} \tag{21}$$

$$\sigma_{r\theta} = -\frac{\sigma_1}{2}\left(1 + 2\frac{R^2}{r^2} - 3\frac{R^4}{r^4}\right)\sin 2\theta \tag{22}$$

$$\sigma_{\theta\theta} = -\frac{\sigma_1}{2}\left\{\left(1 + 3\frac{R^4}{r^4}\right)\cos 2\theta - 1 - \frac{R^2}{r^2}\right\} \tag{23}$$

The Airy stress function ϕ_y for perforated rectangular plate by a circular hole of a radius of R under uniform tension σ_2 in the y-direction can obtain by replacing σ_1 and θ by σ_2 and $\theta \pm \pi/2$, respectively

$$\phi_y = \frac{\sigma_2}{4}\left(r^2 + r^2\cos 2\theta - 2R^2 \ln r + R^4 \frac{\cos 2\theta}{r^2} - 2R^2 \cos 2\theta\right) \tag{24}$$

Accordingly, the stress components become

$$\sigma_{rr} = -\frac{\sigma_2}{2}\left\{\left(1 - 4\frac{R^2}{r^2} + 3\frac{R^4}{r^4}\right)\cos 2\theta - 1 + \frac{R^2}{r^2}\right\} \tag{25}$$

$$\sigma_{r\theta} = \frac{\sigma_2}{2}\left(1 + 2\frac{R^2}{r^2} - 3\frac{R^4}{r^4}\right)\sin 2\theta \tag{26}$$

$$\sigma_{\theta\theta} = \frac{\sigma_2}{2}\left\{\left(1 + 3\frac{R^4}{r^4}\right)\cos 2\theta + 1 + \frac{R^2}{r^2}\right\} \tag{27}$$

3. Stresses for a rectangular plate perforated by a circular hole under bi-axial tensions

Fig. 2 shows a rectangular plate with a circular hole of radius of R under bi-axial tensions σ_1 in the x-direction and σ_2 in the y-direction. The plate is assumed to be very large compared to the hole. By the principle of superposition the Airy stress function ϕ for Eqs. (20) and (24) becomes $\phi = \phi_x + \phi_y$

$$= \frac{\sigma_1}{4}\left(r^2 - r^2\cos 2\theta - 2R^2 \ln r - R^4 \frac{\cos 2\theta}{r^2} + 2R^2 \cos 2\theta\right) + \frac{\sigma_2}{4}\left(r^2 + r^2\cos 2\theta - 2R^2 \ln r + R^4 \frac{\cos 2\theta}{r^2} - 2R^2 \cos 2\theta\right) \tag{28}$$

Consequently, the stress components become

$$\sigma_{rr} = \frac{\sigma_1}{2}\left\{\left(1 - 4\frac{R^2}{r^2} + 3\frac{R^4}{r^4}\right)\cos 2\theta + 1 - \frac{R^2}{r^2}\right\} - \frac{\sigma_2}{2}\left\{\left(1 - 4\frac{R^2}{r^2} + 3\frac{R^4}{r^4}\right)\cos 2\theta - 1 + \frac{R^2}{r^2}\right\} \tag{29}$$

Table 1 Stresses of potential candidates of bi-harmonic functions ϕ

ϕ	σ_{rr}	$\sigma_{r\theta}$	$\sigma_{\theta\theta}$
r^2	2	0	2
$\ln r$	$1/r^2$	0	$-1/r^2$
$r^2\ln r$	$2\ln r+1$	0	$2\ln r+3$
$r^2\cos 2\theta$	$-2\cos 2\theta$	$2\sin 2\theta$	$2\cos 2\theta$
$r^4\cos 2\theta$	0	$6r^2\sin 2\theta$	$12r^2\cos 2\theta$
$\cos 2\theta/r^2$	$-6\cos 2\theta/r^4$	$-6\sin 2\theta/r^4$	$6\cos 2\theta/r^4$
$\cos 2\theta$	$-4\cos 2\theta/r^2$	$-2\sin 2\theta/r^2$	0

Table 2 Displacements of potential candidates of bi-harmonic functions ϕ

ϕ	$2\mu u_r$	$2\mu u_\theta$
r^2	$(\kappa-1)r$	0
$\ln r$	$-1/r$	0
$r^2\ln r$	$(\kappa-1)r\ln r-r$	$(\kappa+1)r\theta$
$r^2\cos 2\theta$	$-2r\cos 2\theta$	$2r\sin 2\theta$
$r^4\cos 2\theta$	$-(3-\kappa)r^3\cos 2\theta$	$(3+\kappa)r^3\sin 2\theta$
$\cos 2\theta/r^2$	$2\cos 2\theta/r^3$	$2\sin 2\theta/r^3$
$\cos 2\theta$	$(\kappa+1)\cos 2\theta/r$	$-(\kappa-1)\sin 2\theta/r$

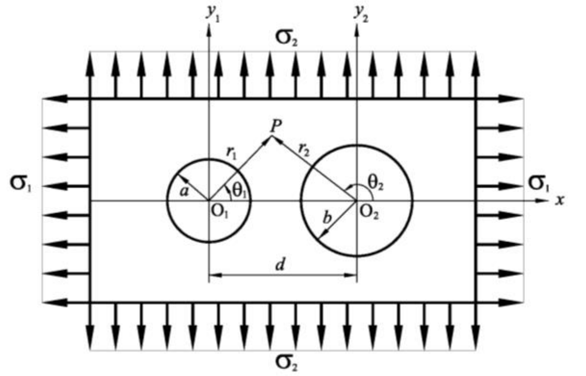

Fig. 3 A rectangular plate perforated by two circular holes of different radii a an b subjected to bi-axial tensions σ_1 and σ_2 in the x and y directions, respectively

$$\sigma_{r\theta} = -\frac{\sigma_1}{2}\left(1+2\frac{R^2}{r^2}-3\frac{R^4}{r^4}\right)\sin 2\theta + \frac{\sigma_2}{2}\left(1+2\frac{R^2}{r^2}-3\frac{R^4}{r^4}\right)\sin 2\theta \tag{30}$$

$$\sigma_{\theta\theta} = -\frac{\sigma_1}{2}\left\{\left(1+3\frac{R^4}{r^4}\right)\cos 2\theta - 1 - \frac{R^2}{r^2}\right\} + \frac{\sigma_2}{2}\left\{\left(1+3\frac{R^4}{r^4}\right)\cos 2\theta + 1 + \frac{R^2}{r^2}\right\} \tag{31}$$

4. Stresses of a rectangular plate perforated by two circular holes under bi-axial tensions

Fig. 3 shows a rectangular plate perforated by two circular holes of radii a and b subjected to bi-axial tension σ_1 in the x-direction and σ_2 in the y-direction, and the rectangular coordinates (x, y_1) and (x, y_2) and the polar coordinates (r_1, θ_1) and (r_2, θ_2). The distance $\overline{O_1O_2}$ between the centers of two circular holes is denoted by d.

Again, applying the principle of superposition from Eq. (28), one can obtain finally the total Airy stress function Φ for an infinite plate perforated by two circular hole of radii a and b under biaxial uniform tensions σ_1 in x-direction and σ_2 in y-direction as below

$$\begin{aligned}\Phi = \frac{\sigma_1}{8}\Bigg[& r_1^2 - r_1^2\cos 2\theta_1 - 2a^2\ln r_1 - a^4\frac{\cos 2\theta_1}{r_1^2} + 2a^2\cos 2\theta_1 \\ & + r_2^2 - r_2^2\cos 2\theta_2 - 2b^2\ln r_2 - b^4\frac{\cos 2\theta_2}{r_2^2} + 2b^2\cos 2\theta_2\Bigg] \\ + \frac{\sigma_2}{8}\Bigg[& r_1^2 + r_1^2\cos 2\theta_1 - 2a^2\ln r_1 + a^4\frac{\cos 2\theta_1}{r_1^2} - 2a^2\cos 2\theta_1 \\ & + r_2^2 + r_2^2\cos 2\theta_2 - 2b^2\ln r_2 + b^4\frac{\cos 2\theta_2}{r_2^2} - 2b^2\cos 2\theta_2\Bigg]\end{aligned} \tag{32}$$

However, two different polar coordinate systems (r_1, θ_1) and (r_2, θ_2) exist in the total Airy stress function Φ in Eq. (32). In order to calculate stress components based on one of the polar coordinate system (r_i, θ_i), the other polar coordinate system (r_j, θ_j) should be transformed into (r_i, θ_i) where $i \neq j$. Using the law of cosines for ΔO_1PO_2 in Fig. 3, the radial coordinates r_1 and r_2 can be expressed in terms of (r_2, θ_2) and (r_1, θ_1), respectively, as below

$$r_1 = \sqrt{r_2^2 + d^2 + 2r_2 d\cos\theta_2}\,, \quad r_2 = \sqrt{r_1^2 + d^2 - 2r_1 d\cos\theta_1} \tag{33}$$

It is also seen from ΔO_1PO_2 by means of the law of sines that

$$\sin\theta_1 = \frac{r_2}{r_1}\sin\theta_2\,, \quad \sin\theta_2 = \frac{r_1}{r_2}\sin\theta_1 \tag{34}$$

where r_1 and r_2 are expressed in terms of (r_2, θ_2) and (r_1, θ_1), respectively, in Eqs. (33). The terms of $\cos 2\theta_1$ and $\cos 2\theta_2$ in Eq. (32) are expressed in terms of (r_2, θ_2) and (r_1, θ_1), respectively, using the formulas of double angle, as below

$$\cos 2\theta_1 = 1 - 2\sin^2\theta_1\,, \quad \cos 2\theta_2 = 1 - 2\sin^2\theta_2 \tag{35}$$

where $\sin\theta_1$ and $\sin\theta_2$ are expressed in terms of (r_2, θ_2) and (r_1, θ_1), respectively, in Eqs. (34). One can now express the total Airy stress function Φ in Eq. (32) in terms of one of the polar coordinate systems, (r_1, θ_1) or (r_2, θ_2). By means of the below relations between stress components and the Airy stress function, the stress components for an infinite plate perforated by two circular holes in terms of one of the polar coordinate systems (r_i, θ_i) (i=1, 2) can be compute as below

$$\sigma_{r_i r_i} = \frac{1}{r_i}\frac{\partial \Phi_i}{\partial r_i} + \frac{1}{r_i^2}\frac{\partial^2 \Phi_i}{\partial \theta_i^2} \tag{36}$$

$$\sigma_{r_i\theta_i} = -\frac{\partial}{\partial r_i}\left(\frac{1}{r_i}\frac{\partial \Phi_i}{\partial \theta_i}\right) \tag{37}$$

$$\sigma_{\theta_i\theta_i} = \frac{\partial^2 \Phi_i}{\partial r_i^2} \tag{38}$$

where Φ_i is the total Airy stress function expressed in terms of (r_i, θ_i).

5. Hoop stresses and stress concentration factors

The circumferential normal stress $\sigma_{\theta_i\theta_i}$ (i=1,2) on the edge (r_1=a or r_2=b) of the circular hole is called the hoop stresses σ_{Hoop}. The stress concentration factor (SCF) is the maximum non-dimensional hoop stress defined by the ratio between the maximum hoop stress and a nominal stress $(\sigma_{\mathrm{Hoop}})_{\max}/\sigma_0$.

Using Eqs. (32) and (38), the non-dimensional hoop stress $\sigma_{\theta_1\theta_1}/\sigma_0$ occurring at the edge of the left circular hole at r_1=a becomes

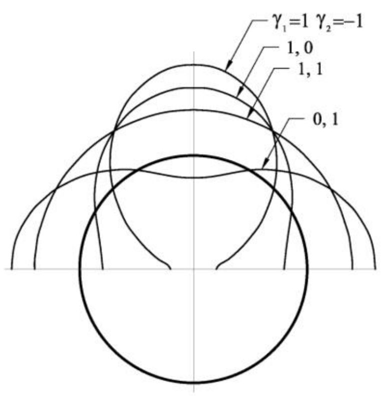

Fig. 4 The non-dimensional hoop stresses $\sigma_{\theta\theta}/\sigma_0$ for an infinite plate perforated by *one* circular hole (b/a=1 and d/a=0) subjected to uni-axial or bi-axial uniform stresses

$$\left.\frac{\sigma_{\theta_1\theta_1}}{\sigma_0}\right|_{r_1=a} = (\gamma_1-\gamma_2)\left[\left[\left\{\beta^4\left(\frac{3-8\delta^2+\delta^4+4\delta^3\cos\theta_1}{2\Gamma^4}\right)\right.\right.\right.$$
$$\left.-\delta\beta^2\left(\frac{\delta^3-3\delta+2\cos\theta_1}{\Gamma^3}\right)+\frac{1}{2}\right\}\sin^2\theta_1$$
$$-\beta^4\left(\frac{3-\delta^2-6\delta\cos\theta_1+4\delta^2\cos^2\theta_1}{4\Gamma^3}\right)$$
$$\left.+\beta^2\left(\frac{1-\delta^2-2\delta\cos\theta_1+2\delta_2\cos^2\theta_1}{4\Gamma^2}\right)+\frac{1}{2}-\cos 2\theta_1\right]$$
$$+\gamma_2\left[\beta^2\left(\frac{1-\delta^2-2\delta\cos\theta_1+2\delta^2\cos^2\theta_1}{2\Gamma^2}\right)+\frac{3}{2}\right] \quad (39)$$

where

$$\beta = b/a\,,\quad \delta = d/a \quad (40)$$

$$\gamma_1 = \sigma_1/\sigma_0\,,\quad \gamma_2 = \sigma_2/\sigma_0 \quad (41)$$

$$\Gamma = \delta^2 + 1 - 2\delta\cos\theta_1 \quad (42)$$

The non-dimensional hoop stress $\sigma_{\theta_2\theta_2}/\sigma_0$ on the edge of the right circular hole at $r_2=b$ results in

$$\left.\frac{\sigma_{\theta_2\theta_2}}{\sigma_0}\right|_{r_2=b} = (\gamma_1-\gamma_2)\left[\left[\left\{\frac{\beta^2}{2\Theta}+\frac{2\beta\delta(\beta+\delta\cos\theta_2)(\delta+\beta\cos\theta_2)}{\Theta^2}\right.\right.\right.$$
$$+\frac{2\beta(\beta^2-\delta^2)(\beta+\delta\cos\theta_2)}{\Theta^4}$$
$$-\frac{\delta(3\beta^2\delta-\delta^3+2\beta^3\cos\theta_2)}{2\Theta^2}\left(\frac{1}{\Theta^2}+1\right)$$
$$\left.-\frac{\beta^2-6\beta^2\delta^2+2\delta^4-4\beta^3\delta\cos\theta_2}{2\Theta^3}\right\}\sin^2\theta_2$$
$$\left.+\frac{1}{4\Theta}\left(\frac{1}{\Theta}-1\right)+\frac{(\beta+\delta\cos\theta_2)^2(\Theta-2)}{2\Theta^3}-\cos 2\theta_2+\frac{1}{2}\right]$$

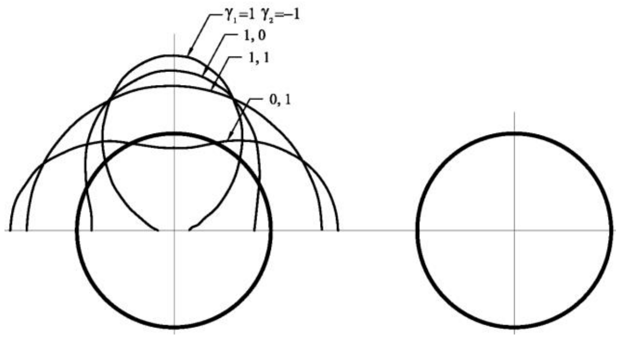

Fig. 5 The non-dimensional hoop stresses $\sigma_{\theta_1\theta_1}/\sigma_0$ for the *left* circular hole of an infinite plate perforated by *two* circular holes subjected to uni-axial or bi-axial uniform stresses with b/a=1 and d/a=3.5

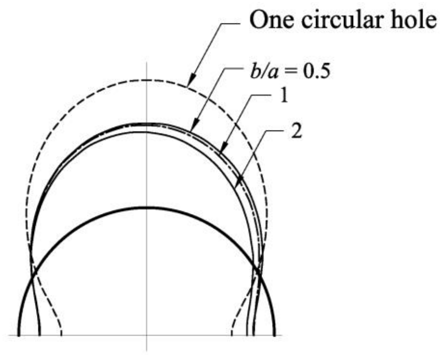

Fig. 6 The non-dimensional hoop stresses $\sigma_{\theta_1\theta_1}/\sigma_0$ for the *left* circular hole of an infinite plate perforated by *two* circular holes subjected uni-axial uniform tension in *x*-direction (γ_1=1 and γ_2=0) for d/a=3.5

$$+\gamma_2\left[\frac{\beta^2-\delta^2+2\beta\delta\cos\theta_2+2\delta^2\cos^2\theta_2}{2\Theta^2}+\frac{3}{2}\right] \quad (43)$$

where

$$\Theta = \beta^2+\delta^2+2\beta\delta\cos\theta_2 \quad (44)$$

The non-dimensional hoop stress for an infinite plate perforated by one circular hole (β=1, δ=0) subjected to bi-axial uniform tensions is reduced to

$$\left.\frac{\sigma_{\theta\theta}}{\sigma_0}\right|_{r=a} = (\gamma_1-\gamma_2)(2\sin^2\theta-\cos 2\theta)+2\gamma_2 \quad (45)$$

Figs. 4-9 show the non-dimensional hoop stresses for infinite rectangular plates perforated by one hole (Fig. 4) or two holes (Figs. 5-9) under uni-axial or bi-axial uniform loadings. Figs. 5-9 indicate them for the left circular hole of the two holes. It is seen in Figs. 4 and 5 that the hoop stresses are for four type of loading conditions (1) γ_1=1, γ_2=0, (2) γ_1=1, γ_2=1, (3) γ_1=1, γ_2=−1, (4) γ_1=0, γ_2=1. It is interesting to note in Figs. 6-9 that the hoop stresses for the plate with one circular hole is larger than those for the plate with two holes irrespective of the values of b/a and loading conditions. Table 3 presents the stress concentration factors corresponding to Figs. 4-9.

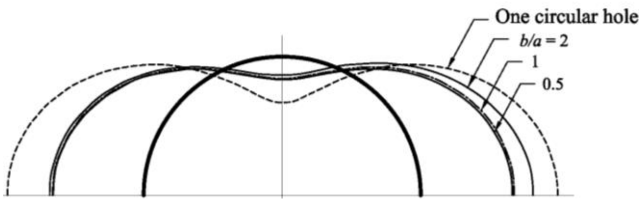

Fig. 7 The non-dimensional hoop stresses $\sigma_{\theta_1\theta_1}/\sigma_0$ for the *left* circular hole of an infinite plate perforated by *two* circular holes subjected to uni-axial uniform tension in *y*-direction (γ_1=0 and γ_2=1) for *d/a*=3.5

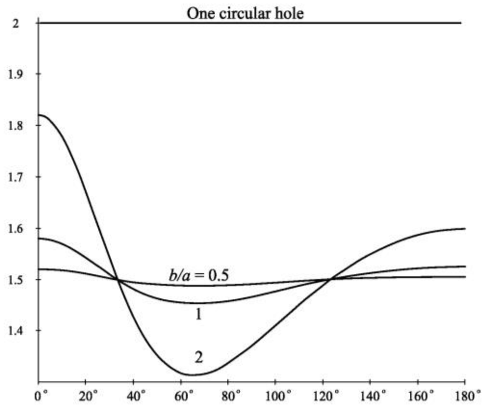

Fig. 8 The non-dimensional hoop stresses $\sigma_{\theta_1\theta_1}/\sigma_0$ for the *left* circular hole of an infinite plate perforated by *two* circular holes subjected to bi-axial uniform tensions (γ_1=γ_2=1) for *d/a*=3.5

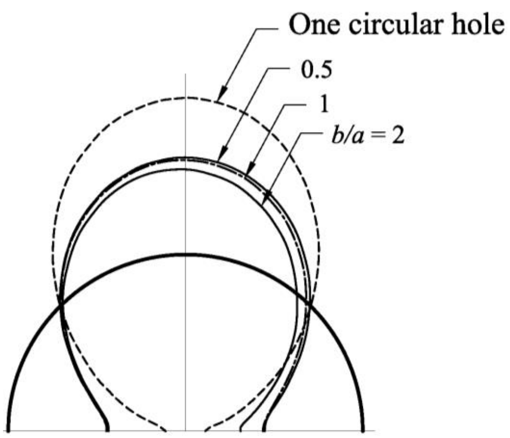

Fig. 9 The non-dimensional hoop stresses $\sigma_{\theta_1\theta_1}/\sigma_0$ for the *left* circular hole of an infinite plate perforated by two circular holes subjected bi-axial uniform stresses (γ_1=1 and γ_2=−1) for *d/a*=3.5

6. Conclusions

Exact solutions for stresses of an infinite rectangular plate perforated by two circular holes of different radii subjected to uni-axial or bi-axial uniform tensions are investigated by two dimensional theory of elasticity using the Airy stress function. The circumferential normal stresses on the edge of the circular hole, which is called the hoop stresses, are calculated and plotted. The stress concentration factors, which is the maximum non-dimensional hoop stresses, are tabulated for various loading conditions. The hoop stresses for the plate with one circular hole is larger than those for the plate with two holes irrespective of the values of *b/a* and loading conditions.

Table 3 Stress concentration factors (SCF) for the *left* circular hole of an infinite plate perforated by two circular holes subjected uni-axial or bi-axial uniform stresses

Loading Condition		SCF (Location) One Hole (*d/a*=0) *b/a*=1	Two holes (*d/a*=3.5) *b/a*=0.5	Two holes (*d/a*=3.5) *b/a*=1	Two holes (*d/a*=3.5) *b/a*=2
Uni-axial uniform stress	σ_1/σ_0 =1 σ_2/σ_0=0	3 (θ=90°)	1.984 (θ_1=90.3°)	1.938 (θ_1=91.1°)	1.780 (θ_1=93.4°)
	σ_1/σ_0=0 σ_2/σ_0=1	3 (θ=0°, 180°)	2.011 (θ_1=0°)	2.059 (θ_1=180°)	2.467 (θ_1=0°)
Bi-axial uniform stresses	σ_1/σ_0=1 σ_2/σ_0=1	2 (any points)	1.520 (θ_1=0°)	1.580 (θ_1=0°)	1.820 (θ_1=0°)
	σ_1/σ_0=1 σ_2/σ_0=−1	±4 (θ=90° or θ=0°, 180°)	−2.502 (θ_1=0°)	−2.538 (θ_1=0°)	−3.114 (θ_1=0°)

The bi-harmonic functions ϕ to satisfy the governing equation $\nabla^4\phi=0$ for plane problems with no body force in elasticity are called the Airy stress functions. Considering multi-valueness and singularity in stresses and displacements and applying the traction boundary conditions, proper bi-harmonic functions for an infinite rectangular plate perforated by one circular hole under uni-axial tension are decided from the table presented by Dundurs (Fu 1996), which contains stresses and displacements of certain bi-harmonic functions. And then using the principle of superposition, the Airy stress functions for an infinite plate perforated one circular hole subjected to bi-axial tension are calculated. Again using the principle of superposition, the Airy stress functions for an infinite plate perforated by two circular holes under bi-axial tension are obtained.

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