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On elastic and plastic length scales in strain gradient plasticity

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Abstract. The Fleck-Hutchinson theory on strain gradient plasticity (SGP), proposed in Adv. Appl Mech 33 (1997) 295, has recently been reformulated by adopting the strategy of decomposing the second order strain presented by Lam et al. in J Mech Pays Solids 51 (2003) 1477. The newly built SGP satisfies the non negativity of plastic dissipation, which is still an outstanding issue in other SGP theories. Furthermore, it explicitly shows how elastic strain gradients and corresponding elastic characteristic length scales come into play in general elastic-plastic loading histories. In this study, the relation between elastic length scales and plastic length scales is investigated by taking wire torsion as an example. It is concluded that the size effects arising when two sets of length scales are of the same order are essentially elastic instead of plastic.

Keywords: strain gradient elasticity (SGE); strain gradient plasticity (SGP); plastic dissipation; length scale; size effect

1. Introduction

SGE and SGP are believed to link closely to each other, and their relation is studied from the viewpoint of elastic and plastic length scales.

Motivated by observed size effects in nonhomogeneous plastic deformations on the micron scale such as tests of microbend (e.g., Stölken and Evans 1998), indentation (e.g., Nix and Gao 1998), and wire torsion (e.g., Fleck *et al*. 1994), various SGP theories were developed by considering the role of plastic strain gradients in plastic hardening, inspired by the Taylor plasticity model that shows that both statistically stored dislocation (SSD) and geometrically necessary dislocation (GND) come into play. There are plenty of SGPs, such as Fleck *et al*. (1994), Fleck and Hutchinson (1997), Gao *et al*. (1999), Fleck and Hutchinson (2001), Gao and Huang (2001), Gudmundson (2004), Fleck and Willis (2009), Bertram and Forest (2014). Despite of numerous successes that SGP has already achieved, some fundamental adjustments are in urgent needs, as discussed by for example, Hutchinson (2000), Evans and Hutchinson (2009) and Hutchinson (2012). Amongst existing issues, one major challenge is that SGP theories are not robustly workable in problems with general loading histories, for example loading-unloading cycles. Other issues announced by Hutchinson (2012), such as unreasonable discontinuous changes in higher order stresses upon certain infinitesimal load changes and failure in guaranteeing nonnegative dissipations, are believed to arise also due to lack of robust SGP theory for complex loading histories.

To eliminate the above thermodynamical issue, Liu and Soh (2014) recently proposed a new SGP reformulation. The essential opinion is that the omission of the strain gradient elasticity effect directly leads to discontinuous changes in *total* strain gradients during complex loading histories, and therefore causes the fundamental issues summarized by Hutchinson (2012). Linking together SGE and SGP has been described in details by Fleck and Hutchinson (1997). Nevertheless, in order to ease the complexity in dealing with 5 elastic material characteristic length scales appearing in the expression of elastic second order strain energy density, Lam *et al*. (2003) proposed a new decomposition strategy of second order strain and, hence, reduced the total number of elastic length scales to 3, based on the conclusion that the anti-symmetric part of the rotational gradient should not enter the expression of the second order strain energy density. Liu and Soh (2014) combined the above-mentioned theories in a compatible manner, and formulated a SGE-SGP framework, which is free of the above-mentioned thermodynamical issues.

In the theory proposed by Liu and Soh (2014), both elastic length scales and plastic length scales are included. Therefore, we aim to explore the relation between them in this study.

The paper is organized as follows. The SGP theory proposed by Liu and Soh (2014) is revisited in Section 2. Then in Section 3, the relation between higher order stresses and length scales is summarized. The wire torsion is taken as an example and solved analytically, based on which the relation between elastic and plastic length scales is discussed in Section 4. This paper ends with conclusions in Section 5.

2. SGP Framework

2.1 Stress-strain relations

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The generalized strain variables are the symmetric strain tensor ε_{ij} and the second order gradient of displacement η_{ijk}, which are expressed, respectively, as

$$\varepsilon_{ij} = \varepsilon_{ji} = \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right), \eta_{ijk} = \eta_{jik} = u_{k,ij} = \partial_{ij} u_k, \quad (1)$$

where u_i is the i-th displacement component and ∂_i is the forward gradient operator.

The major difference between the conventional elasticity and SGE is that the change in elastic strain energy density, $\dot{W}$, depends on the changes in both ε_{ij} and η_{ijk} due to an arbitrary infinitesimal variation of displacement u. In the energetic calculation, in order to maintain consistency with the following SGP formulations, the superscript "e" which denotes "elasticity", is added onto strains and second order strains as well as the strain energy. The change in energy is written as

$$\dot{W}^e = \sigma_{ij}\dot{\varepsilon}_{ij}^e + \tau_{ijk}\dot{\eta}_{ijk}^e, \quad (2)$$

where the symmetric Cauchy stress σ_{ij} ($=\sigma_{ji}$) and the higher-order stress τ_{jik} ($=\tau_{jik}$) are the work conjugates of respectively the elastic strain and second order strain, $\varepsilon_{ij}^e = \int \dot{\varepsilon}_{ij}^e dt$ and $\eta_{ijk}^e = \int \dot{\eta}_{ijk}^e dt$. The higher order stress tensor is composed of both couple stresses and double stresses. The work statement Eq. (2) gives the following elastic constitutive relations

$$\sigma_{ij} = \frac{\partial W^e}{\partial \varepsilon_{ijk}^e}, \tau_{ijk} = \frac{\partial W^e}{\partial \eta_{ijk}^e}. \quad (3)$$

In the following we adopt the SGE formulation by Lam *et al*. (2003).

Firstly, by decomposing the second order strain η into symmetric and anti-symmetric parts via the strategy proposed by Fleck and Hutchinson (1997), we obtain

$$\eta = \eta^S + \eta^A, \eta_{ijk}^S = \frac{1}{3}\left(\eta_{ijk} + \eta_{jki} + \eta_{kij}\right),$$
$$\eta_{ijk}^A = \frac{2}{3}\left(e_{ikp}\chi_{pj} + e_{jkp}\chi_{pi}\right). \quad (4)$$

where χ is the rotational gradient.

Subsequently, new independent second order strain metrics are obtained by splitting the symmetric second order strain η_{ijk}^S into a trace part $\eta_{ijk}^{(0)}$ and a traceless part $\eta_{ijk}^{(1)}$, as follow

$$\eta_{ijk}^S = \eta_{ijk}^{(0)} + \eta_{ijk}^{(1)}, \quad (5)$$

where

$$\eta_{ijk}^{(0)} = \frac{1}{5}\left(\delta_{ij}\eta_{mmk}^S + \delta_{jk}\eta_{mmi}^S + \delta_{ki}\eta_{mmj}^S\right),$$
$$\eta_{ijk}^{(1)} = \eta_{ijk}^S - \eta_{ijk}^{(0)},$$

$$\eta_{mmk}^S = \eta_{kmm}^S = \frac{1}{3}\left(\eta_{mmk} + 2\eta_{kmm}\right). \quad (6)$$

Similarly, the rotational gradient, χ, is also decomposed into a symmetric and an anti-symmetric part as follow

$$\chi_{ij} = \chi_{ij}^S + \chi_{ij}^A, \quad (7)$$

where

$$\chi_{ij}^S = \frac{1}{2}\left(\chi_{ij} + \chi_{ji}\right), \chi_{ij}^A = \frac{1}{2}\left(\chi_{ij} - \chi_{ji}\right). \quad (8)$$

The trace part of the symmetric second order strain is a function of the dilatation gradient and the anti-symmetric part of the rotational gradient, i.e.

$$\eta_{ipp}^S = \varepsilon_{,i} + \frac{2}{3}e_{imn}\chi_{mn}, \quad (9)$$

where ε is the dilatation strain given by

$$\varepsilon = \varepsilon_{mm}. \quad (10)$$

For easy reference, $\varepsilon_{,i}$, $\eta_{ijk}^{(1)}$ and χ_{ij} are called the dilatation gradient, the deviatoric stretch gradient and the rotational gradient respectively. The lower order elastic strain energy rate is the same as the conventional elasticity counterpart, whereas, the second order elastic strain energy rate can be expressed in terms of the rates of three independent second order strains components, $\dot{\eta}_{ijk}^{(0)e}$, $\dot{\eta}_{ijk}^{(1)e}$ and $\dot{\eta}_{ijk}^{A,e}$, where, the superscript "e" denotes "elastic", and the rates of their corresponding work-conjugates $\dot{\tau}_{ijk}^{(0)}$, $\dot{\tau}_{ijk}^{(1)}$ and $\dot{\tau}_{ijk}^{A}$, doubled as follow

$$2\dot{\hat{w}}^e = \dot{\tau}_{ijk}^{(0)}\dot{\eta}_{ijk}^{(0)e} + \dot{\tau}_{ijk}^{(1)}\dot{\eta}_{ijk}^{(1)e} + \dot{\tau}_{ijk}^{A}\dot{\eta}_{ijk}^{A,e}, \quad (11)$$

where

$$\tau_{ijk}^{(0)} = \frac{1}{5}\left(\delta_{ij}\tau_{mmk}^S + \delta_{jk}\tau_{mmi}^S + \delta_{ki}\tau_{mmj}^S\right), \quad (12)$$

$$\tau_{ijk}^{(1)} = \tau_{ijk}^S - \tau_{ijk}^{(0)}, \quad (13)$$

are the trace and traceless parts of the symmetric portion of the higher order stress tensor, respectively. Note that these two tensors, i.e., $\tau_{ijk}^{(0)}$ and $\tau_{ijk}^{(1)}$, are orthogonal to each other.

Liu and Soh (2014) have proven that the anti-symmetric part of the rotational gradient, i.e., χ_{ij}^A is not involved in the calculation of second order elastic strain energy. This supports our hypothesis that the work-conjugate of χ_{ij}^A should vanish, i.e., $m_{ij}^A \equiv 0$, which gives

$$m_{ij}\dot{\chi}_{ij}^e = m_{ij}^S\dot{\chi}_{ij}^{S,e}, \quad (14)$$

since, m_{ij}^S, which is the work conjugate of $\chi_{ij}^{S,e}$, does no mechanical work along $\dot{\chi}_{ij}^{A,e}$, i.e., $m_{ij}^S\dot{\chi}_{ij}^{A,e} = 0$.

Therefore, χ_{ij} can be replaced by χ_{ij}^S in computing second order elastic strain energy density. If the set of second order strain metrics $\varepsilon_{,i}^{e}$, $\eta_{ijk}^{(1)e}$ and $\chi_{ij}^{S,e}$ and the corresponding work-conjugates p_i, $\tau_{ijk}^{(1)}$ and m_{ij}^{S} are adopted, the change in the second order elastic strain energy density can be expressed as

$$\dot{\hat{W}}^{e} = p_i \dot{\varepsilon}_{,i}^{e} + \tau_{ijk}^{(1)} \dot{\eta}_{ijk}^{(1)e} + m_{ij}^{S} \dot{\chi}_{ij}^{S,e}, \tag{15}$$

where

$$p_i = \frac{3}{5} \tau_{mmi}^{S}, \tag{16}$$

$$m_{ij}^{S} = \frac{4}{3} \tau_{ipq}^{A} e_{jpq} - \frac{2}{5} e_{ijk} \tau_{mmk}^{S}. \tag{17}$$

The inverse form is

$$\begin{aligned} \tau_{ijk}^{A} &= \frac{1}{4}\left(e_{ikp} m_{jp}^{S} + e_{jkp} m_{ip}^{S}\right) \\ &\quad - \frac{1}{6}\left(2\delta_{ij} p_k - \delta_{ik} p_j - \delta_{jk} p_i\right), \end{aligned} \tag{18}$$

$$\tau_{ijk}^{(0)} = \frac{1}{3}\left(\delta_{ij} p_k + \delta_{jk} p_i + \delta_{ik} p_j\right). \tag{19}$$

For linear elastic centro-symmetric isotropic solids, the total strain energy rate becomes

$$\begin{aligned} \dot{w}^{e} &= \frac{1}{2}\lambda\left(\dot{\varepsilon}_{kk}^{e}\right)^2 + \mu \dot{\varepsilon}_{ij}^{e} \dot{\varepsilon}_{ij}^{e} + \frac{3}{2} K L_0^2 \dot{\varepsilon}_{,i}^{e} \dot{\varepsilon}_{,i}^{e} \\ &\quad + \mu\left(L_1^2 \dot{\eta}_{ijk}^{(1)e} \dot{\eta}_{ijk}^{(1)e} + L_2^2 \dot{\chi}_{ij}^{S,e} \dot{\chi}_{ij}^{S,e}\right), \end{aligned} \tag{20}$$

and the elastic constitutive relations include the conventional elasticity portion which is given by

$$\sigma_{ij} = \lambda \delta_{ij} \varepsilon_{mm}^{e} + 2\mu \varepsilon_{ij}^{e}, \tag{21}$$

and the second order elasticity portion which is given by

$$p_i = 3KL_0^2 \varepsilon_{,i}^{e}, \tau_{ijk}^{(1)} = 2\mu L_1^2 \eta_{ijk}^{(1)e}, m_{ij}^{S} = 2\mu L_2^2 \chi_{ij}^{S,e}. \tag{22}$$

There are only three elastic length scales, i.e., L_0, L_1 and L_2. Furthermore, no cross terms appear in the expression of the new energy rate in Eq. (20), which greatly facilitates the upcoming SGP formulation to be carried out in the following sections. It is worth noting that the first expression in Eq. (22) is adopted instead of $p_i = 2\mu L_0^2 \varepsilon_{,i}^{e}$ used by Lam *et al*. (2003), in order to achieve a rational correspondence between the higher order measures $\left(L_0 \varepsilon_{,i}^{e}, p_i/L_0\right)$ and the lower order measures (ε_{mm}, σ_{kk}) where σ_{kk}=3$K\varepsilon_{mm}$.

2.2 Flow rule

The flow rule is obtained by generalizing the conventional J2 plasticity into the above-mentioned SGE-SGP framework.

The strain rate, $\dot{\varepsilon}_{ij}$, is divided into a volumetric part $\dot{\varepsilon}_{kk}$ and a deviatoric part $\dot{\varepsilon}'_{ij}$. For the second order strain rate excluding the anti-symmetric part of the rotational gradient, its volumetric part is taken as the dilatation gradient rate $\dot{\varepsilon}_{,i}$, while its deviatoric part is made up of two compositions, i.e., the deviatoric stretch gradient rate $\dot{\eta}_{ijk}^{(1)}$ and the symmetric part of rotational gradient rate, $\dot{\chi}_{ij}^{S}$. The volumetric strain and second order strain rates $\dot{\varepsilon}_{kk}$ and $\dot{\varepsilon}_{,i}$ are purely elastic due to the assumed incompressibility of plastic deformation and are, respectively, related to the hydrostatic stress and higher order stress rates, $\dot{\sigma}_{kk}$ and $\dot{p}_i$, as

$$\dot{\varepsilon}_{kk} = \frac{\dot{\sigma}_{kk}}{3K}, L_0 \dot{\varepsilon}_{,i} = \frac{\dot{p}_i}{3KL_0}. \tag{23}$$

The deviatoric strain and second order strain rates are decomposed into elastic and plastic parts as

$$\begin{aligned} &\dot{\varepsilon}'_{ij} = \dot{\varepsilon}_{ij}^{e'} + \dot{\varepsilon}_{ij}^{p}, \dot{\eta}_{ijk}^{(1)} = \dot{\eta}_{ijk}^{(1)e} + \dot{\eta}_{ijk}^{(1)p}, \\ &\dot{\chi}_{ij}^{S} = \dot{\chi}_{ij}^{S,e} + \dot{\chi}_{ij}^{S,p} \end{aligned} \tag{24}$$

where the elastic deviatoric strain and second order strain rates are proportional to their deviatoric work-conjugate rates respectively, in the form

$$\dot{\varepsilon}_{ij}^{e'} = \frac{\dot{\sigma}'_{ij}}{2\mu}, L_1 \dot{\eta}_{ijk}^{(1)e} = \frac{\dot{\tau}_{ijk}^{(1)}}{2\mu L_1}, L_2 \dot{\chi}_{ij}^{S,e} = \frac{\dot{m}_{ij}^{S}}{2\mu L_2}. \tag{25}$$

The plastic strain and second order strain rates are proportional to their corresponding deviatoric work-conjugates by the associative rule as

$$\dot{\varepsilon}_{ij}^{p} = \dot{\Lambda} \sigma'_{ij}, l_1 \dot{\eta}_{ijk}^{(1)p} = \dot{\Lambda}\left(\frac{\tau_{ijk}^{(1)}}{l_1}\right), l_2 \dot{\chi}_{ij}^{S,p} = \dot{\Lambda}\left(\frac{m_{ij}^{S}}{l_2}\right). \tag{26}$$

where l_1 and $l_2$are material’s plastic characteristic length scales introduced to establish generalized correspondences between plastic strain and second order strain, and between Cauchy stress and “plastic higher order stress”. The newly introduced coefficient $\dot{\Lambda}$ in Eq. (26) is calculated by squaring three equations in Eq. (26) and then adding them together, as

$$\dot{\Lambda} = \left[\frac{\frac{3}{2}\dot{\varepsilon}_e^2 + l_1^2\left(\dot{\eta}_e^{(1)}\right)^2 + l_2^2\left(\dot{\chi}_e^{S}\right)^2}{\frac{2}{3}\sigma_e^2 + \left(\frac{\tau_e^{(1)}}{l_1}\right)^2 + \left(\frac{m_e^{S}}{l_2}\right)^2}\right]^{\frac{1}{2}} = \frac{\dot{\varepsilon}_e^{t}}{\sigma_e^{t}}, \tag{27}$$

where

$$\begin{aligned} &\dot{\eta}_e^{(1)} = \sqrt{\dot{\eta}_{ijk}^{(1)p} \dot{\eta}_{ijk}^{(1)p}}, \dot{\chi}_e^{S} = \sqrt{\dot{\chi}_{ij}^{S,p} \dot{\chi}_{ij}^{S,p}}, \\ &\dot{\varepsilon}_e^{t} = \sqrt{\frac{3}{2}\dot{\varepsilon}_e^2 + l_1^2\left(\dot{\eta}_e^{(1)}\right)^2 + l_2^2\left(\dot{\chi}_e^{S}\right)^2} \end{aligned} \tag{28}$$

are respectively effective rates of the deviatoric stretch gradient, the symmetric part of rotational gradient and the "total" effective plastic strain covering contributions of both conventional plastic strain and plastic second order strain; while

$$\tau_e^{(1)} = \sqrt{\tau_{ijk}^{(1)}\tau_{ijk}^{(1)}},\ m_e^S = \sqrt{m_{ij}^S m_{ij}^S}, \qquad (29)$$
$$\dot{\varepsilon}_e^t = \sqrt{\frac{2}{3}\sigma_e^2 + \left(\frac{\tau_e^{(1)}}{l_1}\right)^2 + \left(\frac{m_e^S}{l_2}\right)^2}$$

are respectively effective values of higher order stresses $\tau_{ijk}^{(1)}$, m_{ij}^S and the effective plastic stress. The occurrence of plastic loading for work-hardening materials requires

$$\dot{\sigma}_e^t = \frac{1}{\sigma_e^t}\left(\frac{2}{3}\sigma_e\dot{\sigma}_e + \frac{\tau_e^{(1)}\dot{\tau}_e^{(1)}}{l_1^2} + \frac{m_e^S\dot{m}_e^S}{l_2^2}\right) > 0. \qquad (30)$$

Combining Eqs. (24)-(27) gives

$$\dot{\sigma}'_{ij} = 2\mu\left(\dot{\varepsilon}'_{ij} - \frac{\dot{\varepsilon}_e^t}{\sigma_e^t}\sigma'_{ij}\right),$$
$$\frac{\dot{\tau}_{ijk}^{(1)}}{L_1} = 2\mu\left(L_1\dot{\eta}_{ijk}^{(1)} - \frac{\dot{\varepsilon}_e^t}{\sigma_e^t}\frac{L_1\tau_{ijk}^{(1)}}{l_1^2}\right), \qquad (31)$$
$$\frac{\dot{m}_{ij}^S}{L_2} = 2\mu\left(L_2\dot{\chi}_{ij}^S - \frac{\dot{\varepsilon}_e^t}{\sigma_e^t}\frac{L_2 m_{ij}^S}{l_2^2}\right).$$

The above equations can be grouped into a more concise form by defining the following 45-component vectors

$$\left\{\dot{\hat{\varepsilon}}_I^{t'}\right\} = \left\{\dot{\varepsilon}_{11}^{t'} \cdots \dot{\varepsilon}_{33}^{t'} \cdots L_1\dot{\eta}_{111}^{(1)} \cdots L_1\dot{\eta}_{333}^{(1)}\ L_2\dot{\chi}_{11}^S \cdots L_2\dot{\chi}_{33}^S\right\},$$

$$\left\{\dot{\varepsilon}_I^{t',e}\right\} = \left\{\dot{\varepsilon}_{11}^{t',e} \cdots \dot{\varepsilon}_{33}^{t',e} \cdots L_1\dot{\eta}_{111}^{(1)e} \cdots L_1\dot{\eta}_{333}^{(1)e}\ L_2\dot{\chi}_{11}^{S,e} \cdots L_2\dot{\chi}_{33}^{S,e}\right\},$$

$$\left\{\dot{\varepsilon}_I^{t,p}\right\} = \left\{\dot{\varepsilon}_{11}^{t,p} \cdots \dot{\varepsilon}_{33}^{t,p} \cdots l_1\dot{\eta}_{111}^{(1)p} \cdots l_1\dot{\eta}_{333}^{(1)p}\ l_2\dot{\chi}_{11}^{S,p} \cdots l_2\dot{\chi}_{33}^{S,p}\right\},$$

$$\left\{\sigma_I^{t',p}\right\} = \left\{\sigma'_{11} \cdots \sigma'_{33}\ \frac{\tau_{111}^{(1)}}{l_1} \cdots \frac{\tau_{333}^{(1)}}{l_1} \cdots \frac{m_{11}^S}{l_2} \cdots \frac{m_{33}^S}{l_2}\right\},$$

$$\left\{\dot{\sigma}_I^{t',e}\right\} = \left\{\dot{\sigma}'_{11} \cdots \dot{\sigma}'_{33}\ \frac{\dot{\tau}_{111}^{(1)}}{L_1} \cdots \frac{\dot{\tau}_{333}^{(1)}}{L_1} \cdots \frac{\dot{m}_{11}^S}{L_2} \cdots \frac{\dot{m}_{33}^S}{L_2}\right\},$$

$$\left\{\hat{\sigma}_I^{t'}\right\} = \left\{\sigma'_{11} \cdots \sigma'_{33}\ \frac{L_1\tau_{111}^{(1)}}{l_1^2} \cdots \frac{L_1\tau_{333}^{(1)}}{l_1^2} \cdots \frac{L_2 m_{11}^S}{l_2^2} \cdots \frac{L_2 m_{33}^S}{l_2^2}\right\}.$$

The total work rate done by the generalized stress rate along the generalized strain rate can be written as

$$\dot{w} = \dot{w}^e + \dot{w}^p = \frac{1}{2}\dot{\sigma}_I^{t',e}\dot{\varepsilon}_I^{t',e} + \frac{1}{2}\dot{\sigma}_I^{t',p}\dot{\varepsilon}_I^{t',p}. \qquad (32)$$

We have $\left(\sigma_e^t\right)^2 = \sigma_I^{t',p}\sigma_I^{t',p}$, and Eq. (31) is reduced to a concise form

$$\dot{\sigma}_I^{t',e} = 2\mu\left(\dot{\hat{\varepsilon}}_I^{t'} - \frac{\dot{\varepsilon}_e^t}{\sigma_e^t}\hat{\sigma}_I^{t'}\right), \qquad (33)$$

Similarly, the generalized plastic stress rate vector can be defined as

$$\dot{\sigma}_I^{t',p} = 2\mu\left(\dot{\varepsilon}_I^{t'} - \frac{\dot{\varepsilon}_e^t}{\sigma_e^t}\sigma_I^{t'}\right), \qquad (34)$$

where

$$\left\{\dot{\varepsilon}_I^{t'}\right\} = \left\{\dot{\varepsilon}_{11}^{t'} \cdots \dot{\varepsilon}_{33}^{t'} \cdots \frac{L_1^2}{l_1}\dot{\eta}_{111}^{(1)} \cdots \frac{L_1^2}{l_1}\dot{\eta}_{333}^{(1)}\ \frac{L_2^2}{l_2}\dot{\chi}_{11}^S \cdots \frac{L_2^2}{l_2}\dot{\chi}_{33}^S\right\},$$
$$\left\{\sigma_I^{t'}\right\} = \left\{\sigma'_{11} \cdots \sigma'_{33} \cdots \frac{L_1^2\tau_{111}^{(1)}}{l_1^3} \cdots \frac{L_1^2\tau_{333}^{(1)}}{l_1^3}\ \frac{L_2^2 m_{11}^S}{l_2^3} \cdots \frac{L_2^2 m_{33}^S}{l_2^3}\right\}.$$

The effective total plastic strain ε_e^t is obtained from the effective total plastic strain rate $\dot{\varepsilon}_e^t$ as

$$\varepsilon_e^t = \int \dot{\varepsilon}_e^t dt. \qquad (35)$$

The uniaxial stress-strain relation is now extended to express in terms of the effective total plastic strain ε_e^t for three-dimension deformations as

$$\sigma^t = \sigma_{ref}^t f_p^t\left(\varepsilon_e^t\right), \qquad (36)$$

where the material parameter σ_{ref}^t and the yield function f_p^t can be different from their uniaxial counterparts f_p due to the inclusion of new kinds of dislocation mechanism like GND, which is produced by inhomogeneous plastic deformation. And we have

$$f_p^t\left(\varepsilon_e^t\right) = f^t\left(\varepsilon^t\right) = f^t\left[\varepsilon_e^t + \frac{\sigma_{ref}^t}{E} f_p^t\left(\varepsilon_e^t\right)\right]. \qquad (37)$$

The yield criterion is

$$\sigma_e^t = \sigma^t. \qquad (38)$$

Differentiating the square of Eq. (38) with respect to time leads to the consistency condition

$$\sigma_e^t\dot{\sigma}_e^t = \sigma_I^{t',p}\dot{\sigma}_I^{t',p} = \left(\sigma_{ref}^t\right)^2 f_p^t\left(f_p^t\right)'\dot{\varepsilon}_e^t. \qquad (39)$$

Inserting Eq. (31) into Eq. (38) yields

$$\dot{\varepsilon}_e^t = \frac{2\mu\sigma_I^{t',p}\dot{\varepsilon}_I^{t'}}{2\mu\sigma_I^{t',p}\sigma_I^{t'}\Big/\sigma_e^t + \left(\sigma_{ref}^t\right)^2 f_p^t\left(f_p^t\right)'}. \qquad (40)$$

Taking account of the possibility of elastic unloading, the total plastic strain rate $\dot{\varepsilon}_e^t$ is given by

$$\dot{\varepsilon}_e^t = \begin{cases} \dfrac{2\mu\sigma_I^{t',p}\dot{\varepsilon}_I^{t'}}{2\mu\sigma_I^{t',p}\sigma_I^{t'}\Big/\sigma_e^t + \left(\sigma_{ref}^t\right)^2 f_p^t\left(f_p^t\right)'} & \text{if } \sigma_e^t = \sigma^t \text{ and } \dot{\sigma}_e^t \geq 0, \\ 0 & \text{if } \sigma_e^t < \sigma^t \text{ or } \dot{\sigma}_e^t < 0. \end{cases} \qquad (41)$$

A brief explanation is made on Eq. (26). In Eq. (26), the evolutions of $\dot{\varepsilon}_{ij}^{p}$, $l_1\dot{\eta}_{ijk}^{(1)p}$ and $l_2\dot{\chi}_{ij}^{S,p}$ are determined by the common coefficient $\dot{\Lambda}$. Thus, plastic evolution happens as long as $\dot{\Lambda}>0$ (or equivalently $\dot{\varepsilon}_e^t>0$), even though there may exist $\dot{\varepsilon}_e=0$ or $\dot{\eta}_e^{(1)}=0$ or $\dot{\chi}_e^S=0$. This inference is drawn based on the generalization that $l_1\dot{\eta}_{ijk}^{(1)p}$ and $l_2\dot{\chi}_{ij}^{S,p}$ play equivalent roles as what the strain $\dot{\varepsilon}_{ij}^{p}$ does in plastic deformation. While $\dot{\varepsilon}_e$, $\dot{\eta}_e^{(1)}$ and $\dot{\chi}_e^S$ are separately not really "effective" in determining the plastic evolution as indicated by their subscript "*e*", their combination $\dot{\varepsilon}_e^t$ is note that the unrecoverable characteristic of plastic deformation requires the non-negativity of $\dot{\varepsilon}_e^t$ as

$$\dot{\varepsilon}_e^t \geq 0 \tag{42}$$

which has been guaranteed by its definition in Eq. (28) for plastic loading cases, and $\dot{\varepsilon}_e^t$ is zero for elastic unloading. The non-negativity of $\dot{\varepsilon}_e^t$ can also be guaranteed by Eq. (41) because

$$\sigma_I^{t',p}\dot{\varepsilon}_I^{t'}=\frac{1}{2\mu}\sigma_e^t\dot{\sigma}_e^t+\dot{\Lambda}\left[\sigma_e^2+\left(\frac{L_1}{l_1}\right)^2\left(\frac{\tau_e^{(1)}}{l_1}\right)^2+\left(\frac{L_2}{l_2}\right)^2\left(\frac{m_e^S}{l_2}\right)^2\right]\geq 0 \tag{43}$$

$$\text{if } \sigma_e^t=\sigma^t \text{ and } \dot{\sigma}_e^t\geq 0.$$

The satisfaction of Ducker's postulate, i.e., non-negative plastic dissipation rule, is studied. According to Ducker's postulate, we should check the plastic dissipation rate, or equivalently the mechanical work rate done by the generalized stress rate $\dot{\sigma}_I^{t',p}$ along the generalized plastic strain rate $\dot{\varepsilon}_I^{t,p}$, which is equal to half of

$$\dot{\sigma}_I^{t',p}\dot{\varepsilon}_I^{t,p}=\dot{\sigma}_{ij}'\dot{\varepsilon}_{ij}'^{,p}+\dot{\tau}_{ijk}^{(1)}\dot{\eta}_{ijk}^{(1)p}+\dot{m}_{ij}^{S}\dot{\chi}_{ij}^{(1)p}=\sigma_e^t\dot{\Lambda}\dot{\sigma}_e^t\geq 0, \tag{44}$$

in view of the fact that $\dot{\Lambda}\dot{\sigma}_e^t$ is always non-negative for hardening materials. An alternative understanding of Eq. (44) is to bear in mind that $\sigma_e^t\dot{\Lambda}\dot{\sigma}_e^t=\dot{\varepsilon}_e^t\dot{\sigma}_e^t$, which is non-negative during plastic loading or neutral loading and is equal to zero due to $\dot{\varepsilon}_e^t=0$ during elastic unloading for workhardening materials. Therefore, *Ducker's postulate is perfectly satisfied*. Hutchinson (2012) has also revived the non-negative dissipation requirement but it was only for the special cases with the full recoverability of the plastic strain gradient-related dislocations; whereas, the present formulation does not have this limitation and is workable in general cases.

3. Which lengths should the higher order stress be related to?

This topic remains open and here we summarize differences among several representatives in the literature, but nothing can be conclusive.

In the mechanism-based strain gradient (MSG) plasticity (Gao *et al.* 1999), the higher order stress is proportional to the mesoscale representative cell size, l_ε, which is believed to be usually on the order of 10~100 nm. From the viewpoint of orders of magnitude, the τ~l_ε relation in MSG happens to be similar as the τ~l_i relation in the present study, but the purpose of introducing l_ε in MSG is completely different. l_ε and l_i have different physical meanings and should not be taken as the same stuff. It is a coincidence that the MSG-type higher stress and the present higher stress get to the same order of magnitude.

In the incremental SGP recently developed by Fleck *et al.* (2014) which is later called FHW theory for convenience, the higher stress is proportional to l_P^2, where l_p is the unique characteristic plastic length scale in the formulation with a purpose to describe features of micron-scale plasticity by introducing the minimum new parameters. l_i appearing in the present theory and l_p adopted by the FHW theory should have the same physical meaning and therefore should be on the same order. Notably, the FHW higher stress is assumed to be fully recoverable. The undergoing physics picture is that strain gradient is dominantly achieved by recoverable dislocations.

In the SGP formulation proposed by Gudmundson (2004), when the higher order stress can be directly determined from the free energy which depends on an isotropic and quadratic form in the "plastic strain gradients", the higher order stress is related to the square of the length scale, $L_e^{(I)}$, which is introduced to calculate the free energy. Furthermore, Gudmundson (2004) pointed out that $L_e^{(I)}$ is usually significantly smaller than the characteristic plastic length scale. Such a set of free-energy-related length scale, $L_e^{(I)}$, is believed to have a physical meaning equivalent to the characteristic elastic length scale, L_i, in the present study.

In the SGP theory by Anand *et al.* (2005), the higher order stress which was called "microstress" by Anand *et al.* (2005) is decomposed into an energetic part which is related to the "energetic length scale" and a dissipative part which is related to the "dissipative length scale". It indicates that the "dissipative length scale" is equivalent to the characteristic plastic length scale. Furthermore, we suppose that the "energetic length scale" must be very close to the characteristic elastic length scale.

Besides above similarities and differences, it is worth noting that the FHW theory, the formulation by Gudmundson (2004), their earlier archetype, i.e., SGP in Fleck and Hutchinson (2001), and the theory by Anand *et al.* (2005) all belong to the "gradient of strain" models where the plastic part of the strain gradient is taken to be the gradient of plastic strain (Forest and Sievert 2003). MSG relatively has a different derivation, but its conventional form in Huang *et al.* (2004) apparently also has the "gradient of strain" nature. However, the present SGP theory has not been built based on the "gradient of strain" hypothesis, and the plastic strain gradient is produced by the elastic-plastic decomposition of strain gradient.

4. Torsion of a circular cylindrical bar

Torsion of a circular wire is selected as an example to illustrate the proposed SGP model.

4.1 Theoretical formulation

Assume that the wire axis is lying along the x_3 axis of a Cartesian co-ordinate system (x_1, x_2, x_3). For convenience, a cylindrical polar co-ordinate system (r, θ, x_3) is also introduced, with the radius of the wire as a. Let κ be the twist per unit length of the wire, which is taken to be positive without loss of generality.

Since the loading and unloading cycles will be studied, we solve the problem by incremental formulations. We begin with assuming the same displacement field as that in the conventional torsion

$$\dot{u}_1 = -\dot{\kappa} x_2 x_3, \dot{u}_2 = -\dot{\kappa} x_1 x_3, \dot{u}_3 = 0. \tag{45}$$

The associated non-vanishing components of strain increment $\dot{\varepsilon}_{ij}$, strain gradient increment $\dot{\eta}_{ijk}$ and rotational gradient increment $\dot{\chi}^S$ are respectively

$$\dot{\varepsilon}_{13} = \dot{\varepsilon}_{31} = -\frac{\dot{\kappa}}{2} x_2, \dot{\varepsilon}_{23} = \dot{\varepsilon}_{32} = -\frac{\dot{\kappa}}{2} x_1; \tag{46}$$

$$\dot{\eta}_{231} = \dot{\eta}_{321} = -\dot{\kappa}, \dot{\eta}_{132} = \dot{\eta}_{312} = \dot{\kappa}; \tag{47}$$

$$\dot{\chi}^S_{11} = \dot{\chi}^S_{22} = -\frac{\kappa}{2}, \dot{\chi}^S_{33} = \dot{\kappa}. \tag{48}$$

It is notable that $\dot{\varepsilon}_{,i} = 0$ and $\dot{\eta}^{(1)}_{ijk} = 0$ for the case of torsion. In every increment $\dot{\kappa}$, both the strain and the rotational gradient are composed of an elastic and plastic part, and the following denotation is adopted

$$\dot{\kappa} = \dot{\kappa}^e_\varepsilon + \dot{\kappa}^p_\varepsilon sign(\dot{\kappa}) = \dot{\kappa}^e_\chi + \dot{\kappa}^p_\chi sign(\dot{\kappa}), \tag{49}$$

with the subscript "ε" denoting "strain" and the subscript "χ" denoting "gradient". Then

$$\dot{\varepsilon}^e_{13} = \dot{\varepsilon}^e_{31} = -\frac{\dot{\kappa}^e_\varepsilon}{2} x_2, \dot{\varepsilon}^e_{23} = \dot{\varepsilon}^e_{32} = -\frac{\dot{\kappa}^e_\varepsilon}{2} x_1, \dot{\varepsilon}_\mathrm{e} = \frac{\dot{\kappa}^p_\varepsilon r}{\sqrt{3}}, \tag{50}$$

$$\dot{\chi}^{S,e}_{11} = \dot{\chi}^{S,e}_{22} = -\frac{\dot{\kappa}^e_\chi}{2}, \dot{\chi}^{S,e}_{33} = \dot{\kappa}^e_\chi, \dot{\chi}^S_e = \sqrt{\frac{3}{2}} \dot{\kappa}^p_\chi, \tag{51}$$

where $\dot{\kappa}^p_\varepsilon$ and $\dot{\kappa}^p_\chi$ are given according to the associative rule as

$$\dot{\kappa}^p_\varepsilon = \frac{2}{\sqrt{3}} \frac{\dot{\Lambda} m^S_e}{l_2^2}, \dot{\kappa}^p_\chi = \sqrt{\frac{2}{3}} \frac{\dot{\Lambda} m^S_e}{l_2^2}. \tag{52}$$

The non-vanishing stress components are

$$\begin{aligned} &\sigma_{13} = \sigma_{31} = -\mu\kappa^e_\varepsilon \chi_2, \sigma_{23} = \sigma_{32} = \mu\kappa^e_\varepsilon \chi_1, \\ &\sigma^2_\mathrm{e} = 3\left(\mu\kappa^e_\varepsilon r\right)^2, \end{aligned} \tag{53}$$

and the non-vanishing couple stress components are

$$\begin{aligned} &m^S_{11} = m^S_{22} = -\mu L_2^2 \kappa^e_\chi, m^S_{33} = -2\mu L_2^2 \kappa^e_\chi, \\ &\left(\frac{m^e}{l_2}\right)^2 = \frac{6\left(\mu L_2^2 \kappa^e_\chi\right)^2}{l_2^2}. \end{aligned} \tag{54}$$

Therefore the total effective stress can be expressed as

$$\sigma^t_e = \mu\sqrt{2\left(\kappa^e_\varepsilon r\right)^2 + \frac{6\left(L_2^2 \kappa^e_\chi\right)^2}{l_2^2}}. \tag{55}$$

We adopt the yield function

$$\sigma = \sigma_Y f^t_p\left(\varepsilon^t_e\right) = \sigma_Y \left(1 + \frac{E\varepsilon^t_e}{\sigma_Y}\right)^N, \tag{56}$$

where E, σ_Y and N are, respectively, the Young's modulus, the initial yield stress and the hardening exponent, and $\dot{\varepsilon}^t_e$ is given as

$$\dot{\varepsilon}^t_e = \sqrt{\frac{1}{2}\left(\dot{\kappa}^p_\varepsilon r\right)^2 + \frac{3}{2}\left(l_2 \dot{\kappa}^p_\chi\right)^2}. \tag{57}$$

The twist decomposition in Eq. (49) can be accomplished by iterative calculations, i.e., checking the effective stress, σ^t_e, in Eq. (55) against the yield surface constraint in Eq. (56).

According to the equilibrium and boundary conditions, by performing some basic mathematical derivations (refer to Liu and Soh 2014 for more details), the torque applied on the wire can be expressed as

$$T = 2\pi\mu \int_0^a \left[\kappa^e_\varepsilon + 3\left(\frac{L_2}{r}\right)^2 \kappa^e_\chi\right] r^3 dr. \tag{58}$$

4.2 Results and discussions

Firstly, size effects related to the ratio of a/l_2 are shown in Fig. 1. A "smaller is stronger" effect is illustrated. Namely, the smaller the wire radius, the bigger yield strength is. This has been a well-known phenomenon and it has been investigated frequently (Fleck *et al.* 1994, Aifantis 1999, Dunstan *et al.* 2009). The inhomogeneity of plastic deformation has often been considered as the physics mechanism behind such size effects. Interestingly, we can find that curves with varying a/l_2 have already become different considerably during the linearly elastic stage. Therefore, one reasonable conclusion is that the observed size effect in plastic hardening is achieved by the inhomogeneities of both elastic and plastic deformations in the specimen where L_2 and l_2 are on the same order of magnitude.

Then, we mainly discuss what the ratio of L_2/l_2 can lead to Fig. 2 shows the torque-twist curves under different values of L_2/l_2. An increase in L_2/l_2 obviously leads to an increase in yield strength. Nevertheless, it is interesting to note that the size effect has become very profound during the initially linearly elastic stage: The slope of the curve for the biggest L_2/l_2 is the biggest among three ratios studied. Therefore, effects of strain gradient elasticity and effects of strain gradient plasticity actually coupled with each other. Thus, it is interesting to differentiate these two effects. Actually, when L_2 and l_2 are of the same order, the size effect in yield hardening is dominantly inherited from the

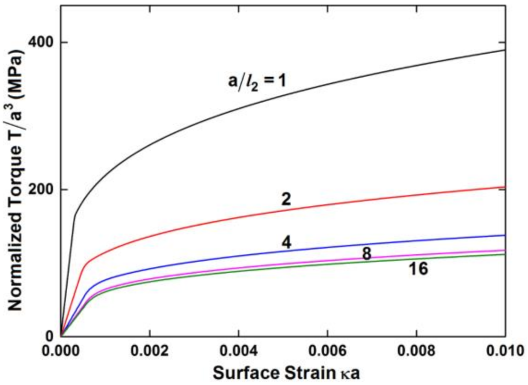

Fig. 1 The normalized torque, T/a^3, versus the normalized surface strain, κa, for various ratios of wire radius to plastic length scale, a/l_2=1,2,4,8,16. The hardening exponent N=0.25, the elastic characteristic length scale L_2=l_2=1μm the shear modulus μ=48 GPa, the yield stress σ_Y=50 MPa, the Poisson's ratio υ=0.34

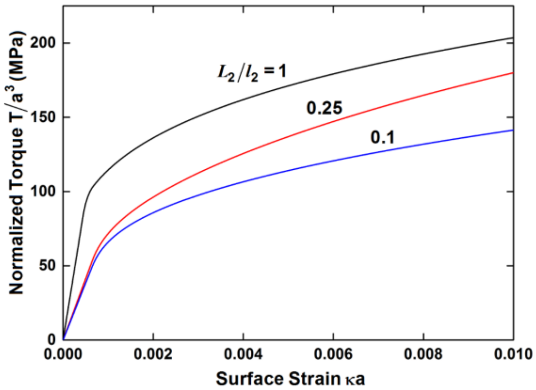

Fig. 2 The normalized torque, T/a^3 versus the normalized surface strain, κa, for various ratios of elastic length scale to plastic length scale, L_2=l_2=1,0.25,0.1. The wire radius a=2μm, the hardening exponent N=0.25, the plastic characteristic length scale l_2=1μm, the shear modulus μ=48 GPa, the yield stress σ_Y=50 MPa, the Poisson's ratio υ=0.34

size effect in elasticity. Notably, the size effect in elastic responses found in both Fig. 1 and Fig. 2 shows the same trends as experimental observations by Lam *et al*. (2003), while that in plastic responses coincides with experimental trends by Fleck *et al.* (1994), Kiener *et al*. (2010), Liu *et al*. (2013).

This topic still seems widely open. The plastic length scale l_2 is usually believed to be of the order of microns. However, the order of the elastic length scale is relatively less conclusive. Lam *et al*. (2003) studied the proxy and found its elastic length scale should be a few microns. But this has been questioned by many other investigators who argued that the elastic length scale should be much smaller than micron. Of course, the answer is essentially material-dependent. For structured materials, the elastic length scale is closely related to the interstructures. For example, for metallic foams, it is expected to be of the same order of the pores. Our future plan is to design appropriate experiments to calibrate the elastic length scales by conducting the purely linear elastic tests, then carry on the elasto-plastic tests on the same specimens. The main challenge that we are still working on is how to separate contributions from inhomogeneities of elastic and plastic deformations, or equivalently contributions of elastic and plastic length scales.

5. Conclusions

The relation between elastic length scales and plastic length scales has been investigated. As already mentioned in Section 3 the definition of higher order stresses varies among different SGPs. Thus, it is worth noting that the discussion is based on the SGP framework recently established by Liu and Soh (2014). When the elastic length scale is of the same order of magnitude as the elastic length scale, considerable size effects have arisen during the initially linear elastic stage. These elastic size effects will be brought into the follow-up elastic-plastic deformation. The relative magnitude of the elastic length scale can significantly determine the intensity of the size effects in plastic hardening.

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References

Aifantis, E. (1999), "Strain gradient interpretation of size effects", *Int. J. Fract.*, **95**(1), 299-314.

Anand, L., Gurtin, M., Lele, S. and Gething, C. (2005), "A one-dimensional theory of strain-gradient plasticity: formulation, analysis, numerical results", *J. Mech. Phys. Solid.*, **53**(8), 1789-1826.

Bertram, A. and Forest, S. (2014), "The thermodynamics of gradient elastoplasticity", *Continuum Mech. Thermodyn.*, **26**(3), 269-286.

Dunstan, D., Ehrler, B., Bossis, R., Joly, S., P'ng, K. and Bushby, A. (2009), "Elastic limit and strain hardening of thin wires in torsion", *Phys. Rev. Lett.*, **103**(15), 155501.

Evans, A. and Hutchinson, J. (2009), "A critical assessment of theories of strain gradient plasticity", *Acta Materialia*, **57**(5), 1675-1688.

Fleck, N. and Hutchinson, J. (1997), "Strain gradient plasticity", *Adv. Appl. Mech.*, **33**, 295-361.

Fleck, N. and Hutchinson, J. (2001), "A reformulation of strain gradient plasticity", *J. Mech. Phys. Solid.*, **49**(10), 2245-2271.

Fleck, N., Hutchinson, J. and Willis, J. (2014), "Strain gradient plasticity under non-proportional loading", *Proc. Royal Soc.: A*, **470**(2170), 20140267-20140267.

Fleck, N., Muller, G., Ashby, M. and Hutchinson, J. (1994), "Strain gradient plasticity: theory and experiment", *Acta Metallurgica Et Materialia*, **42**(2), 475-487.

Fleck, N. and Willis, J. (2009), "A mathematical basis for strain-gradient plasticity theory. part ii: tensorial plastic multiplier", *J. Mech. Phys. Solid.*, **57**(7), 1045-1057.

Forest, S. and Sievert, R. (2003), "Elastoviscoplastic constitutive frameworks for generalized continua", *Acta Mechanica*, **160**(1), 71-111.

Gao, H. and Huang, Y. (2001), "Taylor-based nonlocal theory of plasticity", *Int. J. Solid. Struct.*, **38**(15), 2615-2637.

Gao, H., Huang, Y., Nix, W. and Hutchinson, J. (1999), "Mechanism-based strain gradient plasticity-i. theory", *J. Mech. Phys. Solid.*, **47**(6), 1239-1263.

Gudmundson, P. (2004), "A unified treatment of strain gradient plasticity", *J. Mech. Phys. Solid.*, **52**(6), 1379-1406.

Huang, Y., Qu, S., Hwang, K., Li, M. and Gao, H. (2004), "A conventional theory of mechanism-based strain gradient plasticity", *Int. J. Plast.*, **20**(4-5), 753-782.

Hutchinson, J. (2000), "Plasticity at the micron scale", *Int. J. Solid. Struct.*, **37**(1-2), 225-238.

Hutchinson, J. (2012), "Generalizing j2 flow theory: fundamental issues in strain gradient plasticity", *Acta Mechanica Sinica*, **28**(4), 1078-1086.

Kiener, D., Motz, C., Grosinger, W., Weygand, D. and Pippan, R. (2010), "Cyclic response of copper single crystal micro-beams", *Scripta Materialia*, **63**(5), 500-503.

Lam, D., Yang F., Chong, A., Wang, J. and Tong, P. (2003), "Experiments and theory in strain gradient elasticity", *J. Mech. Phys. Solid.*, **51**(8), 1477-1508.

Liu, D., He, Y., Dunstan, D., Zhang, B., Gan, Z., Hu, P. and Ding, H. (2013), "Toward a further understanding of size effects in the torsion of thin metal wires: An experimental and theoretical assessment", *Int. J. Plast.*, **41**(1), 30-52.

Liu, J. and Soh, A. (2014), "Bridging strain gradient elasticity and plasticity toward general loading histories", *Mech. Mater.*, **78**(16), 11-21.

Nix, W. and Gao, H. (1998), "Indentation size effects in crystalline materials: a law for strain gradient plasticity", *J. Mech. Phys. Solid.*, **46**(3), 411-425.

Stölken, J. and Evans, A. (1998), "A microbend test method for measuring the plasticity length scale", *Acta Materialia*, **46**(14), 5109-5115.

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