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Procedures for determination of elastic curve of simply and multiple supported beams

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Abstract. In this paper two procedures for determination of the elastic curve of the simply and multiple supported beams are developed. Determination of the elastic curve is complex as it requires to solve a strong nonlinear differential equation with given boundary conditions. For numerical solution the initial guess of the slope at the end of the beam is necessary. Two procedures for obtaining of the initial guess are developed: one, based on transformation of the supported beam into a clamped-free one, and second, on the linearization of the problem. Procedures are applied for calculating of elastic curve of a simply supported beam and a beam with three supports. Obtained results are compared. Advantages and disadvantages of both methods are discussed. It is proved that both suggested procedures give us technically accurate results.

Keywords: elastic curve; supported beam; initial guess for slope

1. Introduction

Beams are common elements of structures of different machines, buildings, bridges, and so on. Usually, they are utilized as supporting elements of structures. Due to static loading, the axis of the beam bends. It is of special interest to know the elastic curve of the beam caused by loading. The problem of calculation of the elastic curve of the supported beam exists for a long time (see for example: Timoshenko and Goodier 1952), but, in general, the exact analytic solution is not found, yet. Namely, differential equation which describes the elastic curve $y=y(z)$ is strong nonlinear

$$EI\frac{y''}{(1+y'^2)^{\frac{3}{2}}}=-M, \tag{1}$$

where EI is flexural rigidity of the beam, E is modulus of elasticity, I is moment of inertia of the cross section about its neutral axis, M represents the bending moment of the beam, z is the position coordinate, while (‘)=d/dz and (“)= d^2/dz^2. Closed form analytic solutions of (1) are obtained only for some special cases.

Significant number of investigation is directed toward linearization of the problem. In the

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Table 1 Procedure for numerical solving of (1)

$y''(y'(z),y(z))$	y'	y
$y''_o(y'_o,y_o)$	y'_o	y_o
$y''_1(y'_1,y_1)$	$y'_1=y'_o+y''_o(z_1-z_o)$	$y_1=y_o+y'_1(z_1-z_o)$
$y''_2(y'_2,y_2)$	$y'_2=y_1+y''_1(z_2-z_1)$	$y_2=y_1+y'_2(z_2-z_1)$
$y''_3(y'_3,y_3)$	$y'_3=y'_2+y''_2(z_3-z_1)$	$y_3=y_2+y'_2(z_3-z_2)$
….	….	….

papers of Ramachandra and Roy (2002), Roy and Kumar (2005) transversal linearization methods are presented. Kumar *et al.* (2006) introduced the multi-step linearization technique. Viswanath and Roy (2007) developed the tangential linearization method. Merli *et al*. (2010) compared these linearization procedures. The main disadvantage of these methods is that they are suitable only for the case when the nonlinearity is small. For the case when the nonlinearity is strong numerous approximate analytic procedures for solving (1) are developed (Singh and Sharma 1990, Ramachandra and Roy 2001, Hoseini *et al*. 2009, Shahidi *et al*. 2011, Bayat *et al*. 2013, Pakar *et al*. 2013). However, the most appropriate are the numerical ones. The most efficient and accurate way to integrate the nonlinear equation is to employ some direct integration scheme such Runge-Kutta, Adams-Bashfort or Newmark schemes which are incorporated into some software like the MATLAB finite difference solver, ANSYS and COMSOL with space discretization, Mathematica (see Fertis 2006, Thankane and Stys 2009), etc. In Table 1, common feature of numeric solving procedure is shown.

As it can be seen in Table 1, beside the value of the function *y*, for all of steps of calculation it is necessary to know the first derivative of function *y*', too. The more correct the initial values are, the boundary problem solution problem is more effectively solved. If the conditional initial values are impropriate, the solution would diverge. It means that the problem has to be treated with high accurate initial values. Problem of the beam supported on ends or multiple supported beams, usually, is defined as a boundary value problem and there is not known the adequate number of initial conditions. It gives difficulties in numerical integration of the problem as the boundary problem has to be considered as the conditional initial value problem. The main trouble is how to assume the initial values which would satisfy the boundary conditions. In the paper of Kumar *et al*. (2004) these conditional initial values are optimally selected via a genetic search so as to satisfy all the known boundary conditions. Usually, the shooting method is applied for obtaining of the initial values. The shooting method is basically simple, but is stochastic and selection of the appropriate initial conditions depends on the wit of the investigator. It is the main limitation for its application in practice.

The aim of this paper is to suggest new methods, which are deterministic, for obtaining initial values which would satisfy the prescribed boundary conditions. Two procedures for initial guess of *y*'(0) are suggested: one, based on transformation of the simply supported beam into a clamped-free beam and assuming that the initial slope, as a first step, equals to zero, while the second method is based on the solution of the linearized equation of elastic curve. Namely, it is assumed that for the case of small nonlinearity, the elastic curve determined for the linear model is near to the elastic curve of the real one.

In the paper values of *y*'(0), obtained using the first and the second method, are compared. Two types of structures are considered: beam with two and three supports. Using the suggested

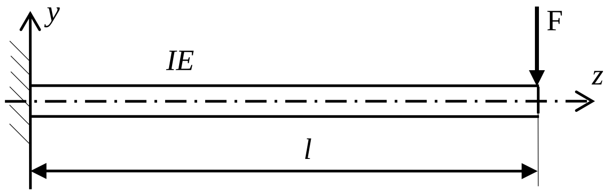

Fig. 1 Clamped-free beam (Cantilever beam)

procedures the elastic curves are calculated. The obtained results are compared. Advantages and disadvantages of the procedures are given. For technical reasons, the optimal method is selected.

2. Procedures for calculation of the slope of the beam near support

2.1 First procedure

For clamped-free (cantilever) beams (Fig. 1), the Eq. (1) with boundary conditions $y(0)=0$ and $y'(0)=0$ can be treated as a simple initial value problem (boundary conditions are at the same time initial conditions) and integration of (1) is done without troubles. Using the procedure given in Table 1, the $y(z)$ solution of (1) is obtained.

However, if the beam with length l is simply supported at its ends, boundary conditions are $y(0)=0$ and $y(l)=0$. For simply supported beam, we have a typical two-boundary value problem whose solution is obtained only if a good initial solution guess is introduced.

Unfortunately, due to these boundary conditions we have no information about the initial slope. Boundary conditions at the end of the beam are not enough instructive to give us the parameters for solving the differential equation. According to the procedure, given in Table 1, for numerical calculation of the problem it is necessary to know the initial slope $y'(0)$, i.e., the initial guess requires the knowledge of $y'(0)$. The basic assumption of the suggested procedure is to give the connection between the simply supported and clamped-free beam. The procedure is developed in the following steps:

1. Support reaction of the simply supported beam, loaded with active forces and torques, are calculated.
2. Simply supported beam with boundary conditions $y(0)=0$ and $y(l)=0$ is considered as a clamped-free beam: the left end of the beam is clamped, while the right end is free without support. Boundary conditions of the obtained clamped-free beam are $y(0)=0$ and $y'(0)=0$. These values are the initial ones for future calculation.
3. Applying these initial conditions the elastic curve of the clamped-free beam are determined.
4. Maximal displacement of the free end of the beam $y(l)$ is calculated.
5. Let us treat the elastic curve – visually – as a “rigid wire”. By rotation of this “rigid wire” around the axis of the left support for 180°, the left end behaves as the free one and the right is fixed. The slope of the “rigid wire” is $y(l)/l$ and with negative sign.
6. Applying this value, approximate initial conditions of the problem become: $y(0)=0$ and $y'(0)=-y(l)/l$. These values are the initial guess for further numerical calculation.

7. Obtained result is checked, by predicting of boundary conditions. If boundary conditions are satisfied, the initial guess is correct and the calculated elastic curve is correct.
8. If the boundary conditions are not accurately satisfied, the procedure has to be repeated until satisfaction of the needed accuracy.

2.2 Second procedure

The second procedure of initial guess of $y'(0)$ is based on the solution of the linearized equation of elastic curve (1). Namely, the exact analytical solution of the linearized equation exists and is applied for calculating of the slope at the left end of the beam. This value is accurate only for the case when the nonlinearity is small, while for strong nonlinear system needs to be improved. The procedure for determination of initial slope is as follows:
1. Using the linearized equation

$$y'' = \frac{M(z)}{EI}, \tag{2}$$

analytical solution for elastic curve of the beam is obtained. Boundary conditions are satisfied.
2. Derivation of the elastic curve $y(z)$ gives the slope $y'(z)$. Slope for z=0 is calculated and we obtain $y'(0)$.
3. The obtained value $y'(0)$ is used for calculation of the elastic curve using (1), where boundary conditions are $y(0)$ and $y(l)$=0.

3. Simply supported beam

Let us consider the beam of length l with a distributed load at its left half (Fig. 2). It is known that the connection of elastic curve and moment function is

$$0 \le z \le \frac{l}{2} \qquad y_1'' = -\frac{q}{2EI}\left(\frac{3}{4}lz - z^2\right)(1 + y'^2)^{\frac{3}{2}}, \tag{3}$$

$$\frac{l}{2} \le z \le l \qquad y_2'' = \frac{ql}{8EI}(z - l)(1 + y'^2)^{3/2}, \tag{4}$$

with the following real boundary conditions

$$y_1(0) = 0,\ y_1\left(\frac{l}{2}\right) = y_2\left(\frac{l}{2}\right),\ y_1'\left(\frac{l}{2}\right) = y_2'\left(\frac{l}{2}\right),\ y_2(l) = 0, \tag{5}$$

where q is the distributed load of the beam. Reaction forces are $F_A=3ql/8$ and $F_B=ql/8$. In Fig. 3, the moment function $M(z)$ is given. The maximal moment is $M(3l/8)=ql^2/128$.

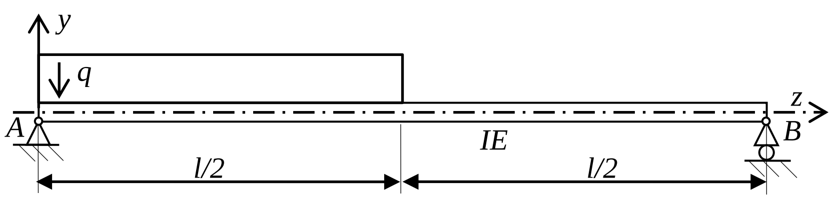

Fig. 2 Simply supported beam

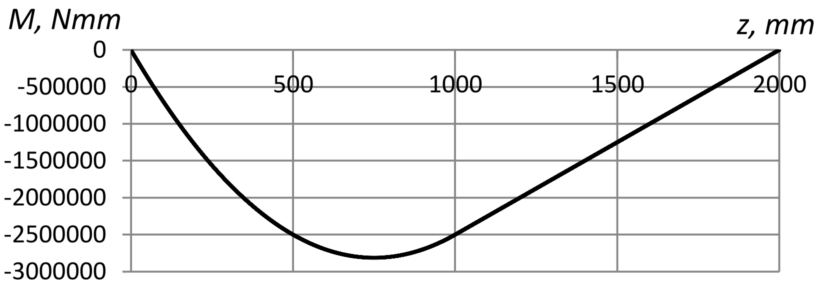

Fig. 3 Moment function of simply supported beam

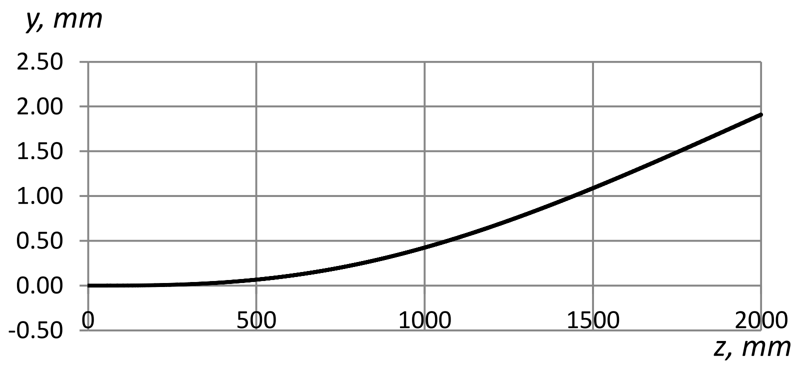

Fig. 4 Elastic curve of the beam for $y'(0)=0$

The problem of determination of elastic curve of the beam given in Fig.2 represents a two point boundary value problem. For its numerical solution, the initial guess for $y'(0)$ is determined using both procedures developed in this paper. Numerical data for the beam are: q=-10 N/mm, l=2000 mm, I=935 cm^4, E=210 GPa.

3.1 Procedure 1

Let us transform the simply supported beam into the clamped-free one and assume the boundary (initial) conditions $y(0)=0$ and $y'(0)=0$. The obtained y-z curve is plotted in Fig. 4. Maximal deflection on the right end is calculated as the result of double numerical integration and yields $y(l) = 1.90985\ \mathrm{mm}$.

Using this value the approximate initial slope of the simply supported beam is obtained

$$y'(0) = -\frac{y(l)}{l} = \ -9.54928 \cdot 10^{-4}\, rad\,. \tag{6}$$

Applying the initial values $y(0)=0$ and $y'(0)= -9.54928 \cdot 10^{-4}$, relations (3) and (4) and algorithm given in Table 1, the elastic curve is numerically calculated with software MS Excell.

Reviewing the obtained result we calculate the deflection value at the end of the beam $y(l)$. Numerically obtained value is $y(l)$=-1,01791*10^{-6} mm. The result differs from zero, but is appropriate for technical consideration.

3.2 Procedure 2

Linearizing relations (3) and (4) we have

$$0 \leq z \leq \frac{l}{2} \qquad y_1'' = -\frac{q}{2EI}\left(\frac{3}{4}lz - z^2\right), \tag{7}$$

$$\frac{l}{2} \leq z \leq l \qquad y_2'' = \frac{ql}{8EI}(z - l), \tag{8}$$

with boundary conditions (5). Integrating relations (7) and (8), we have

$$y_1' = -\frac{q}{2}\frac{1}{EI}\left(\frac{3}{8}lz^2 - \frac{z^3}{3}\right) + C, \qquad y_1 = -\frac{q}{2}\frac{1}{EI}\left(\frac{3}{24}lz^3 - \frac{z^4}{12}\right) + Cz + D, \tag{9}$$

and

$$y_2' = \frac{ql}{8EI}\left(\frac{z^2}{2} - lz\right) + Q, \;\; y_2 = \frac{ql}{8EI}\left(\frac{z^3}{6} - l\frac{z^2}{2}\right) + Qz + K\,, \tag{10}$$

where C, D, Q and K are constants of integration. Using the boundary conditions (5) and after rearrangements, we have

$$D = 0, \;\; C = \frac{9}{384}\frac{ql^3}{IE}, \quad Q = \frac{17}{384}\frac{ql^3}{IE}, \quad K = -\frac{1}{384}\frac{ql^4}{IE}. \tag{11}$$

Substituting (11) into (9) and (10), equations of the elastic curve is obtained

$$y_1 = -\frac{q}{384}\frac{1}{EI}(-16z^4 + 24lz^3 - 9l^3z), \tag{12}$$

$$y_2 = -\frac{ql}{384}\frac{1}{EI}(8z^3 - 24lz^2 + 17l^2z - l^3). \tag{13}$$

Using numerical data for the beam, we obtain the initial slope

$$y_{an}'(0) = \frac{9}{384}\frac{ql^3}{IE} = -9.54927 \cdot 10^{-4}\,rad. \tag{14}$$

Comparing the results for initial slopes (6) and (14), it is seen that the difference between them is negligible.

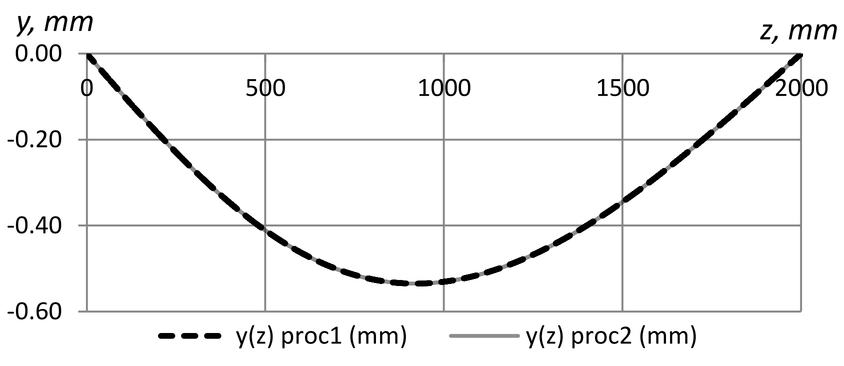

Fig. 5 Elastic curves of the beam obtained by Procedure 1 and 2

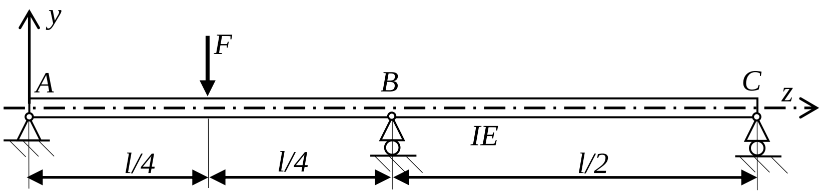

Fig. 6 Beam with three-supports

Due to this fact, there is a good agreement between *y*-*z* diagrams (see Fig. 5) obtained with the Procedure 1 and Procedure 2. Namely, there is no real difference between elastic curves obtained by these two methods.

4. Beam with three supports

In Fig. 6, a beam with three supports is loaded with a force *F* which acts on *l*/4 from the left end of the beam. The flexural rigidity of the beam *EI* is constant. Our aim is to determine the elastic curve of the beam shown in Fig. 6.

Elastic curve is divided into following sections

$$0 \leq z \leq l/4,\ \ y_1'' = -\frac{39}{96}\frac{Fz}{EI}(1+y'^2)^{3/2}, \tag{15}$$

$$l/4 \leq z \leq l/2,\ \ y_2'' = \frac{F}{96\,IE}(57z-24l)(1+y'^2)^{3/2}, \tag{16}$$

$$l/2 \leq z \leq l,\ \ y_3'' = \frac{9F}{96\,IE}(l-z)(1+y'^2)^{\frac{3}{2}}, \tag{17}$$

with boundary conditions

$$y_1(0)=0, \quad y_1\left(\frac{l}{4}\right) = y_2\left(\frac{l}{4}\right), \quad y_1'\left(\frac{l}{4}\right) = y_2'\left(\frac{l}{4}\right),$$

$$y_2\left(\frac{l}{2}\right) = y_3\left(\frac{l}{2}\right) = 0, \quad y_2'\left(\frac{l}{2}\right) = y_3'\left(\frac{l}{2}\right), \quad y_3(l) = 0. \tag{18}$$

The supported beam is indeterminate structure and to calculate the constraint forces beside the static equilibrium equations we need to apply linearized deformation equations. For numerical values: l=2000 mm, E=210 GPa, I=328 cm^4, F=-20000 N, without going into details the calculated constraint forces are $F_A = \frac{39}{96}F,\ F_B = \frac{66}{96}F,\ F_C = -\frac{9}{96}F$.

Based on the force distribution, the moment function is plotted in Fig. 7. Two procedures for determination of the elastic curve are considered.

4.1 Procedure 1

At first the initial conditions for the continuous beam are *y*(0)=0 and *y'*(0)=0 again. The end of the beam *y*(*l*) has the maximal deflection 2.72213 mm (see Fig. 8).

Applying the above proposed method the initial slope is obtained as

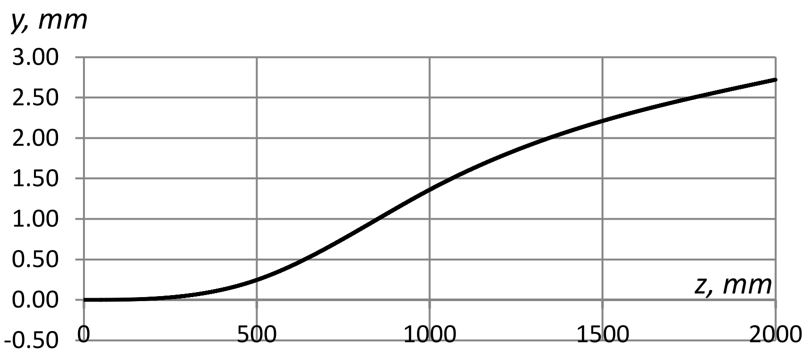

Fig. 8 Elastic curve of the beam in case of $y'(0)=0$

$$y'(0) = -\frac{y(l)}{l} = -\frac{2.72213\,mm}{2000\,mm} = -1.361065 \cdot 10^{-3}\,rad. \tag{19}$$

For initial values $y(0)=0$ and $y'(0)=-1.36106\ 10^{-3}$, we obtain $y_B=y(l/2)=-1.08454\ 10^{-6}$ mm and $y_C=y(l)=-\ 2.08936\ 10^{-6}$mm. Boundary conditions are fully satisfied.

4.2 Procedure 2

Let us introduce the linearization into (15)-(17), and determine the functions and their first derivatives

$$y'_1 = -\frac{13}{64}\frac{F}{EI}z^2 + C_1, \quad y_1 = -\frac{13}{192}\frac{F}{EI}z^3 + C_1 z + C_2,$$
$$y'_2 = \frac{F}{192\,EI}(57z^2 - 48lz) + C_3, \quad y_2 = \frac{F}{192\,EI}\left(57\frac{z^3}{3} - 24lz^2\right) + C_3 z + C_4,$$
$$y'_3 = \frac{9F}{192\,EI}(-z^2 + 2lz) + C_5, \quad y_3 = \frac{9F}{192\,EI}\left(-\frac{z^3}{3} + lz^2\right) + C_5 z + C_6\,. \tag{20}$$

According to (20) and boundary conditions (18), six unknown constants are determined

$$C_1 = \frac{3}{256}\frac{Fl^2}{IE}, \ \ C_2 = 0, \ C_3 = \frac{11}{256}\frac{Fl^2}{IE}, \ C_4 = -\frac{1}{384}\frac{Fl^3}{IE}, \ C_5 = -\frac{11}{256}\frac{Fl^2}{IE}, \ C_6 = \frac{3}{256}\frac{Fl^3}{IE}. \tag{21}$$

Substituting (21) into (20), equations of the elastic curve are obtained:

$$y_1 = -\frac{1}{768}\frac{F}{EI}(52z^3 - 9l^2 z),$$
$$y_2 = \frac{1}{768}\frac{F}{EI}(76z^3 - 96lz^2 + 33l^2 z - 2l^3),$$
$$y_3 = \frac{1}{256}\frac{F}{EI}(-4z^3 + 12lz^2 - 11l^2 z + 3l^3). \tag{22}$$

In this case the initial slope is $y'(0)=y'_1(0)=C_1$. Substituting numerical data the initial slope is

$$y'_{an}(0) = C_1 = \frac{3}{256}\frac{Fl^2}{IE} = -1.3610627 \cdot 10^{-3}\,rad. \tag{23}$$

Comparing initial slopes (19) and (23), it is evident that the difference between can be omitted.

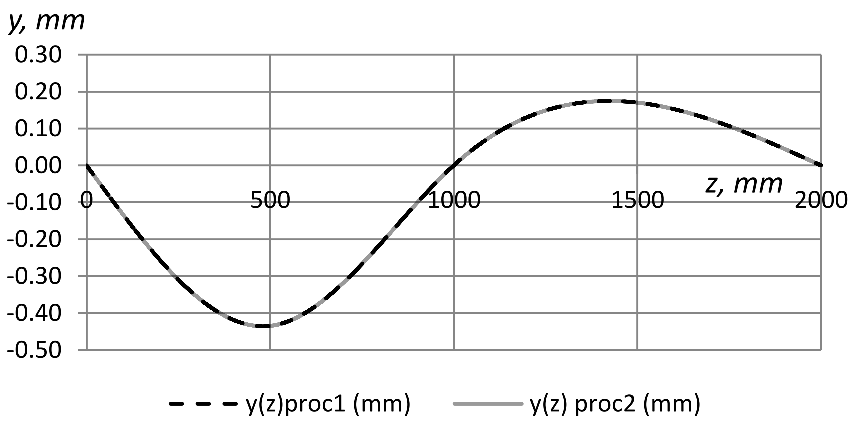

Fig. 9 Elastic curves of the beam with three supports obtained by Procedure 1 and Procedure 2

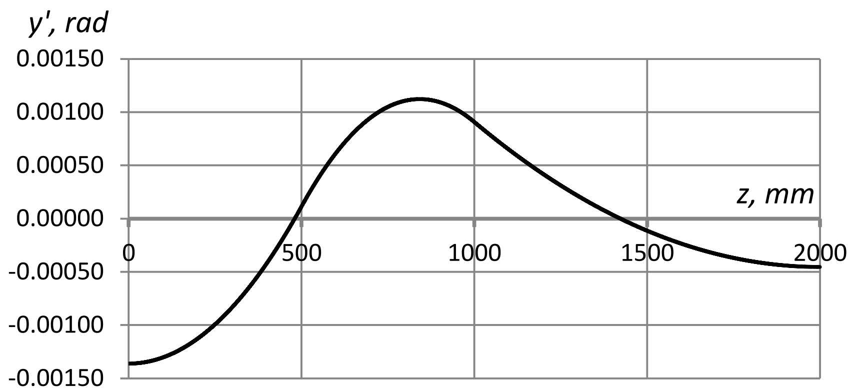

Fig. 10 Slope function of the beam with three supports

Elastic curves determined by both procedures are almost equal (see Fig. 9). In Fig. 10 the slope function of the beam is plotted.

5. Conclusions

Followings can be concluded:

• Numerical calculation of the elastic curve of the simply and multiple supported beam requires the knowledge of the value of the slope in one end-point. Namely, the boundary problem is modified into the initial problem whose calculation is relatively simple and stable.

• Procedures for the initial guess of the slope at the support, given in this short communication, are deterministic. Initial guess does not depend on the knowledge and wit of the person who is calculating. Procedures are strongly defined and repeatable.

• Difference between initial guesses obtained with these two methods is negligible. Elastic curves obtained with both procedures are almost equal. Elastic curve satisfies the boundary

conditions with very high accuracy.
• Both procedures are suitable for practical use due to their simplicity.

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