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# Direct implementation of stochastic linearization for SDOF systems with general hysteresis

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**Abstract.** The first and second moments of response variables for SDOF systems with hysteretic nonlinearity are obtained by a direct linearization procedure. This adaptation in the implementation of well-known statistical linearization methods, provides concise, model-independent linearization coefficients that are well-suited for numerical solution. The method may be applied to systems which incorporate any hysteresis model governed by a differential constitutive equation, and may be used for zero or non-zero mean random vibration. The implementation eliminates the effort of analytically deriving specific linearization coefficients for new hysteresis models. In doing so, the procedure of stochastic analysis is made independent from the task of physical modeling of hysteretic systems. In this study, systems with three different hysteresis models are analyzed under various zero and non-zero mean Gaussian White noise inputs. Results are shown to be in agreement with previous linearization studies and Monte Carlo Simulation.

**Key words:** linearization; hysteresis; asymmetric models; random vibrations.

## 1. Introduction

The analysis of structural members under intense random (i.e., earthquake) excitation, requires the incorporation of nonlinear stress-strain relationships into solution methods for random vibration. Exact solutions for the response statistics of nonlinear systems are very limited, particularly when the material behavior is hysteretic, having a multi-valued force-deformation pattern with nonconservative energy dissipation. The lack of closed form solutions for hysteretic random oscillators necessitates the use of accurate methods for approximate solution.

Since the first random vibration study of an inelastic system, performed by Caughey (1960), many researchers have applied various approximation methods to nonlinear stochastic analysis. These include amongst others, moment closure techniques (Iyengar and Dash 1978, Crandall 1980, Noori and Davoodi 1990), stochastic averaging (Roberts 1987, Roberts and Spanos 1986), equivalent nonlinear systems (Nielsen *et al.* 1990), and an energy dissipation balancing procedure by Cai and Lin (1990). While these methods offer alternative means for the analysis of hysteretic or generally nonlinear systems, the most widely published of

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approximation methods has been that of Equivalent Linearization. Formulations of the linearization procedure can be found in Spanos (1981), Iwan (1973) and Caughey (1963). Application of linearization for multi-degree of freedom hysteretic structural simulation, include work by Pradlwater and Schuëler (1992), while the majority of researchers working in this field have utilized linearization for the study of SDOF hysteretic systems.

Some of the more popular hysteresis models for SDOF systems follow a rate-type format, where the governing equations are as follows:

$$\ddot{u} + 2\zeta\omega\dot{u} + \alpha\omega^2 u + (1-\alpha)\omega^2 z = f(t) \tag{1}$$

$$\dot{z} = g(\dot{u}, z) \tag{2}$$

where $u(t)$ is system displacement, $z(t)$ is proportional to the hysteretic restoring force, $\zeta$ and $\omega$ are damping and natural frequency parameters, respectively; $\alpha$ is a ratio of post-yield to pre-yield stiffness, and $g(\dot{u}, z)$ is a general nonlinear constitutive equation that defines the hysteretic behavior.

Equivalent Linearization has been used to analyze the system of Eqs. (1) and (2) using several different mathematical models for hysteresis (Wen 1980, Baber and Wen 1981, Baber 1984, Baber and Noori 1985, 1986, and Foliente 1993), with approximate solutions comparing favorably to results obtained by pure Monte Carlo Simulation. While the procedure of linearization for each of these models has been identical, following the format presented by Atalik and Utku (1976), each unique constitutive equation has required a unique analytical derivation for the equivalent linear system parameters. Based upon examination of the derivations associated with these models, it is reasonable to conclude that as hysteresis models have evolved into more sophisticated analytical forms to describe progressively more generalized stress-strain behavior, the analytical effort of linearization has become increasingly complex and tedious.

The focus of this research is to present a procedure for obtaining the response statistics (first and second moments) of the SDOF system, that is generally applicable to any rate-type constitutive model for the hysteretic nonlinearity $g(\dot{u}, z)$. The expressions for equivalent linear system parameters will be posed in a form that is convenient for numerical solution, so as to allow for a fully automated linearization process, to be accomplished by computer code.

## 2. Models for hysteresis

Statistical linearization shall be performed upon three different rate-type analytical models for hysteresis. Bouc (1967) introduced a model in the differential format of Eq. (2), which has since been expanded by Wen (1976), Baber and Wen (1981) and Baber and Noori (1985) to include material degradation and loop pinching properties. The original form of the Bouc-Wen model is given by:

$$\dot{z} = A\dot{u} - \beta|\dot{u}||z|^{n-1}z - \gamma\dot{u}|z|^n \tag{3}$$

where $A$, $\beta$, $\gamma$ and $n$ are parameters which regulate the shape of the hysteresis loop. Wen (1976) showed the versatility of this model by altering the values of the shape parameters. An example of a Bouc-Wen hysteresis loop pattern is shown in Fig. 1.

A second hysteresis model is presented by Baber and Noori (1986), and offers stiffness reduction in the unloading regimes of the stress-strain cycle. The unloading stiffness reduction

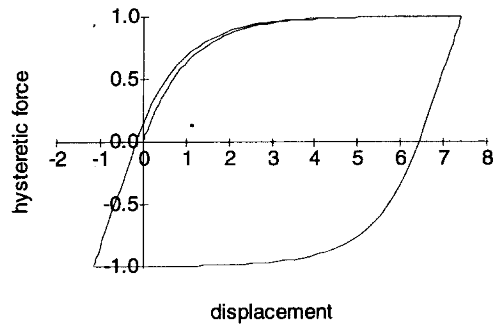


Fig. 1 Bouc-Wen hysteresis

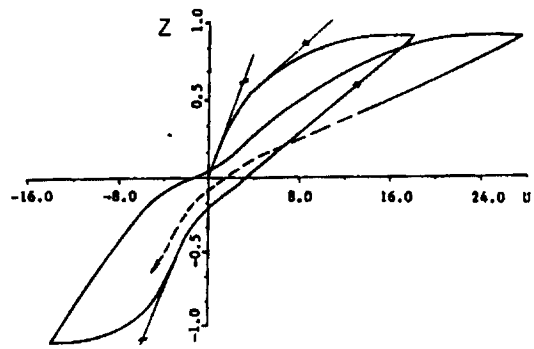


Fig. 2 USR model hysteresis (Barber and Noori 1986)

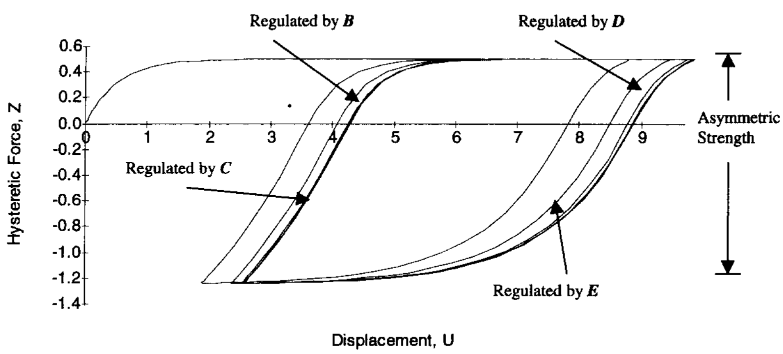


Fig. 3 Asymmetric hysteresis from general asymmetric model with $A$=1.0, $B$= − 0.5, $C$=0.1, $D$= − 0.4 and $E$=0.2

(USR) model is given by the following differential equation:

$$\dot{z}=(\frac{A}{4})\cdot\left\{\begin{array}{l}\left[(.5+\frac{1}{\pi}\arctan(\frac{z-z_+}{z_{ch}}))\cdot(1-(\frac{z}{z_{ult}})^n)+(.5-\frac{1}{\pi}\arctan(\frac{z-z_+}{z_{ch}})\cdot\mu\right]\cdot(\dot{u}+|\dot{u}|)\\ +\left[(.5+\frac{1}{\pi}\arctan(\frac{z-z_-}{z_{ch}}))\cdot\mu+(.5-\frac{1}{\pi}\arctan(\frac{z-z_+}{z_{ch}}))\cdot(1+\left|\frac{z}{z_{ult}}\right|^{n-1}(\frac{z}{z_{ult}}))\right]\cdot(\dot{u}-|\dot{u}|)\end{array}\right\} \quad (4)$$

Like the previous model, a variety of loop patterns may be generated by adjusting the shape parameters $A$, $z_-$, $z_+$, $z_{ch}$, $z_{ult}$ and $\mu$. Furthermore, the USR model is capable of generating asymmetric hysteresis patterns. An example of USR hysteresis is shown in Fig. 2.

Finally, a recently proposed model (Dobson 1984), is given by the following:

$$\begin{aligned}\dot{z} =& A\dot{u} + B(\dot{u} + |\dot{u}|)(z + |\dot{z}|)^n + C(\dot{u} + |\dot{u}|)(z - |\dot{z}|)^n \\ &+ D(\dot{u} - |\dot{u}|)(z + |\dot{z}|)^n + E(\dot{u} - |\dot{u}|)(z - |\dot{z}|)^n\end{aligned} \tag{5}$$

Similar to the BW and USR models, the above defines the hysteretic pattern in discrete "branches". Also similar to piecewise-linear models (Zaiming *et al.* 1991), the functions containing absolute values act as switches; all but one of which vanish depending upon the signs of the state variables. The advantage of this particular form is that the hysteretic pattern can be "custom built", since each of the shape parameters B, C, D and E uniquely regulates one portion of the loading cycle, and the entire loop pattern is specified in one concise differential equation. Depending upon the signs of these parameters, linear, softening or hardening curves maybe generated independently, to form the complete cycle. By making these parameters time or system dependent, the loop pattern may become dynamic, to account for material degradation.

The General Asymmetric Model offers a broad range of loop patterns, and is particularly well-suited for systems with dramatic asymmetry. Such a system under sinusoidal loading is shown in Fig. 3.

## 3. Statistical equivalent linearization

The method of equivalent linearization has been proven effective for providing first and second moments of response for rate-type hysteretic systems (Barber 1984, 1985, 1986, 1981, Dobson 1984, Foliente 1993, Park *et al.* 1986, Wen 1980). The procedure calls for writing the system of Eqs. (1) and (2) in state space format; letting $q_1=u$, $q_2=\dot{u}$ and $q_3=z$ be non-zero mean random variables, then from Eqs. (1) and (2):

$$\dot{q}_1 = q_2 \tag{6a}$$

$$\dot{q}_2 = -\alpha\omega^2 q_1 - 2\zeta\omega q_2 - (1-\alpha)\omega^2 q_3 + f(t) \tag{6b}$$

$$\dot{q}_3 = g(q_2, q_3) \tag{6c}$$

Taking the expected value of each equation above, with $\mu_i=E(q_i)$ and $\mu_f=E(f(t))$:

$$\dot{\mu}_1 = \mu_2 \tag{7a}$$

$$\dot{\mu}_2 = -\alpha\omega^2\mu_1 - 2\zeta\omega\mu_2 - (1-\alpha)\,\omega^2\mu_3 + \mu_f \tag{7b}$$

$$\dot{\mu}_3 = E(g(q_2, q_3)) \tag{7c}$$

Subtracting Eqs. (7) from (6) yields zero-mean random variables, $y_i=q_i-\mu_i$ and $F=(f-\mu_f)$ such that:

$$\dot{y}_1 = y_2 \tag{8a}$$

$$\dot{y}_2 = -\alpha\omega^2 y_1 - 2\zeta\omega y_2 - (1-\alpha)\,\omega^2 y_3 + F \tag{8b}$$

$$\dot{y}_3 = g(q_2, q_3) - E(g) \tag{8c}$$

The nonlinear Eq. (8) involving the constitutive hysteresis model $g(q_2, q_3)$ maybe replaced with the linear form:

$$\dot{y}_3 = C_e y_2 + K_e y_3 \tag{9}$$

where $C_e$ and $K_e$ are unknown equivalent damping and stiffness parameters, respectively. This substitution allows for the generation of a linear system of ODE's whose solution provides response covariances,

$$S_{ij} \equiv E(y_i y_j):\ \{S\} \equiv \begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \\ S_5 \\ S_6 \end{bmatrix} \equiv \begin{bmatrix} S_{11} \\ S_{12} \\ S_{13} \\ S_{22} \\ S_{23} \\ S_{33} \end{bmatrix},\ \begin{bmatrix} \dot{S}_{11} \\ \dot{S}_{12} \\ \dot{S}_{13} \\ \dot{S}_{22} \\ \dot{S}_{23} \\ \dot{S}_{33} \end{bmatrix} \tag{10}$$

$$= \begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 \\ -\alpha\omega^2 & -2\zeta\omega & -(1-\alpha)\omega^2 & 1 & 0 & 0 \\ 0 & C_e & K_e & 0 & 1 & 0 \\ 0 & -2\alpha\omega^2 & 0 & -4\zeta\omega & -2(1-\alpha)\omega^2 & 0 \\ 0 & 0 & -\alpha\omega^2 & C_e & K_e - 2\zeta\omega & -(1-\alpha)\omega^2 \\ 0 & 0 & 0 & 0 & 2C_e & 2K_e \end{bmatrix} \cdot \{S\} + \begin{bmatrix} 0 \\ E(y_1F) \\ 0 \\ 2E(y_2F) \\ E(y_3F) \\ 0 \end{bmatrix}$$

Excitation of Gaussian White Noise simplifies $E(y_1F)=E(y_3F)=0$, $E(y_2F)=\pi W_0$, where $W_0$ is power spectral density. Together, Eqs. (7) and (10) govern the means and covariances of system response. They may be solved simultaneously if a suitable choice for $C_e$ and $K_e$ can be made.

For closure, $C_e$ and $K_e$ are chosen to minimize the error of the linear approximation in Eq. (9), in a statistical sense. Following the results of Atalik and Utku (1976) under certain conditions, including the assumption that the state variables of velocity and hysteretic force are jointly Gaussian, the choice which minimizes error is given by the following:

$$C_e = E\left[\frac{\partial g(\dot{u}, z)}{\partial \dot{u}}\right] \tag{11}$$

$$K_e = E\left[\frac{\partial g(\dot{u}, z)}{\partial z}\right] \tag{12}$$

These equations for equivalent stiffness and damping coefficients are obtained from an expression that is applicable to MDOF systems with general (not necessarily hysteretic) nonlinearity. It should be noted that the values of these coefficients are system dependent, and must be continually updated as the system of ODE's for means and covariances is solved numerically.

The Bouc-Wen model of Eq. (3) and its various extensions have been studied under Gaussian White Noise input using the procedure outlined above (Baber 1984, 1985, 1986,

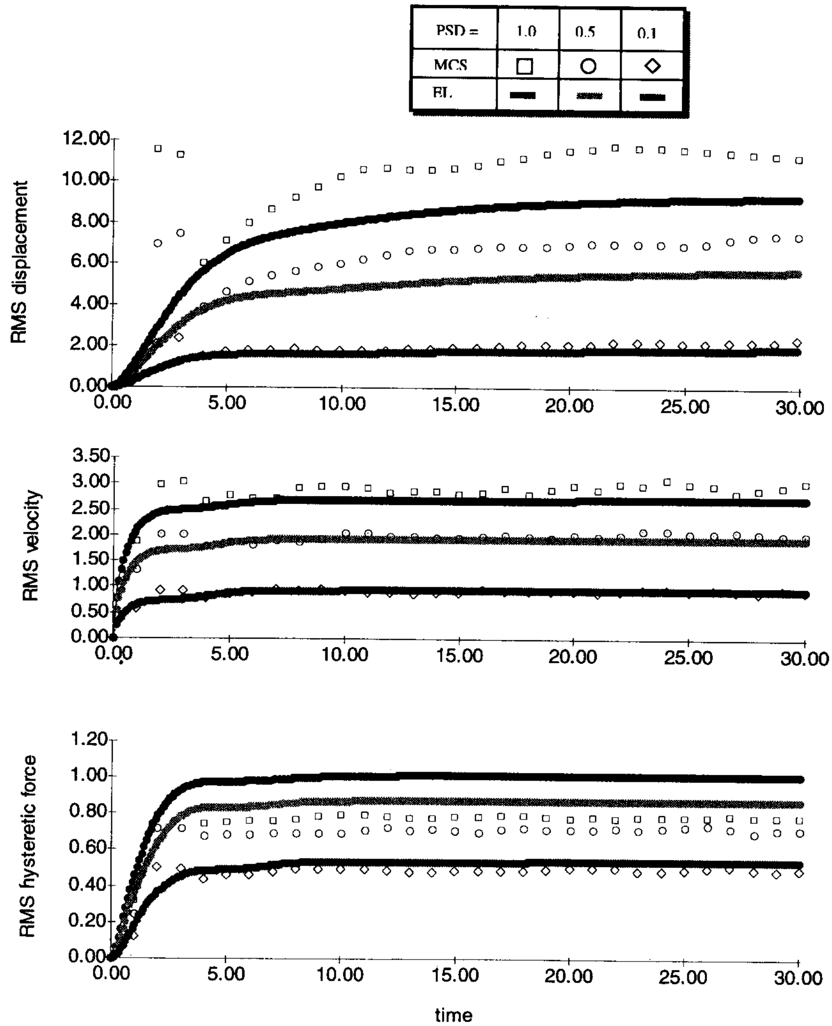


Fig. 4 Zero-mean analysis of USR model using previous linearization

1981, Wen 1976). It must be emphasized however, that any changes introduced to any hysteretic model, which alter the partial derivatives contained in Eqs. (11) and (12), requires analytical effort to re-derive new expected value expressions. For the USR model of Eq. (4), the derivation of equivalent linear coefficients, using the approach of Eqs. (11) and (12), is presented by Dobson *et al.* (1997). The process is tedious and time consuming, even under the simplification of zero-mean random variables. Taking the partial derivatives of the nonlinear model expands the function into twenty-four different additive sub-functions for equivalent damping, and forty additional for stiffness. The expected value of each of these functions must be evaluated at each time step of a numerical routine which solves the system response statistics.

The result of this procedure is shown in Fig. 4, as the RMS responses of displacement, velocity and hysteretic force under three levels of Power Spectral Density, are shown and compared to results of approximately 700 samples of Monte Carlo Simulation. Parameter values for the system are as follows: $\zeta$=0.10, $\omega$=1.0, $\alpha$=0.05, $A$=1, $n$=1, $\mu$=1, $z_+=-z_-=0.1$, $z_{ult}=1$, $z_{ch}=0.005$. Overall results are good, especially at low levels of PSD, as would be expected. However, the linearization progressively underestimates displacement and overestimates hysteretic force as excitation intensity increases.

## 4. A different implementation of stochastic linearization

The hysteretic system of Eqs. (1) and (2) is an example of a system in which there is only one state variable that is governed by a nonlinear differential equation. This is an advantage to a system with multiple nonlinearities; and leads to the development of linearization coefficients that differs from the definitions given in Eqs. (11) and (12). Another issue which contributes to this development is the fact that although a linear system is used as an approximation to the equations governing means and covariances, the response dependence of this linear system prevents its analytical solution. This system dependence of the linearization coefficients influences their new definition.

Consider the error of linearization; the difference between the nonlinear system and the approximation of Eq. (9):

$$\{e\} = \begin{Bmatrix} 0 \\ 0 \\ g - E(g) - C_e y_2 - K_e y_3 \end{Bmatrix} \tag{13}$$

Only the third variable is nonzero, therefore, the covariances of the error terms are all zero, except for the term:

$$E(ee^T) = E[(g - E(g) - C_e y_2 - K_e y_3)^2] \tag{14}$$

Since $C_e$ and $K_e$ are response dependent, and require continual re-evaluation in the numerical solution of Eq. (10), they may be treated as constants at each discrete time step, and brought outside the expected value operator. Taking partial derivatives of Eq. (14) with respect to $C_e$ and $K_e$, and equating to zero, leads to the following:

$$\frac{\partial e^2}{\partial C_e} = -2E(gy_2) + 2E(y_2 E(g)) + 2K_e S_5 + 2C_e S_4 = 0 \tag{15}$$

$$\frac{\partial e^2}{\partial K_e} = -2E(gy_3) + 2E(y_3 E(g)) + 2K_e S_6 + 2C_e S_5 = 0 \tag{16}$$

If $E(g)$ is assumed to be known at each time step, then $E(y_2E(g))=E(y_2)E(g)=0=E(y_3E(g))$, since $y_2$ and $y_3$ are zero-mean random variables. Also noting that $E(gy_2)=E(gq_2)-\mu_2E(g)$, $E(gy_3)=E(gq_3)-\mu_3E(g)$, leads to the solution of Eqs. (15) and (16), and the specific definition of the linearization coefficients:

$$C_e = \frac{S_6 E(g\dot{u}) - S_5 E(gz) - S_6 E(g)\mu_2 + S_5 E(g)\mu_3}{S_4 S_6 - S_5^2} \tag{17}$$

$$K_e = \frac{S_4 E(gz) - S_5 E(g\dot{u}) - S_4 E(g)\mu_3 + S_5 E(g)\mu_2}{S_4 S_6 - S_5^2} \tag{18}$$

In essence, both the classical and currently proposed method of linearization are founded upon the same principle, the mean-square minimization of expected error. The difference is in the way the process is implemented, with the current method being better suited for computational methods. The advantage to the above definition over that of Eqs. (11) and (12) is that it is posed directly in terms of the desired response statistics and only three expected values, which also do not contain partial derivatives of the hysteresis model. With the excitation being Gaussian white noise, $\dot{u}$ and $z$ can be assumed to have a jointly Gaussian probability density function, thereby allowing the evaluation of the three unknown expected values $E(g)$, $E(g\dot{u})$ and $E(gz)$, and closure of the system. Using the mean and covariance response solutions of Eqs. (7) and (10), the expected values may be obtained through Gauss-quadrature integration. Whatever the choice of hysteretic model, Eqs. (7), (10), (17) and (18) remain unchanged, and the effort of linearization is reduced to simply changing the function of $g(\dot{u}, z)$ in the three critical expected values. Often, this redefinition, or introduction of a new

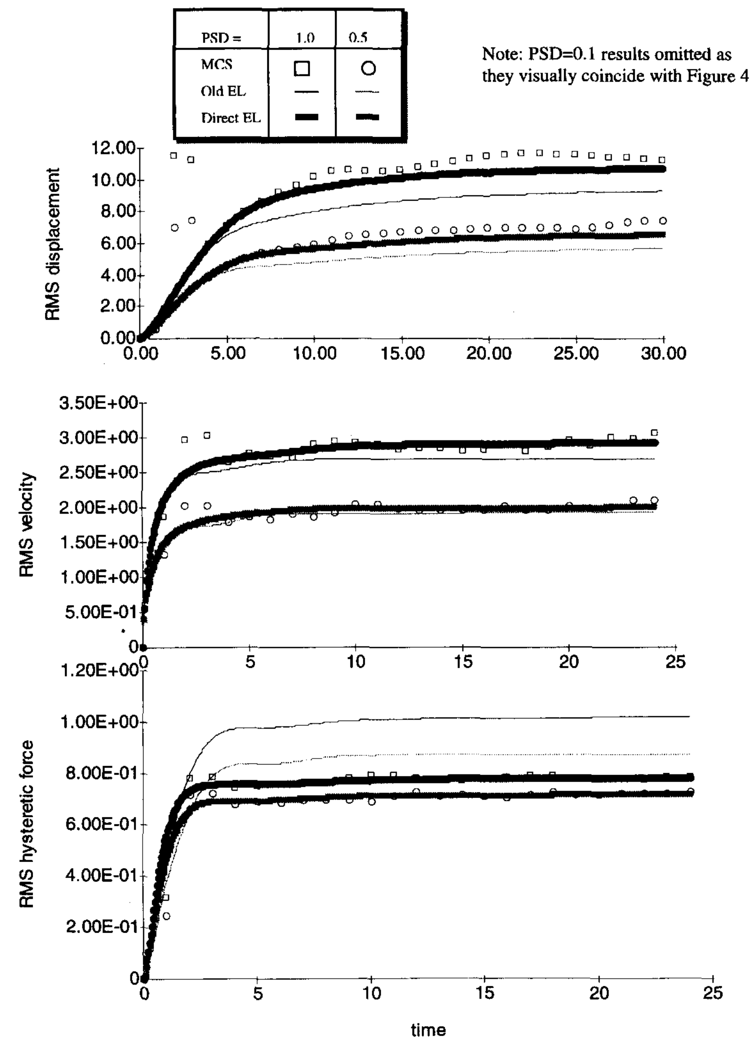


Fig. 5 Comparison of linearization results for USR model

model, requires the altering of only one line of computer code, and most importantly, completely avoids the type of analytic effort previously associated with equivalent linearization.

## 5. Numerical results

A computer program written in BASIC solves the system of Eqs. (7) and (10) with a fourth-order Runge-Kutta subroutine, and also continually updates the linearization coefficients defined by Eqs. (17) and (18). The first system studied, incorporates the USR model of Eq. (4) under zero-mean excitation, and is a comparison between the results provided by each implementation of linearization. Fig. 5 transposes the time histories of RMS displacement, velocity and hysteretic restoring force found by the current method, with the results of Fig. 4. The newly proposed implementation shows a more accurate result when judged by MCS. The reason for this, may be due to the reduced number of numerically evaluated expected values, and additive truncation error. There is a distinct improvement in the results for displacement

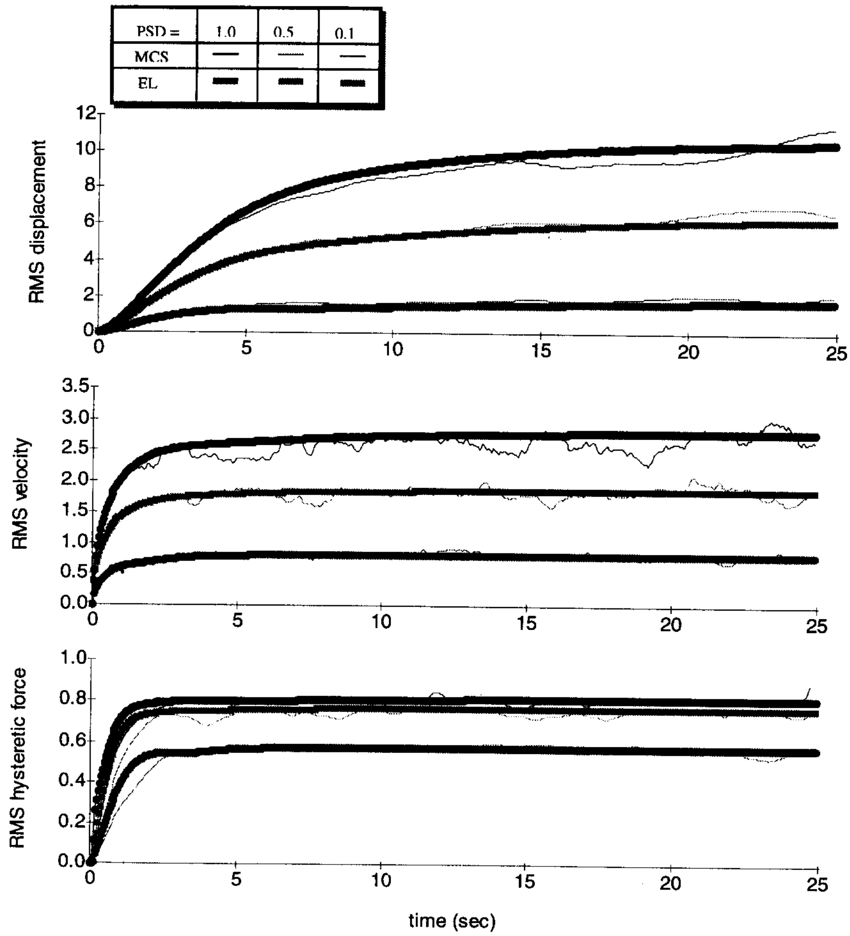


Fig. 6 Zero-mean analysis of general asymmetric model

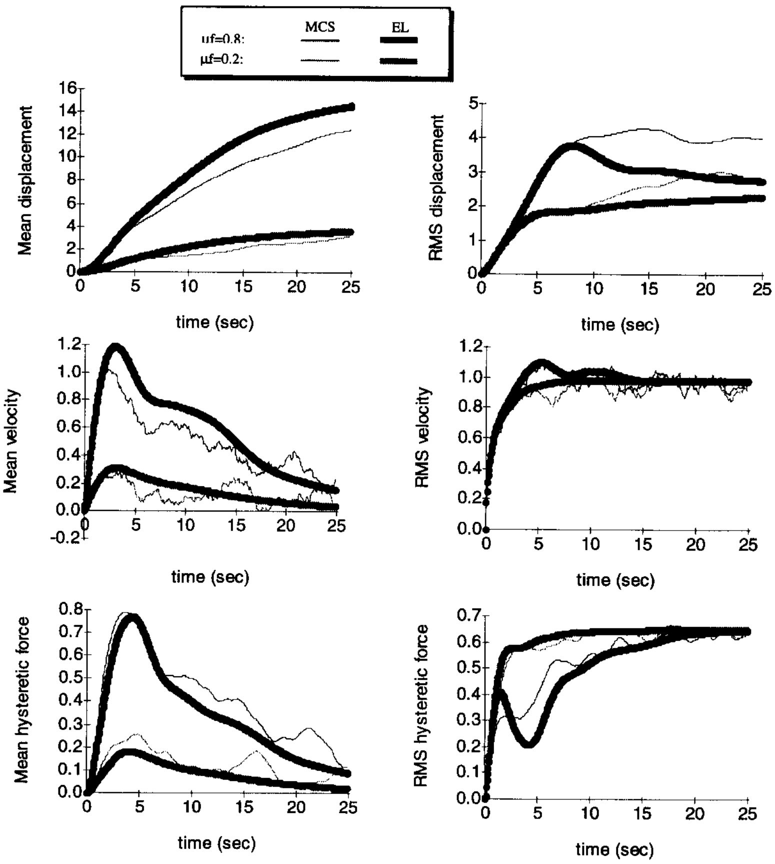

Fig. 7 Nonzero-mean analysis of Bouc-Wen model

and hysteretic force, as the new linearization nearly coincides with purely simulated response.

The next study conducted with the new implementation of linearization was a zero-mean analysis with the General Asymmetric model of Eq. (5). Parameter values defining the system are given by: $\zeta$=0.10, $\omega$=1.0, $\alpha$=0.05, $A$=1, $n$=1, $B$=-.25, $C$=0, $D$=0, $E$=0.25. Fig. 6 shows the time histories of RMS responses for each system variable, and compares the linearization results to 200 samples of MCS. Again, there is excellent agreement between linearization and simulation, even at the most intense excitation.

The last analytical model to be studied using the direct linearization approach is the original Bouc-Wen model. The system is under nonzero-mean excitation and is defined by the following: $\zeta$=0.01, $\omega$=1.0, $\alpha$=0.05, $A$=1, $n$=1, $\beta$=$\gamma$=0.5. Mean excitation values of $f$=0.2 and $f$=0.8 are utilized, and the response statistics are compared to MCS. Fig. 7 presents the time profiles of mean and RMS state variables along with the result of 200 samples of MCS. There is excellent agreement between this linearization and MCS, and furthermore, the reader may verify the agreement between the current results and the previous work of Baber (1984), who utilized classical linearization for the same analysis.

## 6. Conclusions

A procedure has been presented for obtaining first and second moments of response variables for a SDOF system incorporating a general hysteretic nonlinearity. The method is an adaptation of classical linearization procedures, and eliminates the effort of analytical derivations associated with each hysteresis model. Case studies using three different forms of rate-type hysteresis models, have shown the method to be accurate when judged by Monte Carlo results, and possibly more accurate then classical linearization routines, due to reduced numerical error. It is possible that with the analytical effort removed from the task of random vibration analysis, this generally applicable procedure may promote the introduction of more sophisticated hysteresis models for simulation of real-world systems.

Future work involving this linearization method is focused upon extensions to MDOF systems, by the direct minimization of the variance of approximation error for each DOF; and extension to Non-Gaussian response, as the assumption of Gaussian state variables was not mandated in the derivation of equivalent linear coefficients.

## Acknowledgements

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