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A finite element yield line model for the analysis of reinforced concrete plates

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Abstract. This paper concerns the development and implementation of an orthotropic, stress resultant elasto-plastic finite element model for the collapse load analysis of reinforced concrete plates. The model implements yield line plasticity theory for reinforced concrete. The behaviour of the yield functions are studied, and modifications introduced to ensure a robust finite element model of cases involving bending and twisting stress resultants (M_x, M_y, M_{xy}). Onset of plasticity is always governed by the general yield-line-model (YLM), but in some cases a switch to the stress resultant form of the von Mises function is used to ensure the proper evolution of plastic strains. Case studies are presented, involving isotropic and orthotropic plates, to assess the behaviour of the yield line approach. The YLM function is shown to perform extremely well, in predicting both the collapse loads and failure mechanisms.

Key words: yield line model; orthotropic plasticity; stress-resultant yield function; reinforced concrete plates; plastic node method.

1. Introduction

Most finite element studies of reinforced concrete plates have concentrated on the effects of cracking on nonlinear response e.g., Hand *et al.* (1973), Lin and Scordelis (1975) and Gilbert and Warner (1978). Finite element analyses of plastic collapse are less prominent, even though most concrete structures are ductile because ultimate limit states design provisions are plasticity based. The reasons relate to the difficulty in the implementation of a general orthotropic yield function capable of predicting the correct collapse loads and kinematics in reinforced concrete, and ironically to the success of yield line method of analysis.

In that regard, Munro and da Fonseca (1978) presented a plastic limit analysis of plates through a direct implementation of the yield line method within a finite element discretisation. They allowed hinges to form along (fixed) element boundaries, and used linear programming to find the optimal yield line layout; if actual yield lines do not correspond to chosen mesh lines, the collapse load is an upper bound. Hence, without sophisticated remeshing as part of the optimisation process, the method cannot automatically capture yield line patterns in

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complex cases, and does not predict the evolution of plasticity, as would be expected from an elasto-plastic analysis.

There have been many studies of continuum plasticity in concrete, where in fact high order theories are now available (de Borst and Mühlhaus 1992). For plate and shell problems, most analyses use the layered approach, originally developed for reinforced concrete by Hand *et al.* (1973) and Lin and Scordelis (1975), and adopted for general anisotropic plasticity by e.g., Owen and Figuerias (1983) and de Borst and Feenstra (1990). In that approach, biaxial stress conditions apply in each of the layers and the stress resultants are formed by integration across the plate thickness. Given that most plasticity models are von Mises based, the analysis of orthotropic plates requires specific consideration of the layers to simulate a reinforced section where the reinforcement directions dominate.

The substance of this paper is to present an elasto-plastic finite element model for the collapse load analysis of reinforced concrete, using *orthotropic* yield functions written directly in terms of the *stress-resultants*. An important consideration in developing this model is that plasticity in reinforced concrete is strongly influenced by the reinforcement directions, and does not exhibit the same shear dominated response as a traditional elasto-plastic material, like mild steel for example.

For reinforced concrete panels/plates subjected to in-plane forces only, or bending and twisting moment only, closed form yield criteria which embody yield line theory have been derived by several investigators, among these, Baker (1991), Bræstrup (1970), Nielsen (1984) and Morley (1966); Rajendam and Morley (1974) also confirmed these yield criteria experimentally. The objective of this paper is to present the implementation of these 'stress resultant' yield conditions, that are known to be particularly suited to the analysis of reinforced concrete, into a continuum plasticity finite element model. The behaviour of the yield functions are studied, and modifications required to generate a robust model are discussed. A variety of slabs are analysed and the results are compared with known solutions and von Mises analyses to demonstrate the accuracy and relevance of the finite element yield line model to limit analysis of reinforced concrete plates. Nothwithstanding the value of an automated (finite element) limit analysis of RC slabs, this work was also motivated by the development of a general yield model for plates, shells and box structures, and that work will be presented in a sequel.

Finite element implementations of elasto-plastic analysis generally employ the standard continuum concept, where plastic deformations are assumed to spread over a finite element. Ueda and Tetsuya (1982) developed an alternative approach, named the *Plastic Node Method*, in which it is assumed that the plastic deformations are concentrated in the nodes of the finite element, and the remainder of the element is elastic. Ueda and Fujikubo (1991) generalized the theory, so that the checking points for plasticity are not restricted to the element nodes, but can be any set of points within the element or on the element boundary. Once checking points become plastic, plastic deformations are accumulated in the element nodes, and affect the change of strains and stresses in the checking point for plasticity. Ueda and Fujikubo (1991) have shown that hinges, hinge lines (yield lines) or plastic strip lines can be developed with select elements. The method is thus advantageous when studying the plastic collapse of reinforced concrete plates, since the failure mechanisms exhibit these yield lines patterns. The aim of this work has been to develop a model able to predict the ultimate capacity, spread of plasticity (evolution of plasticity) and failure mechanisms. Thus, serviceability effects such as cracking, tension stiffening and loss of bond, have not been implemented in this model.

2. Plastic node method with DKL elements

An elasto-plastic computational algorithm consists of a numerical iterative method for solving the nonlinear equations, and a numerical algorithm for the state determination of the material relations; only material nonlinearity is considered. The state dermination algorithm solves the problem to obtain an increment of force for a given increment of deformation using the constitutive equations. The elasto-plastic framework adopted is a basic initial stress, modified Newton Raphson solution, using the 'plastic node' method of analysis (Ueda and Fujikubo 1991) on a macro 'discrete Kirchoff Loof' (DKL) element (Ristic *et al.* 1993).

Given an incremental approach, the elastic nodal displacement increment for one element $\Delta\{u^e\}$ is related to the element nodal force increment $\Delta\{P\}$ by

$$\Delta\{P\} = [K^e]\ \Delta\{u^e\} \tag{1}$$

where $[K^e]$ is the elastic stiffness matrix. Plastic yielding of an element occurs when the stress state in the i-th checking point reaches the yield surface, and irreversible plastic deformations occur. In Ueda and Fujikubo (1991) it has been shown that the stress vector can be represented as a function of nodal force vectors at j nodes in the element, with $j \leq n$, where n is the total number of nodes in the element. The number j depends on the displacement function assumed in the element. The plasticity condition for the i-th checking point can now be written as

$$F_i(P_1, P_2, \cdots, P_j) = 0, \quad j \leq n \tag{2}$$

Analogous to the continuum strain decomposition, the total nodal displacement increment $\Delta\{u\}$ can be expressed as the sum of elastic and plastic parts as

$$\Delta\{u\} = \Delta\{u^e\} + \Delta\{u^p\} \tag{3}$$

If yielding occurs at the i-th checking point, the plastic nodal displacement increment can according to the associative flow rule be written as

$$\Delta\{u^p\} = \Delta\lambda_i\ \{\phi_i\} \tag{4}$$

where $\Delta\lambda_i$ is a positive scalar (plastic multiplier) and for n nodes in the element

$$\{\phi_i\} = \left\{\frac{\partial F_i}{\partial\{P\}}\right\} = \left\{\frac{\partial F_i}{\partial\{P_1\}}, \frac{\partial F_i}{\partial\{P_2\}}, \cdots, \frac{\partial F_i}{\partial\{P_n\}}\right\} \tag{5}$$

and the plastic nodal displacement increment is normal to the yield surface. If k checking points in one element yield, Eq. (4) can be rewritten as

$$\Delta\{u^p\} = \sum_{i=1}^{k} \Delta\lambda_i\ \{\phi_i\} = [\Phi]\ \Delta\{\lambda\} \tag{6}$$

where $[\Phi]=[\phi_1, \phi_2, \cdots, \phi_k]$ and $\Delta\{\lambda\}=\{\Delta\lambda_1, \Delta\lambda_2, \cdots, \Delta\lambda_k\}$.

In the context of the plastic node method, the consistency conditions can now be written as

$$[\Phi]\ \{P\} = 0 \tag{7}$$

Rearranging (3, 4, 6, 7), the plastic multiplier can be found as a function of the total nodal displacement increment

$$\Delta\{\lambda\} = ([\Phi]^T \ [K^e] \ [\Phi])^{-1} \ [\Phi]^T \ [K^e] \ \Delta\{u\} \tag{8}$$

Since plastic work according to the Kuhn-Tucker condition is positive, the plastic multiplier must be positive if the node is under loading. Finally, the nodal force increment in terms of imposed nodal displacement increment can be found as

$$\Delta\{P\} = [K^{ep}] \ \Delta\{u\} \tag{9}$$

where the elasto plastic stiffness matrix $[K^{ep}]$ is given by

$$[K^{ep}] = [K^e] - \frac{[K^e][\Phi][\Phi]^T [K^e]}{[\Phi]^T [K^e][\Phi]} \tag{10}$$

Eq. (10) shows that in the determination of the elasto-plastic stiffness matrix, the plastic node method does not involve numerical integration, in contrast to the classical approach. This procedure for determining the elasto-plastic constitutive relations can be used with the modified Newton-Raphson method, and results in the widely used 'initial stress method' developed by Zienkiewicz *et al.* (1969). The return mapping algorithm always brings the resulting stress back to the yield surface, and if unloading occurs the process is elastic.

Since the authors' initial interest was the analysis of plastic behaviour of box girders, a flat shell element has been adopted. A triangular flat faceted shell element which utilizes the linear strain triangle (LST) for membrane representation and a Discrete Kirchoff plate-bending element with Loof Nodes (DKL) has been implemented. This shell element was developed by Meek & Tan (1986). Since the element is flat, a plane stress analysis is used with in-plane displacements and a bending analysis with deflections and rotations. Although this element has shown to be accurate and convenient, in applications to flat folded structures many degrees of freedom are required. To reduce the number of degrees of freedom, four triangular shell elements can be (statically) condensed into one rectangular macro-element, Ristic *et al.* (1993). This macro shell element, named the 4TS element, has 32 degrees of freedom around the edges: three deflections at each corner and mid side node, and one normal rotation along the edges at Loof nodes; the element has been shown to have excellent convergence properties Ristic *et al.* (1993).

Corresponding to the findings of Ueda and Fujikubo (1991) for triangular elements, each macro 4TS shell element will contain four checking points for plasticity each positioned in the centroid of the triangular elements. Since plastic deformations are concentrated in the nodes, Loof nodes that only have a rotational degree of freedom can be regarded as providing hinges along the element edge.

3. Yield conditions for orthogonally reinforced plates

The orthotropic yield function used is based on the yield conditions for a reinforced concrete slab derived from yield line theory for rigid-sections. For comparison, the isotropic von Mises yield function is included in the assessment. Since the YLM model is implemented in a generalized stress resultant form, the von Mises function is also given in a general stress resultant form, in the appendix.

We mainly consider concrete plates reinforced orthogonally with layers of steel in the (x, y)

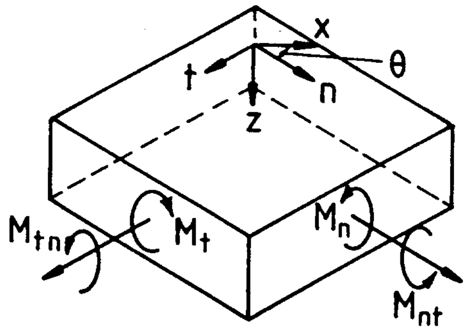

Fig. 1 Stress resultants

directions parallel to the plate mid-plane. The sign convention for the stress resultants per unit width is shown in Fig. 1, for rectangular coordinates (n, t) rotated θ clockwise from (x, y) direction. It is assumed that the transverse shear forces and stresses normal to the plate have negligible effect on the strength, and that local stress variations, and effects such as bond failure and cracking have no influence overall, so only mean stresses and strains are considered. The reinforcement is treated as a plate of constant thickness at the level of the bar centres. The concrete is assumed as an isotropic rigid-plastic (no-tension) material. If the materials are rigid-plastic, then the entire body can be treated as rigid-plastic, and the yield surface derived in stress resultant space will obey the flow rule.

Classical yield lines for rigid-slab elements have one non-zero deformation rate - the rotation, $\dot{\theta}_n$. That is, it is assumed that the in-plane displacement, $\dot{\delta}_n$, normal to a yield line at the mid-depth of the shell, and the shearing displacement along the yield line are zero. It is known that in-plane "dome effects" (e.g., Bræstrup and Morley 1980) have an effect on the ultimate capacity. However, an analysis which includes these in-plane actions requires the development of yield conditions for shell elements, and that is the subject of a sequel to this paper. For a linear strain distribution, the strain rates at a distance z from the median plane, $\dot{e}_n$, are given by

$$\dot{e}_n = \dot{\varepsilon}_{nn} + \dot{\kappa}_{nn} \cdot z \tag{11}$$

the corresponding stresses, σ_n, in the concrete at depth z are written as nonlinear functions of the strain rates, and the contribution of the concrete to the stress resultants can then be found from the usual integral:

$$M_n^c = \int_{-h/2}^{h/2} \sigma_n \, z \, dz \tag{12}$$

The contribution of the reinforcment, M_n^s, can be added, making use of standard transformations of reinforcement forces from the Cartesian system to the local axes. The yield condition then requires simply that the normal applied moment, M_n, reaches the total capacity, M_{pn}:

$$-(M_n^c + M_n^s) + M_n = 0 \qquad \text{for} \quad \dot{\theta}_n \geq 0 \tag{13}$$

$$(M_n^c + M_n^s) - M_n = 0 \qquad \text{for} \quad \dot{\theta}_n < 0 \tag{14}$$

Morley (1966) transformed the yield criterion into (M_x, M_y, M_{xy}) space, and expressed M_n in

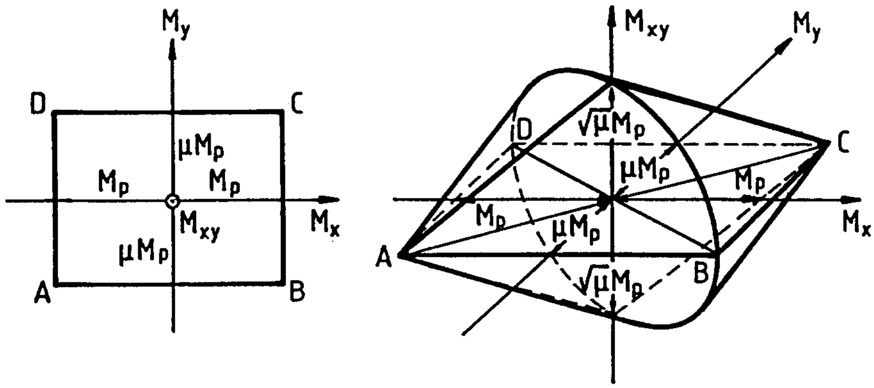

Fig. 2 Yield function for a plate with same top and bottom steel

terms of $(M_x, \cdots, M_{xy}, \theta)$ to give a function F=0. The defined yield surface can then be obtained by eliminating θ between the equations for F=0 and $dF/d\theta$=0; the latter is necessary since F must also be a minimum at that θ, i.e., the plate is just yielding.

Given the normal plastic capacities for ductile concrete sections, found from a rectangular stress block, one finds from Eq. (13) for a positive yield line, the yield function:

$$F = -(M_{px} - M_x) \cdot (M_{py} - M_y) + M_{xy}^2 \tag{15}$$

where M_{px}, M_{py} are the ultimate moments for simple bending in the x and y directions respectively.

The corresponding expression for negative bending, reducing Eq. (14), becomes

$$F = -(M'_{px} + M_x) \cdot (M'_{py} + M_y) + M_{xy}^2 \tag{16}$$

where M'_{px}, M'_{py} are the ultimate hogging moments for simple bending in the x and y directions. These are the well known yield criteria for plates under bending and twisting alone; see e.g., Bræstrup (1970) and Morley (1966). Eqs. (15, 16) represent two cones in a Cartesian coordinate system, as illustrated in Fig. 2. The curve of intersection between this surface and the (m_x, m_y) plane is naturally rectangular with the sides of $2M_p$ and $2\mu M_p$, where μ is the ratio of x and y capacities. Points on the cones given by Eqs. (15) and (16) correspond to positive and negative yield lines respectively. Moreover, the apeces A and C of the yield surface, see Fig. 2, correspond to the intersection of yield lines of the same sign, and points on the intersection of the two cones correspond to the intersection of yield lines of opposite sign.

Baker (1991) derived the expressions more directly by showing that yield corresponds to a zero eigenvalue of the tensor of excess moment capacity, written in arbitrary axis system. Since the transformations from the Cartesian system are orthogonal, the yield condition corresponds to a zero eigenvalue, or zero determinant, of the excess capacity in the Cartesian system itself:

$$\det \begin{bmatrix} M_{px} - M_x & -M_{xy} \\ -M_{xy} & M_{py} - M_y \end{bmatrix} = 0 \tag{17}$$

whence Eq. (15) follows. By the same reasoning, one finds the yield condition for skew reinforcement immediately. Let the x-axis be parallel to one set of reinforcement, and let the

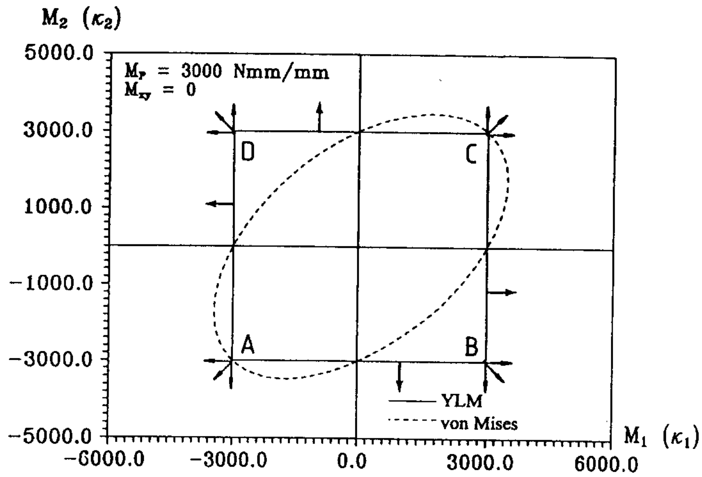

Fig. 3 Yield surfaces for plates, M_{xy}=0

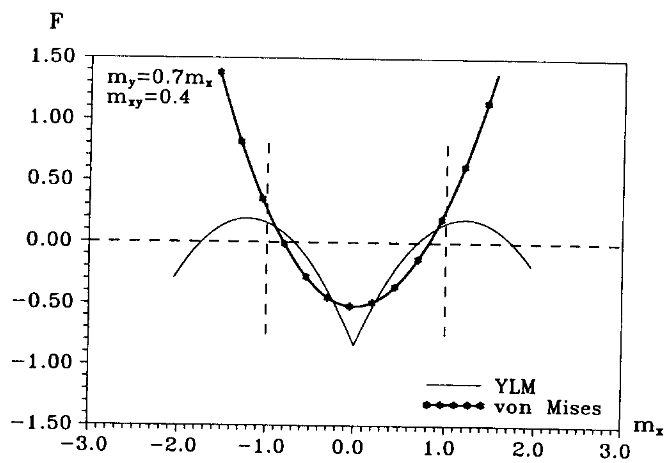

Fig. 4 YLM and von Mises yield functions

other be placed at an angle, α, to that axis. Then, with capacities, M_{px} and $M_{p\alpha}$, one finds for a positive yield line:

$$F = -[(M_{px} + M_{p\alpha}\cos^2\alpha) - M_x].\ [M_{p\alpha}\sin^2\alpha - M_y] + M_{xy}^2 \tag{18}$$

In the orthotropic case with M_{px}=M'_{px}=M_p and M_{py}=M'_{py}=μM_p, where μ is a constant, Eq. (15) is valid if $M_y \geq -\mu M_x$ and Eq. (16) is valid if $M_y \leq -\mu M_x$. In Fig. 3, the isotropic von Mises yield function and the isotropic case of the yield line model (YLM) functions are depicted in the principal moment space for a given value of M_p when no twisting moment occurs, i.e., M_{xy}= 0. It can be seen that if the two bending moments are equal, the two yield functions coincide.

4. Implementation of the yield line model (YLM)

The importance of a yield line rationale to the failure of reinforced concrete plates cannot be overstated, but its implementation in a finite element context presents certain mathematical anomalies which will now be discussed, as will the modifications designed to make the YLM robust. The yield functions, Eqs. (15) and (16) are arranged such that F is negative when stress resultants lie within the yield surface and zero when on the surface. However, positive values of F imply that stresses lie outside the surface and must then redistributed.

Firstly, the isotropic case of Eqs. (15) and (16) will be discussed i.e., $M_{px}=M_{py}=M_p$. Fig. 4 shows the variation of F with dimensionless moments. That is, the yield functions (15) and (16) have been reduced to a generalized stress resultant form by dividing by M_p, i.e., $m_x=M_x/M_p$, $m_y=M_y/M_p$ and $m_{xy}=M_{xy}/M_p$. For this particular example with $m_y=.7m_x$ and a constant twisting moment. For comparison the von Mises yield function is also depicted.

It is clear that the von Mises yield function will always increase with continuous loading, whereas the YLM function has a peak, beyond which the value of the yield function decreases. In fact F becomes less than zero (elastic) which physically can not happen. Yield line theory gives an orthotropic yield function which relates to the difference between moment capacity, M_{pn}, and applied moment, M_n, at an angle, θ (Baker 1991). As load increases, the excess becomes zero (F=0) at some angle, θ, and a yield line results perpendicular to the n-axis. If load is increased further (M_x, M_y increase proportionally as in Fig. 4), the shape of the variation with θ cannot change for either M_{pn} or M_n, and create F=0 at another angle in a second yield line. Hence, the mathematical excess becomes negative at some point. This is physically impossible and at this point continued use of the equations is improper.

Another difficulty arises when the value of the yield function gets beyond the peak. Although F is still positive, complications arise in the return mapping algorithm as moments are redistributed, since some of the derivatives have changed sign. From Eq. (2)-(6), it can be seen that a negative plastic displacement increment would result (which is inadmissible) even though F is larger than zero. Hence, it is essential that the value of the yield function continues to increase with loading.

A close examination of yield functions, Eqs. (15) and (16) shows that a peak will almost always occur. Only in special cases can a peak be avoided, for example in the isotropic case where $m_x=m_y=|m_{xy}|$, in which the value of the yield function will increase linearly with e.g., m_x. A simple determination of where the peak occurs is not possible, though it can be shown that the peak will happen after one of the parentheses in Eq. (15) or Eq. (16) changes sign. For example, for the isotropic case, the peak occurs when $m_x=m_y=1$, and $F=m_{xy}^2$. If in this case, no twisting moment occurs, the value of the yield function is F=0, and hence a load increment that normally would lead to $F>0$ will instead result in $F<0$. So even with very small load increments, the yield function will not be valid if the twisting component is very small.

We note, briefly, that the same issue arises with skew reinforcement, since Eq. (18) has the same form as (15), and a peak will result unless the twisting moments are large.

It is evident that the YLM function has to be modified to avoid a decreasing value of the yield function. A variety of models might be proposed as an extension to the YLM beyond the peak, but it has been decided to use the von Mises function. As already discussed, the value of the YLM function will always increase as long as either $|m_x|\leq 1$ or $|m_y|\leq 1$. Where the principal curvature is positive, the change to the von Mises function is invoked when either $m_x>1$ or $m_y>1$, and the value of von Mises function is larger than the YLM function; for a negative principal curvature similar conditions apply.

For small or no twisting moments, the composite YLM function will be a continuous but not differentiable function, whereas in the case of a larger magnitude of twisting moments, as in the case in Fig. 4, a slight jump in the yield function will occur. This is results from a larger twisting term in the von Mises compared with the YLM function. However, a small discontinuity in the yield function will not influence the results significantly, since the return mapping algorithm used only operates with the slope of the yield function and, in addition, a jump in the yield function does not result in jumps in the values of the stress resultants.

In the special case of pure twisting of the plate, i.e., $m_x=m_y=0$, the YLM function will always be an increasing function, and hence no shift to the von Mises function is necessary. Normally, the level of plasticity in a checking point is relatively high when shifting to a von Mises yield function occurs, and hence it could be argued that the plastic behaviour at a high plastic level will be similar for a von Mises and YLM based material. In order words, the plastic mechanism changes to a (transverse) shear flow at high load. Also, if no twisting moment occurs, the von Mises and the YLM functions will predict similar results if the two bending moments are of the same magnitude since plasticity is essentially isotropic.

The same condition that the peak occurs after one of the parentheses changes sign also applies to the orthotropic case, i.e., after one of the bending capacities has been reached. The isotropic von Mises yield function is again used as to extend the orthotropic YLM function. If the orthotropic version of the von Mises yield function, i.e., the Hill criterion, had been used, it would only have been after one of the bending capacities had been exceeded. Moreover, as discussed in de Borst & Feenstra (1990), the implementation of the orthotropic Hill criterion presents considerable computational difficulties compared to the von Mises model. In the von Mises yield function the smaller of M_{px} and M_{py} is to be used. However, if one bending capacity was significantly higher than the other, it was found that the use of an average value in the extended von Mises yield function is appropriate.

An associated flow rule is used, so that the plastic deformations can be found explicitly from the derivatives of the yield conditions.

6. Plastic limit analysis of RC plates

In this section we present a number of numerical case studies carried out to assess the viability and accuracy of the 'yield-line model' (YLM) in analysing plates up to failure, both in terms of the failure loads and mechanisms. Comparisons will be given with von Mises solutions to highlight the importance of a 'reinforced concrete' yield function, and to demonstrate the influence of twisting moments in the two functions.

6.1. Simply supported plate under uniformly distributed load

We begin with a simply supported orthotropic square plate under uniformly distributed load; the exact plastic solution (Prager 1952) for a square isotropic plate with $M_p=M_p'$ is $p_o=24M_p/a^2$, where a is the side length; the solution is exact in this case since the lower and upper bound solutions are equal (uniqueness theroem). Due to the double symmetry only one quarter of the plate (800×800 mm) has to be considered. A convergence analysis was carried out using a 2×2, 4×4, and 8×8 meshes, which showed that the collapse load using either yield criteria converges towards the exact plastic solution when finer mesh is used. Convergence occurs from above because the checking points of plasticity are moved closer to the exact yield line for a finer mesh.

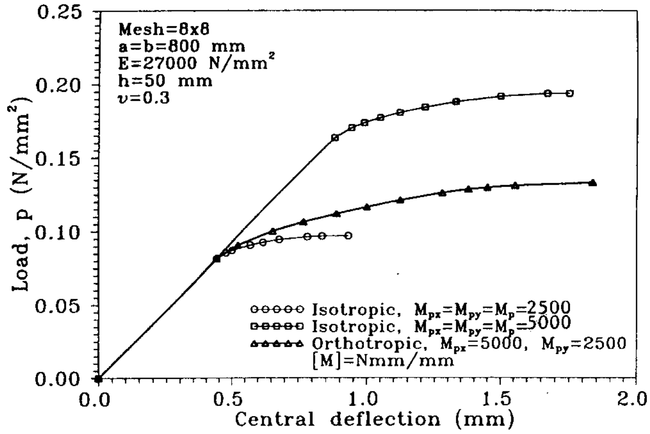

Fig. 5 Load deflection for square plate - YLM criterion

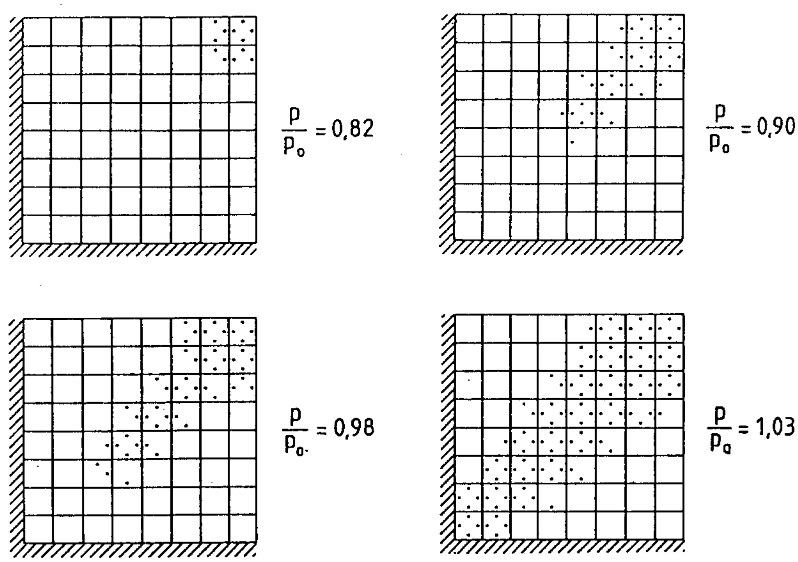

Fig. 6 Spread of plasticity for YLM analysis

Fig. 5 shows load versus deflection curves for two isotropic and one orthotropic plate using the YLM criterion. The bending capacities chosen and the material and geometric properties are given in the figure. For the isotropic case, the YLM and von Mises functions collapse load values agreed to within 1%, these values being just 2% above the exact solution. Fig. 6 shows the spread of plasticity for the YLM analysis of the isotropic case, over a sequence of load steps, where the dots indicate plasticity in element checking points. Firstly we note that the plastic node method is well suited to predicting the growth of yield line patterns. Moreover, when using the YLM yield function, plasticity occurs firstly around the centre of the plate, and spreads along the diagonal towards the corner as load is applied, as predicted by yield line theory and the exact solution. However, the von Mises yield function gives first yielding near the corner of the plate, mainly because the biggest twisting moments occur near the corners and hence the large twisting term in Eq. (20) predicts first yield near the corners.

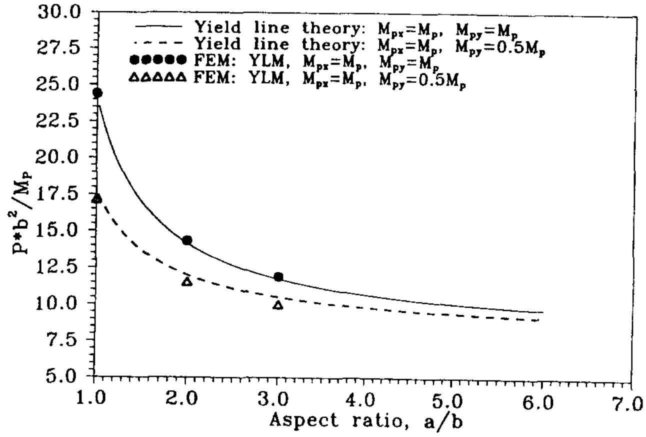

Fig. 7 Collapse load variation with aspect ratio

The influence of orthotropy is clear: the yield load for the orthotropic case corresponds to the yield load in the isotropic case if the smaller of the bending capacities is used, but the collapse load is significantly higher as expected. To determine the actual collapse load for this orthotropic plate a comparison with the yield line theory is carried out. Making use of the *affinity theorem*, solutions applicable to isotropic plates can be transformed into solutions for orthotropic plates. Thus, the upper bound solution for a simply supported rectangular orthotropic plate is:

$$p = \frac{24M_p}{b^2\left[\sqrt{3+\mu\left(\frac{b}{a}\right)^2}-\sqrt{\mu}\,\frac{b}{a}\right]^2} \qquad \text{for } b \le \frac{a}{\sqrt{\mu}} \tag{19}$$

where the bending capacities are M_p and μM_p for the longer, a, and shorter, b, sides respectively; μ is a constant ratio. It can be seen that Eq. (19) reduces to the solution for a isotropic square plate, i.e., when μ=1 and a/b=1. When comparing the orthotropic collapse load of the example in Fig. 5, i.e., μ=0.5 and a=b, with the yield line theory given by Eq. (19) one finds an excellent agreement with only 2% deviation.

In addition to the square plate, finite element predictions for two other aspect ratios, a/b=2 and a/b=3, with M_p=M_p', were carried out with meshes of 8×16 and 8×24 respectively, and compared with the yield line solution given by Eq. (19); the results are given in Fig. 7. The upper bound solution given by Eq. (19) is only exact for a=b and M_p=M_p'. However, for the case of an isotropic rectangular plate, Wood (1955) found a lower bound solution that was only 1.5% from Eq. (19), and hence the YLM is expected to predict collapse loads close to that given by Eq. (19) for all aspect rations in the isotropic case. Fig. 7 confirms that the YLM analysis in the isotropic case predicts excellent results for all three aspect rations.

In the orthoptropic case, the YLM gives an excellent result compared with Eq. (19) for the square plate where Eq. (19) is exact. However, the YLM analyses in the orthotropic case start to fall below the upper bound solution Eq. (19) as aspect ratio increases. Hence, the upper bound Eq. (19) obviously weakens as aspect ratio increases. It should be mentioned that the spread of plasticity according to the finite element calculation is identical to the yield line pattern for the rectangular simply supported plate upon which Eq. (19) is based. The von Mises criterion gave

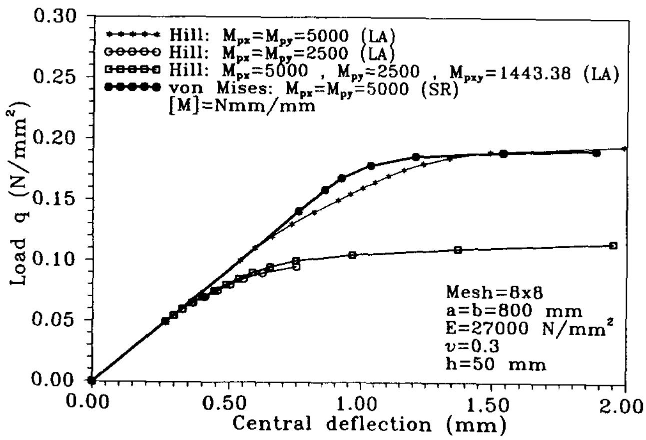

Fig. 8 Load deflection for square plate - Hill criterion

similar results in the isotropic case but cannot of course capture othotropy at all.

The YLM function is assessed further through a comparison (Fig. 8) with results obtained using LUSAS (1991) finite element software, which implements the anisotropic Hill yield criterion and the layered approach developed by de Borst and Feenstra (1990). A semi-loof rectangular shell element, QSL8, with 32 degrees of freedom has been used. Since the Hill criterion reduces to the von Mises criterion in the isotropic cases a direct comparison is possible between the layered (LA) and the stress resultant (SR) approaches. The comparison is shown in Fig. 8 for a plastic bending moment of M_p=5000 Nmm/mm. The Hill criterion implementation requires that equivalent yield stresses are given for each of the six stress components. To calculate these values the plastic moment of M_p=5000 Nmm/mm was converted to equivalent uniaxial yield stresses assuming fully plastic stress block, e.g., in this example corresponding to a yield stress of 8 MPa of a 50 mm thick plate. Fig. 8 shows that the non-linear response starts to occur at a lower load level using the layered approach. This is because the plasticity can spread gradually though the thickness of the plate, in contrast to the stress resultant model, where the nonlinear response will not begin until a fully plastic section has developed. However, the two models predict the same ultimate capacity. Since the same mesh size and similar elements have been used, the LUSAS calculations confirm the accuracy of the plasticity technique adopted.

No obvious equivalence exists between the anisotropic Hill and the YLM criteria for orthotropy. The shear yield capacities required for the Hill function can be chosen randomly. However, if the logic from the isotropic case is used, as well as the principle used by de Borst and Feenstra (1990) for shells, the yield stresses corresponding to a twisting capacity of $M_{pxy}=M_{py}/(3)^{1/2}$ should be used. Fig. 8 shows the results of the calculation when μ=0.5, i.e., M_{px}=5000 Nmm/mm, M_{py}=2500 Nmm/mm, M_{pxy}=1443.38 Nmm/mm. Comparing this result with those obtained using the orthotropic YLM function (Fig. 5), one finds that the Hill criterion gives a collapse load which is 9% lower than the exact YLM result. Again, the smaller twisting capacity embedded in the Hill criterion is the reason for this deviation.

6.2. Cantilever square plate under twisting loads

We also consider the example of a square isotropic plate clamped along one side and sub-

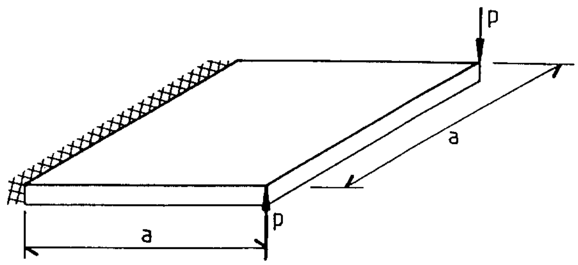

Fig. 9 Square cantilever plate under twisting loads

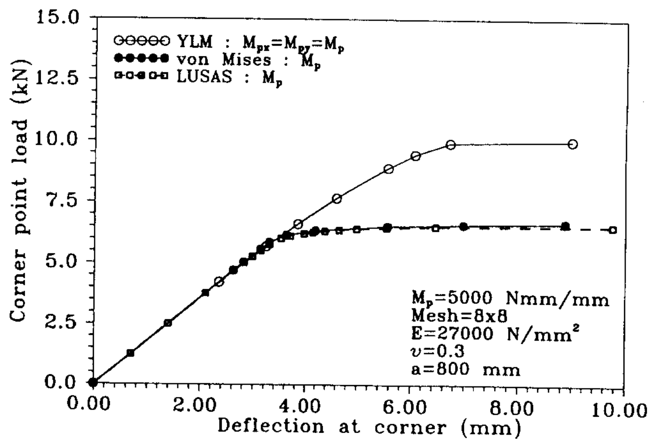

Fig. 10 Load deflection for cantilever plate

jected to two concentrated loads, P, in opposite directions at the two free corners (Fig. 9) to highlight the influence of twisting terms in the yield functions. Fig. 10 shows the load-deflection curves for both the YLM and von Mises models. An upper bound solution has been developed based on diagonal yield lines, with one load rising by a deflection, δ, and the other dropping by the same deflection. The mid-point between the loads does not deflect, nor does the centre point of the slab. The analysis yields simply $P=2M_p$, where M_p is the fully plastic moment. It can be seen from Fig. 10 that the numerical results using the YLM criterion predict this value to within 1%. However, the von Mises yield function yields a capacity which is 33% lower. Since the magnitude of twisting is significant, the collapse load according to the YLM criterion was reached without a shift to a von Mises yield function at any stage. Using the von Mises yield function the diagonal yield lines developed mainly from the point loads to the clamped edge, whereas the diagonal yield lines for the YLM function developed from the clamped edge towards the point loads, which seems more plausible.

7. Concluding remarks

This paper has presented the details of an orthotropic, stress resultant yield function particularly suited to the plastic limit analysis of reinforced concrete plates. The yield criteria derived from yield line concepts have been implemented into a finite element code using discrete Kirchoff Loof elements, and based on the plastic node method for evolution of plasticity.

The behaviour of the 'yield line' rationale has been evaluated and a revision to the parametric yield function made to ensure stability within a finite element code. First yield is always predicted by the orthotropic yield line function (YLM), but in certain cases, particularly with small twisting moments, the growth of plastic deformations after yield is traced with a von Mises function, since this exhibits continuous growth in plastic strain.

A number of test cases on isotropic and orthotropic plates have been presented, from which the importance of the yield line appproach has been demonstrated, in terms of the ability to predict both collapse loads and failure mechanisms.

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Appendix: Von Mises yield criterion for isotropic plates

The exact isotropic yield function for thin plates which obeys the von Mises function can be derived as (Ueda and Fujikubo 1991):

$$F = m_x^2 - m_x\, m_y + m_y^2 + 3m_{xy}^2 - 1 \tag{20}$$

where m_x, m_y, m_{xy} are generalized bending and twisting moments, obtained by dividing the stress resultants by the fully plastic moment, M_p, e.g., $m_{xy}=M_{xy}/M_p$.