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Multicracks identification in beams based on moving harmonic excitation

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Abstract. A method of damage detection based on the moving harmonic excitation and continuous wavelet transforms is presented. The applied excitation is used as a moving actuator and its frequency and speed parameters can be adjusted for an amplified response. The continuous wavelet transforms, CWT, is used for cracks detection based on the resulting amplified signal. It is demonstrated that this identification procedure is largely better than the classical ones based on eigenfrequencies or on the eigenmodes wavelet transformed. For vibration responses, free and forced vibration analyses of multi-cracked beams are investigated based on both analytical and numerical methodological approaches. Cracks are modeled through rotational springs whose compliances are evaluated using linear elastic fracture mechanics. Based on the obtained forced responses, multi-cracks positions are accurately identified and the CWT identification can be highly improved by adjusting the frequency and the speed excitation parameters.

Keywords: multi-cracks identification; wavelet transform; differential quadrature method; moving harmonic excitation; free and forced vibration

1. Introduction

Vibration-based structural health monitoring consists on identifying health of structures or mechanical systems by using many methods based on dynamic behavior caused by damage. Modal information has long been used for damage identification Doebling *et al.* (1996), many methods were developed in this regard, using either mode shapes, or natural frequencies as detailed in Salawu (1997), the aim of vibrations based methods is to combine experimental data to vibration models for cracks detection, location, characterization, and quantification. During the last decades, modeling vibrations of damaged structures gained increasing attention from structural health monitoring researchers. In fact, a wealth of analytical, numerical, and experimental investigations now exists. Fracture mechanics has long been interested in propagation of cracks in materials. It

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was first founded by Griffith who gave a criterion for crack propagation in brittle materials and was succeeded by other demonstrations and generalizations enabling modeling different configurations of cracked materials Papadopoulos (2008), Fracture mechanics approach considers a crack as local flexibility that depends on crack dimensions and applied loads, this was modeled as a rotational spring linking two neighboring half-beams Dimarogonas (1996), The rotational spring model gives relatively better results in finding natural frequencies, and then it is largely used in vibration problems of cracked structural elements. Vibration-based methods in structural health monitoring mainly use these models, to better identify damages in structures.

For structure's health monitoring, free and forced vibrations signals are analyzed using signal processing techniques. Wavelet transforms are widely used. Uses of wavelet transforms for detection of structure's defects have been investigated in many research works Ramon *et al.* (2012), Ovasenova and Suàrez (2004) demonstrated strength of wavelet transforms as a tool for detecting cracks in structures. In another work, two wavelet-based approaches for the detection of cracks locations based on beam's response are developed in Khorram *et al.* (2012), the beam was subjected to a concentrated moving load, contained one crack and the Gaussian 4 wavelet has been used. Loutridis *et al*. (2004) used mode shapes wavelet transformed to develop an intensity factor for estimating the relative depth of the cracks. Their positions are obtained from the sudden changes in wavelet coefficients. The same idea was adopted by Zhu and Law (2006) using forced response of the one cracked beam instead of mode shapes. Results are obtained from analytical method, where one has to solve an 8x8 algebraic system for one crack and a 12×12 one for a double cracked beam. Various methods are used to solve cracked beams vibration problem. Recently Nassar *et al.* (2013) make use of the differential quadrature method to study free vibrations of an Euler Bernoulli beam. The beam is considered to contain one crack, made of a functionally graded material and rests on a Winkler Pasternak foundation. Orhan (2007) studied free and forced vibrations of one cracked beam subjected to harmonic load by finite element method and showed that the forced vibration better describes changes in crack depth and locations than the free vibration. The effects of crack depth and position on the natural frequency of the beam using the analytical solution and the finite element method have been elaborated by Al-Waily (2013), Based on the two-dimensional finite elements, the dynamics of cantilever beams with a breathing crack, simulated as a frictionless contact problem, has been investigated by Andreaus *et al*. (2007), The combination of the finite element method and some optimization techniques such as Artificial Neural Networks, Genetic Algorithm and Particle Swarm Optimization, was used to investigate the identification of cracks locations and depths on beams by Abolbashari *et al*. (2014), Vosoughi (2015) .

Recently, more attention was given to multi-cracked beams. Khorram *et al*. (2013) presented a multi-crack detection of beams subjected to a moving load. The classical analytical method for free vibrations problem was used which limited their study to three cracks because of complexity of the resulting algebraic equation. Shifrin and Ruotolo (1999) used Dirac's delta function to express the governing equation for free vibration of multi-cracked beam. The frequency equation can be determined from a system of (n+2) linear equations. Otherwise, numerical approach based on differential quadrature element method (DQEM) was proposed by Torabi *et al.* (2014) in order to solve the free vibration problem of multi-cracked non-uniform Timoshenko beams with general boundary conditions.

This paper aims to provide a new procedure of cracks identification in beams containing an arbitrary number of cracks, based on the forced response of a multi-cracked beam subjected to harmonic travelling excitation. The beam is modeled as multi span beam and Euler-Bernoulli model is adopted for each span. The cracks are modeled as an equivalent torsional springs. A numerical procedure based on the differential quadrature method is elaborated for multi-cracked beams as well as analytical approaches. The forced vibration response is obtained for beams subjected to a harmonic moving excitation that is used here to get an amplified signal to be analyzed. The identification procedure is based on the continuous wavelet transform of eigenmodes and time forced responses. With a judicious choice of the excitation frequency and speed parameters, the locations of the cracks can be better identified.

2. Mathematical modelling

2.1 Crack modelling

Let us consider a cracked homogeneous uniform and viscoelastic isotropic beam with thickness *h*, width *b*, and length *L*, subjected to transverse vibrations. According to fracture mechanics, a crack introduces a local flexibility due to the strain energy concentration in the vicinity of the crack tip. The rotational spring model was used to quantify, in a macroscopic way, the relation between the applied load and the strain concentration Dimarogonas (1996), In the presence of a crack, additional displacement resulted and calculated by applying the Castigliano theorem

$$\theta^* = \int_0^a J(z)dz \tag{1}$$

where $J(z)$ is the Strain Energy Release Rate (SERR),'a' is the depth of the crack and z is the beam height. $J(z)$ depends on the crack depth and applied generalized forces that are responsible for the modes of fracture (opening, shearing or tearing), For the considered problem, the cracked beam is subjected to bending moment M. The strain energy density can thus be written as

$$J(z) = \frac{1-\vartheta^2}{E}(K_I^2 + K_{II}^2 + K_{III}^2) \tag{2}$$

where E is the Young modulus and ϑ is the Poisson ratio, K_I, K_{II}, K_{III} are the stress intensity factors corresponding to the opening, shearing and tearing modes of fracture. For a single edge cracked beam under pure bending specimen, $J(z)$ is reduced to Tada *et al.* (2000)

$$J(z) = \frac{1-\vartheta^2}{E} K_I^2 \tag{3}$$

$$K_I = \sigma_0 \sqrt{\pi a} F_1\left(\frac{z}{h}\right) \tag{4}$$

$$F_1(\frac{z}{h}) = 1.12 - 1.40\left(\frac{z}{h}\right) + 7.33\left(\frac{z}{h}\right)^2 - 13.1\left(\frac{z}{h}\right)^3 + 14.0\left(\frac{z}{h}\right)^4 \tag{5}$$

where σ_0 is the applied stress due to the bending moment and $F_1(z)$ is the configuration correction factor for the stress intensity factor. According to fracture mechanics analysis, the additional rotation θ^* is given by Tada *et al.* (2000)

$$\theta^* = \left(\frac{4\sigma_0(1-\vartheta^2)}{E}\right)\left(\frac{\frac{a}{h}}{1-\frac{a}{h}}\right)^2 f\left(\frac{a}{h}\right) \tag{6}$$

$$f\left(\frac{a}{h}\right) = (5.93 - 19.69\left(\frac{a}{h}\right) + 37.14\left(\frac{a}{h}\right)^2 - 35.84\left(\frac{a}{h}\right)^3 + 13.12\left(\frac{a}{h}\right)^4) \tag{7}$$

By considering the moment of inertia of the beam $I = \frac{bh^3}{12}$ and $\sigma_0 = \frac{M_f}{I}\frac{h}{2}$, expression of additional rotation related to the bending moment is written as

$$\theta^* = \frac{2hM_f}{EI}\left(\frac{\frac{a}{h}}{1-\frac{a}{h}}\right)^2 f\left(\frac{a}{h}\right) \tag{8}$$

It should be noted that the rotational spring model relates the additional rotation to the applied bending moment. Thus, compliance of the rotational spring is Tada *et al.* (2000)

$$\overline{C} = \left(\frac{2h}{EI}\right) f\left(\frac{a}{h}\right) \tag{9}$$

This rotational spring model will be used here for free and forced vibration analyses of multi-cracked beams under a moving harmonic excitation, as well as for cracks identification. The multi-cracks identification will be first based on the free vibration characteristics and wavelet transforms. The investigation is then focused on the time and space responses due to the harmonic moving load, in order to elaborate a powerful cracks identification procedure.

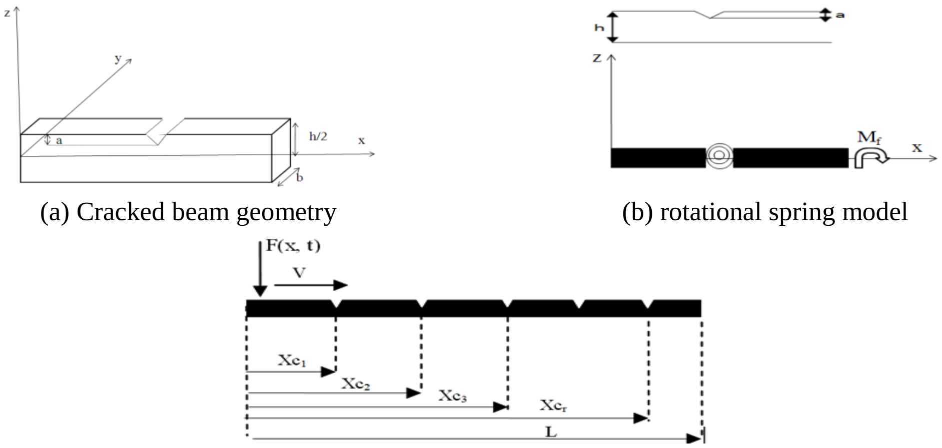
2.2 Governing equation

A homogeneous and uniform continuous beam with '*r*' cracks and subjected to a travelling excitation $F(x,t)$ is considered. Euler-Bernoulli beam model is used for the beam, which is subdivided into $(r+1)$ segments joined by r rotational springs as shown in Fig. 1. The transverse displacement of each segment of the beam is denoted by $y_i(x,t)$ and the partial differential equation governing the motion of each sub-beam is given by

$$EI\frac{\partial^4 y_i(x,t)}{\partial x^4} + c\frac{\partial y_i(x,t)}{\partial t} + \rho A\frac{\partial^2 y_i(x,t)}{\partial t^2} + \alpha\frac{\partial^5 y_i(x,t)}{\partial x^4 \partial t} = F(x,t) = F_0\sin(\Omega t)\delta(x - Vt) \qquad i = 1,2,...,\ r+1. \tag{10}$$

where ρ, A, c and α are mass density, cross sectional area, damping coefficient and viscoelastic coefficient respectively. $F(x,t)$ is a harmonic travelling excitation force along the beam with a constant speed V and frequency Ω. δ denotes the Dirac delta function.

The main objectives of this paper are on one hand, the free and harmonic moving forced vibration analyses of beams with an arbitrary number of open cracks. On the other hand, a multicracks identification procedure is elaborated based on space and time responses, with



(a) Cracked beam geometry (b) rotational spring model

(c) Multi-cracked beam carrying moving harmonic excitation

Fig. 1 Multi-cracked beam under moving harmonic excitation and crack modeling

particular choices of the excitation frequency and speed. The harmonic moving excitation is adopted here in order to elaborate an accurate and powerful identification crack procedure with adjusted frequency **Ω** and speed *V* of the applied excitation force. The free vibration of beams with an arbitrary number of cracks will be first investigated based on the classical and reduced analytical approaches as well as on a numerical procedure.

3. Free vibration analysis

The natural frequencies and eigenfunctions are widely used to find the cracks locations in damaged continuous beams. To determine the natural eigenmodes and eigenfrequencies of the considered multi-cracked beam, the forcing, damping and viscoelastic effects are disregarded and the following boundary value problem has to be solved in each sub beam "*i*"

$$\begin{cases} \dfrac{\partial^4 w_{ij}(x)}{\partial x^4} - \mu_j^4 w_{ij}(x) = 0 \ ; \ \mu_j^4 = \dfrac{\rho A}{EI}\omega_j^2 \qquad \mathrm{i} = 1,2,3,...,\mathrm{r}+1; \quad \mathrm{j} = 1,2,3,....,\mathrm{N} \\ \qquad\qquad + \ \text{Boundary} \quad \text{Conditions} \end{cases} \tag{11}$$

where w_{ij} is the local transverse displacement associated to the j^{th} vibration mode of the i^{th} sub-beam and ω_j is the j^{th} eigenfrequency.

Based on the presented rotation massless spring model, the displacement, bending moment, and shear force at boundaries of two neighboring segments are continuous

$$\begin{cases} w_{ij}(x_i) = w_{(i+1)j}(x_i) \\ w''_{ij}(x_i) = w''_{(i+1)j}(x_i) \\ w'''_{ij}(x_i) = w'''_{(i+1)j}(x_i) \end{cases} \tag{12-a}$$

At the crack location x_i, the model of massless rotational spring is adopted with flexibility $\overline{C}_i$

that depends on the depth of the crack and given by Eq. (9), The slope has thus a jump given by

$$\begin{cases} w'_{(i+1)j}(x_i) = w'_{ij}(x_i) + \overline{C}_i w''_{ij}(x_i) \\ \overline{C}_i = \dfrac{2h}{EI} f\left(\dfrac{a_i}{h}\right) \end{cases} \tag{12-b}$$

where a_i is the depth of the i^{th} crack. The j^{th} eigenmode of the whole beam is then given by the following compact relationship

$$w_j(x) = \sum_{i=1}^{r+1} w_{ij}(x)[H(x - x_{i-1}) - H(x - x_i)] \tag{13}$$

where x_0 corresponds to the right hand boundary and $H(x)$ is the Heaviside step function. Various procedures can be used to compute the sub-beams vibration modes $w_{ij}(x)$, the commonly used approach for cracked beams vibration, called here classical approach is presented in the appendix. Let us note that for a beam containing n cracks, this procedure will lead to solve a highly nonlinear $(4n+4)\times(4n+4)$ algebraic system. This drawback is thus limiting this classical analytical method to a small number of cracks. Other methods allowing the vibration analysis of beams with a larger number of cracks should be used.

3.1 Reduced analytical method

This method allows one to have a factorized algebraic equation system of order 2×2 for any number of cracks by using initial boundary conditions and fundamental solutions to express mode shapes. The general solution of Eq. (11) corresponding to the i^{th} sub-beam is rewritten as Li (2002)

$$w_{ij}(x) = w_{1j}(0)S_{1j}(x) + w'_{1j}(0)S_{2j}(x) + w''_{1j}(0)S_{3j}(x) + w'''_{1j}(0)S_{4j}(x) \tag{14}$$

where:

$$S_{1j}(x) = \frac{1}{2}(\cosh(\mu_j x) + \cos(\mu_j x)) \qquad S_{2j}(x) = \frac{1}{2\mu_j}(\sinh(\mu_j x) + \sin(\mu_j x))$$

$$S_{3j}(x) = \frac{1}{2\mu_j^2}(\cosh(\mu_j x) - \cos(\mu_j x)) \qquad S_{4j}(x) = \frac{1}{2\mu_j^3}(\sinh(\mu_j x) - \sin(\mu_j x))$$

and $w_{ij}(0)$, $w'_{ij}(0)$, $w''_{ij}(0)$ and $w'''_{ij}(0)$ are initial parameters at $x=0$.

Considering the continuous conditions of displacements, bending moments and shear forces as well as the jump of the slope at the boundaries of the i^{th} and the $(i+1)^{th}$ segments, the solution is expressed as

$$w_{(i+1)j}(x) = w_{1j}(x) + \overline{C}_i w''_{ij}(x_i) S_{2j}(x - x_i) H(x - x_i) \tag{15}$$

Having $w_{1j}(x)$, the j^{th} mode shape of the first segment, the j^{th} eigenmode of the $(r+1)^{th}$ segment is given by

$$w_{(r+1)j}(x) = w_{1j}(x) + \sum_{i=1}^{r} \overline{C}_i w''_{ij}(x_i) S_{2j}(x - x_i) H(x - x_i) \tag{16}$$

For the sake of clarity, a multi-cracked cantilever beam will be explicitly developed. Substituting the boundary conditions for the right end of the beam ($x=L$) into Eq. (11) and using some mathematical manipulations, one gets the following 2×2 algebraic system

$$\begin{cases} w_{1j}^{''}(0)\ A(\mu_j, L) + w_{1j}^{'''}(0)\, B(\mu_j, L) = 0 \\ w_{1j}^{''}(0)\, C(\mu_j, L) + w_{1j}^{'''}(0)\, D(\mu_j, L) = 0 \end{cases} \tag{17}$$

Note that the coefficients A, B, C and D depend on the frequency parameter μ_j. For natural frequencies, one has to solve the following simplified nonlinear transcendental equation

$$F(\mu_j) = A(\mu_j, L)\, D(\mu_j, L) - B(\mu_j, L)\, C(\mu_j, L) = 0 \tag{18}$$

For a double cracked cantilever beam (r=2), explicit expression of $A(\mu_j,L)$, $D(\mu_j,L)$, $B(\mu_j,L)$ and $C(\mu_j,L)$ are given by:

$$A(\mu_j, L) = S_{3j}^{''}(L) + C_1 S_{3j}^{''}(x_1) S_{2j}^{''}(L - x_1) + C_2 S_{3j}^{''}(x_2) S_2^{''}(L - x_2) + C_2 C_1 S_{3j}^{''}(x_1) S_{2j}^{''}(x_2 - x_1) S_{2j}^{''}(L - x_2)$$

$$B(\mu_j, L) = S_{4j}^{''}(L) + C_1 S_{4j}^{''}(x_1) S_{2j}^{''}(L - x_1) + C_2 S_{4j}^{''}(x_2) S_{2j}^{''}(L - x_2) + C_2 C_1 S_{4j}^{''}(x_1) S_{2j}^{''}(x_2 - x_1) S_{2j}^{''}(L - x_2)$$

$$C(\mu_j, L) = S_{3j}^{'''}(L) + C_1 S_{3j}^{''}(x_1) S_{2j}^{''}(L - x_1) + C_2 S_{3j}^{''}(x_2) S_{2j}^{'''}(L - x_2) + C_2 C_1 S_{3j}^{''}(x_1) S_{2j}^{''}(x_2 - x_1) S_{2j}^{'''}(L - x_2)$$

$$D(\mu_j, L) = S_{4j}^{'''}(L) + C_1 S_{4j}^{''}(x_1) S_{2j}^{'''}(L - x_1) + C_2 S_{4j}^{''}(x_2) S_{2j}^{'''}(L - x_2) + C_2 C_1 S_{4j}^{''}(x_1) S_{2j}^{''}(x_2 - x_1) S_2^{'''}(L - x_2)$$

The j^{th} eigenmode of the whole beam is finally given by

$$w_j(x) = \sum_{i=1}^{r+1} w_{ij}(x)[H(x - x_{i-1}) - H(x - x_i)] \tag{19}$$

For the considered boundary conditions, the eigenfrequencies can be obtained by numerically solving the resulting nonlinear algebraic transcendental equation. Note that for a large number of cracks, this equation is highly nonlinear. A numerical procedure based on the Newton-Raphson algorithm has been elaborated for numerical results. Based on this method, one is able to obtain the eigenfrequencies and explicit expressions of eigenmodes for beams containing a moderate number of cracks in a reduced CPU time and space in comparison with the classical method. Numerical simulations are performed to show the effectiveness of this procedure. It should be noted that for a large number of cracks this methodological approach is too heavy to handle. As an alternative, a numerical procedure based on the differential quadrature method is elaborated herein for beams with an arbitrary number of cracks.

3.2 Numerical procedure based on Differential Quadrature Method

To analyze vibration of beams with a large number of cracks, a well-adapted numerical procedure is needed. It should to be noted that for thin structures such beams, plates, and shells, the finite element method is the mostly used one. But, cracks modeled by rotation jumps, as presented in Eq. (12-b), cannot be correctly handled by the FEM unless a local very refined mesh is used around each crack. For the considered cracks model and used boundary connections, the differential quadrature method is adopted here for numerical investigations.

The DQM requires to discretize the domain of the problem into N points. The derivatives at any point are approximated by a weighted linear summation of all the functional values along the discretized domain, as follows Zhang and Zhi (2008)

$$\left.\frac{d^m f(x)}{dx^m}\right|_{x=\bar{z}_k} = \frac{d^m}{dx^m}\begin{bmatrix} f(\bar{z}_1) \\ f(\bar{z}_2) \\ f(\bar{z}_3) \\ . \\ . \\ . \\ f(\bar{z}_N) \end{bmatrix} = c_{kj}^{(m)} \begin{bmatrix} f(\bar{z}_1) \\ f(\bar{z}_2) \\ f(\bar{z}_3) \\ . \\ . \\ . \\ f(\bar{z}_N) \end{bmatrix} \tag{20}$$

where 'm' is the order of the highest derivative appearing in the problem, $f(\bar{z}_k)$ are the values of the function at the sampling points $\bar{z}_k$ relating the m^{th} derivative to the functional values at $\bar{z}_k$. These coefficients can be determined by making use of Lagrange interpolation formula as follows

$$f(x) = \sum_{i=1}^{N} \frac{L(x)}{(x-\bar{z}_i)L_1(\bar{z}_i)} \tag{21}$$

where: $$L(x) = \prod_{j=1}^{N}(x-\bar{z}_j), \quad L_1(\bar{z}_i) = \prod_{j=1}^{N}(\bar{z}_i - \bar{z}_j)$$

The weighting coefficients for the first order derivative to the functional values at $\bar{z}_k$ can be obtained as

$$c_{kj}^{(1)} = \begin{cases} \dfrac{L_1(\bar{z}_k)}{(\bar{z}_k - \bar{z}_j)L_1(\bar{z}_j)} & k \neq j \\ -\displaystyle\sum_{j=1,\, j\neq k}^{N} c_{kj}^{(1)} & k = j \end{cases} \tag{22}$$

The second, third and higher derivatives can be calculated as

$$c_{kj}^{(m)} = \sum_{l=1,\, j\neq k}^{N} c_{kl}^{(1)} c_{lj}^{(m-1)} \tag{23}$$

For accurate results, we adopt the Chebychev-Gauss-Lobatto mesh distribution given for interval $[x_i, x_{i+1}]$ by

$$\bar{z}_{ik} = \frac{z_{ik}}{(x_{i+1} - x_i)} = \frac{1}{2}[1 - \cos(\frac{k-1}{N-1}\pi)] \quad k = 1,2,\ldots,N \tag{24}$$

These discretization points $\bar{z}_{ik}$ are presented in Fig. 2. The multi-cracked beam is divided into sub-beams where the i^{th} crack at x_i is located between the i^{th} and the $(i+1)^{th}$ sub-beams.

For the i^{th} sub-beam, the equation of motion is defined on the interval $[x_i, x_{i+1}]$ by

$$\frac{d^4 w_i(x)}{dx^4} - \mu^4 w_i(x) = 0 \qquad x \in [x_i, x_{i+1}] \tag{25}$$

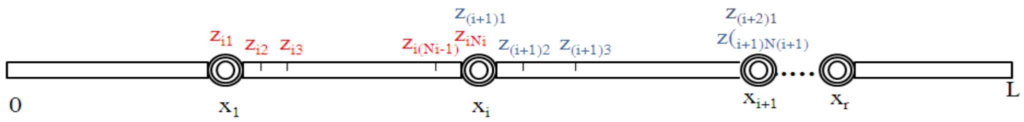


Fig. 2 Gauss Lobatto-Chebychev discretization points of sub-beams

This can be rewritten using DQM as

$$c_{kj}^{(4)} w_i(\bar{z}_{ij}) - \mu^4 w_i(\bar{z}_{ij}) = 0 \qquad i = 1,2,...,r \ \text{ and } \ k, j = 1,2,...N \tag{26}$$

Using DQM, every *i* sub-beam is discretized into *N* sampling points. The compatibility equations at crack location x_i given by equations Eq. (12-a), Eq. (12-b) are written, using DQM, as follows

$$\begin{cases} w_i(\bar{z}_{iN}) = w_{(i+1)}(\bar{z}_{(i+1)N}) \\ \left(\dfrac{c_{kj}^{(1)} w_i(\bar{z}_{iN})}{l_i}\right) - \left(\dfrac{c_{kj}^{(1)} w_{i+1}(\bar{z}_{(i+1)N})}{l_{i+1}}\right) = \left(\bar{C}_i \dfrac{c_{kj}^{(2)} w_i(\bar{z}_{iN})}{l_i^2}\right) \\ \dfrac{c_{kj}^{(2)} w_i(\bar{z}_{iN})}{l_i^2} = \dfrac{c_{kj}^{(2)} w_{i+1}(\bar{z}_{(i+1)N})}{l_{i+1}^2} \\ \dfrac{c_{kj}^{(3)} w_i(\bar{z}_{iN})}{l_i^3} = \dfrac{c_{kj}^{(3)} w_{i+1}(\bar{z}_{(i+1)N})}{l_{i+1}^3} \end{cases} \tag{27}$$

where l_i is the length of the i^{th} sub-beam and $\bar{C}_i$ is the flexibility at the i^{th} location of crack. In addition, the following boundary conditions are also incorporated.

For example, for a simply supported beam:

$$\begin{cases} w_1(\bar{z}_{11}) = 0 \\ (c_{1j}^{(2)} w_1(\bar{z}_{11}) / l_1^2 = 0 \end{cases} \qquad \begin{cases} w_{r+1}(\bar{z}_{(r+1)N}) = 0 \\ (c_{Nj}^{(2)} w_1(\bar{z}_{(r+1)N}) / l_{r+1}^2 = 0 \end{cases} \tag{28}$$

For Clamped-Clamped beam:

$$\begin{cases} w_1(\bar{z}_{11}) = 0 \\ (c_{1j}^{(1)} w_1(\bar{z}_{11}) / l_1 = 0 \end{cases} \qquad \begin{cases} w_{r+1}(\bar{z}_{(r+1)N}) = 0 \\ (c_{Nj}^{(1)} w_1(\bar{z}_{(r+1)N}) / l_{r+1} = 0 \end{cases} \tag{29}$$

For Clamped-Simply supported beam

$$\begin{cases} w_1(\bar{z}_{11}) = 0 \\ (c_{1j}^{(1)} w_1(\bar{z}_{11}) / l_1 = 0 \end{cases} \qquad \begin{cases} w_{r+1}(\bar{z}_{(r+1)N}) = 0 \\ (c_{Nj}^{(2)} w_1(\bar{z}_{(r+1)1}) / l_{r+1}^2 = 0 \end{cases} \tag{30}$$

For Clamped-Free beam

$$\begin{cases} w_1(\bar{z}_{11}) = 0 \\ (c_{1j}^{(1)} w_1(\bar{z}_{11}) / l_1 = 0 \end{cases} \qquad \begin{cases} (c_{Nj}^{(2)} w_{r+1}(\bar{z}_{(r+1)N}) / l_{r+1}^2 = 0 \\ (c_{Nj}^{(3)} w_{r+1}(\bar{z}_{(r+1)N}) / l_{r+1}^3 = 0 \end{cases} \tag{31}$$

Based on these mathematical developments the free vibration of a cracked beam with an arbitrary number of cracks is reduced to the following eigenvalue problem.

$$\begin{cases} [A]\{w\} - \mu^4 \{w\} = 0 \\ \{w\} = \{w_1(\bar{z}_{11}), w_1(\bar{z}_{12}),..., w_1(\bar{z}_{1N_1}), w_2(\bar{z}_{21}), w_2(\bar{z}_{22}),..., w_2(\bar{z}_{2N_2}),..., w_{r+1}(\bar{z}_{(r+1)1}), w_{r+1}(\bar{z}_{(r+1)2}),..., w_{r+1}(\bar{z}_{(r+1)N_{r+1}})\} \end{cases} \tag{32}$$

Numerical solution of the resulting eigenvalue problem allows one to get numerically the eigenfrequencies and associated eigenmodes of multi-cracked beams with a large number of cracks. Various beams boundary conditions can be easily considered.

The strength and accuracy of this methodological approach is numerically tested herein by comparing the obtained numerical results with those obtained by analytical methods. For multi-cracked beams with a large number of cracks, the presented numerical approach is a powerful tool to get easily eigenfrequencies and associated eigenmodes.

Based on the presented methodological approaches, the free vibration characteristics of multi-cracked beams can be obtained. The variation of eigenfrequencies and associated eigenmodes with respect to the depth, location, and number of cracks can be easily analyzed. The cracks identification, elaborated by many authors, is based on the wavelet transforms and the free vibration characteristics. This identification procedure and analysis can be easily treated here.

4. Forced time response

Using the modal expansion theory, the forced response of the system can be expressed as

$$y(x,t) = \sum_{j=1}^{\bar{N}} w_j(x) q_j(t) \tag{33}$$

where $w_j(x)$ is the j^{th} eigenmode of the multi-cracked beam that can be obtained either from analytical methods for a small number of cracks or from DQM for more general cases. $q_j(t)$ is the generalized modal coordinate for either small or large number of cracks. Substituting Eq. (33) into the main partial differential equation (Eq. (10)), multiplying both of sides by $w_j(x)$ and integrating from 0 to L, the following decoupled second order differential equations are resulted

$$\ddot{q}_j(t) + \left(\frac{c}{\rho A} + \alpha \frac{\omega_j^2}{EI} \right) \dot{q}_j(t) + \omega_j^2 q_j(t) = \frac{F_0}{M_{jj}} \sin(\Omega t) w_j(Vt) \qquad j = 1,2,...., \bar{N} \tag{34}$$

where M_{jj}, and $w_j(Vt)$ are respectively the modal mass coefficients, and the j^{th} eigenmode of the multi-cracked beam. These decoupled equations are obtained based on the orthogonal property of considered eigenmodes. It has to be pointed out that the right hand side of Eq. (34) is piecewise defined in $[0, \frac{L}{V}]$. This type of equation has been solved analytically using convolution method in Khorram *et al.* (2013) for a moving force. For a large number of cracks, the excitation term is too complicated to be handled analytically. Numerical method is adopted herein. To do so, Eq. (34) had split in sub-domains with the adjusted initial conditions as follows

$$\begin{cases} \ddot{q}_{j1}(t) + \left(\dfrac{c}{\rho A} + \alpha \dfrac{\omega_j^2}{EI}\right)\dot{q}_{j1}(t) + \omega_j^2 q_{j1}(t) = \dfrac{F_0}{M_{jj}} \sin(\Omega t) w_{j1}(Vt) \qquad 0 \prec t \prec \dfrac{x_1}{V} \\ q_{j1}(0) = 0;\ \dot{q}_{j1}(0) = 0 \end{cases}$$

$$\begin{cases} \ddot{q}_{j(i+1)}(t) + \left(\dfrac{c}{\rho A} + \alpha \dfrac{\omega_j^2}{EI}\right)\dot{q}_{j(i+1)}(t) + \omega_j^2 q_{j(i+1)}(t) = \dfrac{F_0}{M_{jj}} \sin(\Omega(t + \dfrac{x_1}{V})) w_{j1}(V(t + \dfrac{x_1}{V})) \qquad 0 \prec t \prec \dfrac{x_{i+1} - x_i}{V} \\ q_{j(i+1)}(0) = q_{ji}(\dfrac{x_1}{V});\ \dot{q}_{j(i+1)}(0) = \dot{q}(\dfrac{x_1}{V}) \end{cases} \tag{35}$$

Numerical solution of Eq. (35) allows one to determine the piecewise solution $q_{ji}(t)$ for j=1,2,..., $\overline{N}$ and i=1,2,3,...,r

A closed form of the time modal response $q_j(t)$, associated to fixed parameters F_0, **Ω** and V, is thus given by

$$q_j(t) = q_{j1}(t) + \sum_{i=1}^{r} [H(\frac{x_{i+1}}{V}) - H(\frac{x_i}{V})] q_{j(i+1)}(t) \tag{36}$$

This multi-modal solution can be used on one hand to analyze the effects of the moving excitation parameters, F_0, **Ω** and V on the forced response of multi-cracked beams. On the other hand, the obtained forced response can be used for cracks identification.

5. Multi-crack identification procedure

5.1 Moving harmonic excitation technique

The basic idea of this procedure is to subject the beam, to a moving harmonic excitation $F(x,t)=F_0\sin(\Omega t)\delta(x\text{-}Vt)$. This excitation acts on the structure by two effects: the velocity and harmonic frequency. Each of those effects is represented by the parameters β and γ, which are introduced here.

$$\beta \text{ is the speed parameter represented by } \beta = \frac{V}{V_{cr}} \tag{37}$$

where V_{cr} is the critical speed defined by: $V_{cr} = \dfrac{\omega_j L}{\pi}$

The effect of excitation frequency Ω is represented by the frequency ratio $\gamma = \dfrac{\Omega}{\omega_j}$ where ω_j the j^{th} natural frequency of the considered beam.

The main objective of using this kind of excitation is to improve the crack detection, for every position or depth of existing cracks. The moving harmonic excitation which acts on the whole considered beam as a moving actuator leading to an amplified response. The speed and the frequency parameters can be adjusted for a better signal amplification. The continuous wavelet transform data analysis is adopted here for signal analysis. The forced harmonic moving response associated to various frequency and speed parameters are used for detecting cracks positions with various depths.

5.2 Continuous wavelet transforms (CWT)

The continuous wavelet transform of a function $g(x)$ is Ovanesova (2004)

$$C(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} g(x)\psi^{*}\left(\frac{x-b}{a}\right)dx \tag{38}$$

where $\psi^{*}(x)$ denotes the complex conjugate of wavelet $\psi(x)$. It has to fulfill the admissibility condition, which implies that the wavelet must change sign at least once along its support. It can be expressed by the following equation: $C(a,b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega \prec \infty$ where $\hat{\psi}(\omega)$ is the Fourier transform of the wavelet. The function $\psi(x)$ is called, analyzing wavelet. The parameter, '*a*' is called scale, controls the width of the wavelet. A high value of '*a*' corresponds to wavelets with large support, so that low-frequency components can be looked through, while a low value of '*a*' corresponds to "small" wavelets, suitable for the analysis of high-frequency components. Analyzing wavelets can be shifted using parameter b to cover the range over which function *g* is defined (or the signal is sampled), In other words, a multiscale and well-localized analysis of the function is possible looking at the function's interesting features through a wavelet window.

The continuous wavelet transform (CWT) is then, the sum over all time (or space) of the signal multiplied by a scaled and shifted version of a mother wavelet where the scale '*a*' and the position '*b*' real numbers. The results of the transform are wavelet coefficients that show how well a wavelet function correlates with the signal analyzed. Hence, sharp transitions in $g(x)$ create wavelet coefficients with large amplitudes and this precisely is the basis of the proposed identification method.

6. Numerical results and discussion

6.1 Free vibration analysis of multi-cracked beams

The differential quadrature method aims to solve differential equations, using a discretizing procedure, by means of coefficients c_{ij}. This method makes the solution much easier than finite elements method, which is the most commonly used method. It has been applied here for free and forced vibration analysis of multi-cracked beams. To test the performance and efficiency of this method, the free vibration of uncracked beam is first analyzed. A slender beam with the following characteristics is considered: Young modulus E=210 GPa, material mass density ρ=7800 Kg/m^3, height h=0.01 m, thickness b=0.01 m, and length L=1 m. Eigenmodes and associated natural frequencies are obtained by the presented analytical approaches as well as by the DQM method.

For the sake of comparison, the finite elements method based on ANSYS software, is used where and the adopted finite element is BEAM189 with 6 degrees of freedom per node. The first three natural frequencies μ_j of an uncracked cantilever beam, obtained by the analytical approach,

Table 1 First three natural frequencies of uncracked cantilever beam using analytical, DQM and FEM (ANSYS)

Natural frequencies	Analytical approach	DQM			FEM (ANSYS)		
		N=8	N=10	N=15	N=8	N=15	N=100
$\omega_1/2\pi$	8,38190255	8,38177708	8,381903	8,38190254	8,392	8,3844	8,3813
$\omega_2/2\pi$	52,5284866	52,4777909	52,5286085	52,5284866	54,323	53,014	52,511
$\omega_3/2\pi$	147,081283	149,385947	146,901307	147,081292	162,39	151,14	146,98

Table 2 Relative positions and depths of the four cracks

Crack number	Crack position	Crack depth
1	x_1=0.2	a_1/h=0.2
2	x_2=0.4	a_2/h=0.15
3	x_3=0.6	a_3/h=0.1
4	x_4=0.8	a_4/h=0.1

Table 3 First three natural frequencies of beams containing four cracks with various boundary conditions

Boundary conditions	Reduced analytical approach			DQM (N=150)			Error %	
	μ1	μ2	μ3	μ1	μ2	μ3	μ1	μ2
Pinned-Pinned	3,1340997	6,2652589	9,3978741	3,1340999	6,2652588	9,3978742	6,7324E-08	-7,661E-09
Clamped-Free	1,8701409	4,6874925	7,8405544	1,8701414	4,6874920	7,8405548	2,513E-07	-1,051E-07
Clamped-Clamped	4,7255210	7,8408701	10,968782	4,7255209	7,8408700	10,968782	-1.227E-08	-1,300E-08
Clamped-pinned	3,9215767	7,0563020	10,1870383	3,9215768	7,0563018	10,1870386	3,57E-08	-2,522E-08

the DQM with 8, 10, and 15 points as well as with the FEM-ANSYS with 8, 15 and 100 nodes are presented in Table 1. It is clearly demonstrated that DQM converges to analytical solutions with a largely reduced number of discretization points in comparison with the FEM. Moreover, this method is suited to cracked beam problem because it handles correctly the rotational spring model in opposition to the finite element method that requires a finite element node must be placed at the crack location. Indeed, the finite element method has been largely developed and widely used for many engineering problems and particularly for static and dynamic analysis of thin structures. For cracked structures, some models are developed such as reduction of stiffness, or simply by removing elements from the structure Friswell and Penny (2002),

To test the efficiency and the accuracy of the DQM for multi-cracked beams, a beam with four cracks with positions and depths, given in Table 2, is considered. The following classical boundary conditions, Pinned-Pinned, Clamped-Free, Clamped-Clamped, Clamped-pinned, are considered. The beam is divided into five sub-beams and each sub-beam is discretized by 30 nodes. The DQM is thus used with 150 nodes.

Table 3 shows the first three natural frequencies μ_j for the considered depth values and positions of the cracks given in Table 2, and for various boundary conditions. Exact solutions are

Table 4 Relative positions and depths of the seven cracks

Crack number	Crack position	Crack depth
1	x_1=0.1	a_1/h=0.2
2	x_2=0.2	a_2/h=0.1
3	x_3=0.3	a_3/h=0.1
4	x_4=0.4	a_4/h=0.1
5	x_5=0.6	a_5/h=0.1
6	x_6=0.7	a_6/h=0.1
7	x_7=0.8	a_7/h=0.1

Table 5 First three eigenfrequencies of an eight equally spaced cracked beam

	$\mu 1$	$\mu 2$	$\mu 3$
Simply supported	3.1113694	6.2257783	9.3442528
Clamped-Clamped	4.7046164	7.8005252	10.915135
Clamped-Free	1.8601738	4.6558802	7.790633
Clamped-Pinned	3.8957558	7.0155609	10.1375497

also performed based on the presented reduced analytical approach. Comparing the DQM results and exact solutions, the applicability, accuracy, and convergence of the differential quadrature method are confirmed.

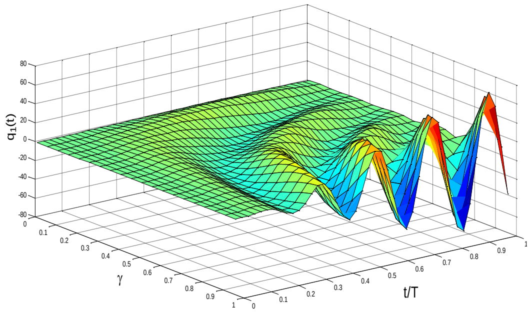
It should to be noted that for four cracks, a system of (4×(4+1)=20) algebraic equations have to be solved for the classical analytical method and only a 2×2 system for the reduced one. This leads to a large mathematical developments reduction. However, a hardly nonlinear transcendental equation has to be solved even for the reduced method.

Thus, for a large number of cracks, the classical analytical approach is not adequate since high size matrices have to be manipulated. The reduced analytical method is an alternative for a moderate number of cracks. For a high number of cracks, the presented DQM is more suited. These three methods are tested here for free vibration of beams containing seven cracks. Their relative depths and positions are presented in Table 4. The DQM shows superiority, effectiveness by its ease and reduced CPU time. The obtained numerical results based on the DQM are presented in Table 5.

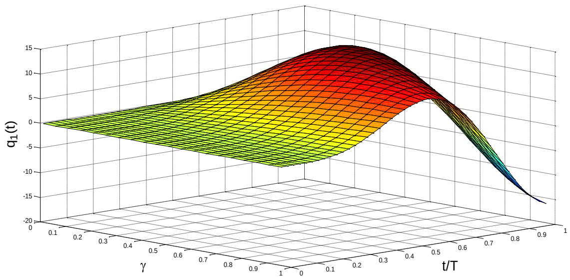
6.2 Forced vibration under harmonic moving load

The previously obtained eigenmodes are used to solve the forced vibration of the multi-cracked beam subjected to a moving harmonic excitation. The harmonic excitation is travelling along the beam at constant velocity V and with excitation frequency Ω. Figs. 3 and 4 depict the forced responses $q_1(t)$ and $q_2(t)$ with respect to normalized time t/T, ($\mathrm{T} = \frac{\mathrm{L}}{\mathrm{V}}$), and γ for different values of β and γ associated to a simply supported one cracked beam at mid span.

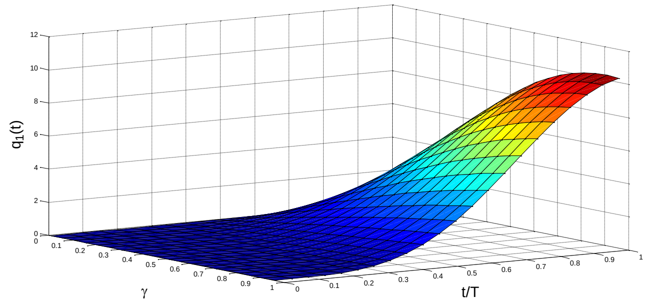
It is obvious that for high values of β (high travelling speed V), intervals of time will be reduced. In other hand, for small values of β, frequency effect of the applied excitation force seems to be predominant. In fact the less is the speed the more the beam is oscillating with high



(a) β=0.1

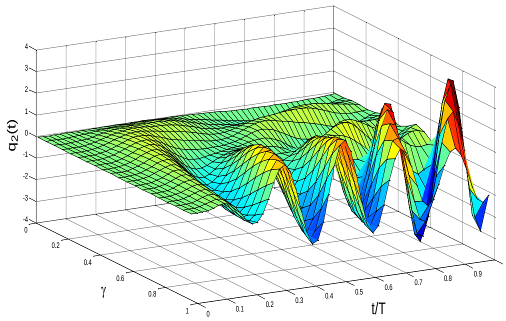


(b) β=0.5

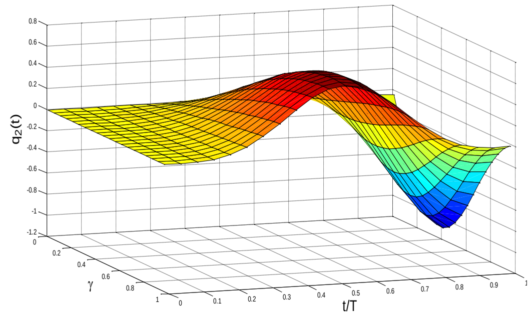


(c) β=1

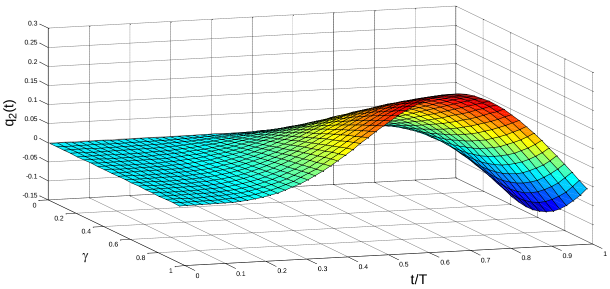
Fig. 3 Time response $q_1(t)$ of a simply supported beam at the mid span with respect to normalized time t/T ($T=L/V$) and frequency ratio γ



(a) β=0.1

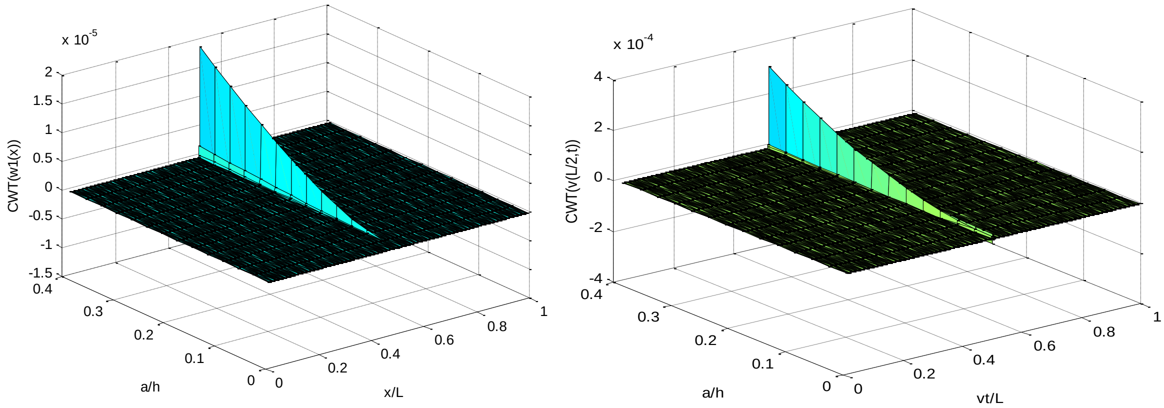


(b) β=0.5



(c) β=1

Fig. 4 Time response $q_2(t)$ of a simply supported beam at the mid span with respect to normalized time t/T ($T=L/V$) and frequency ratio γ



(a) Mode shape (b) Forced response at mid span β=0.1; γ=0.5

Fig. 5 Wavelet transform of the mode shape and forced response at mid-span

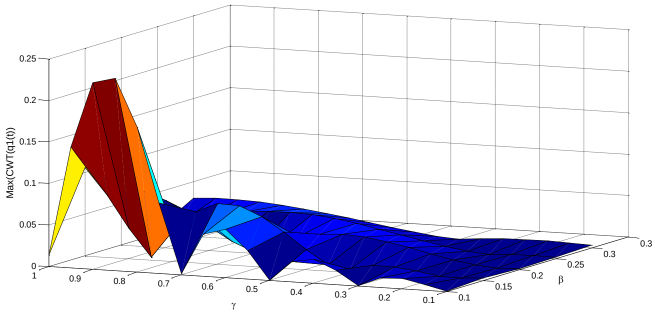


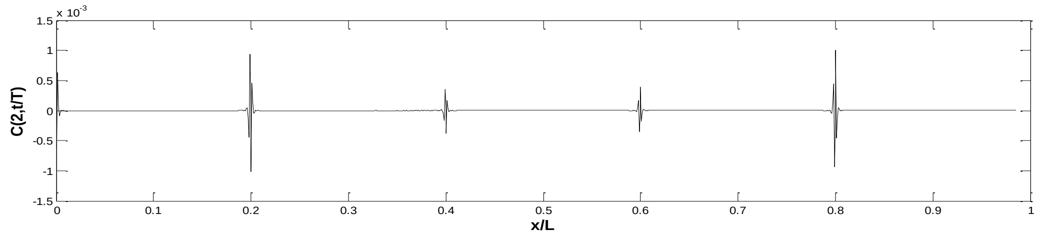
Fig. 6 Maximum of continuous wavelet transform coefficients of $q_1(t)$ with respect to β and γ

amplitudes. This kind of excitation allows the choosing three parameters F_0, β and γ for an amplified signal response that can be used for cracks detection.

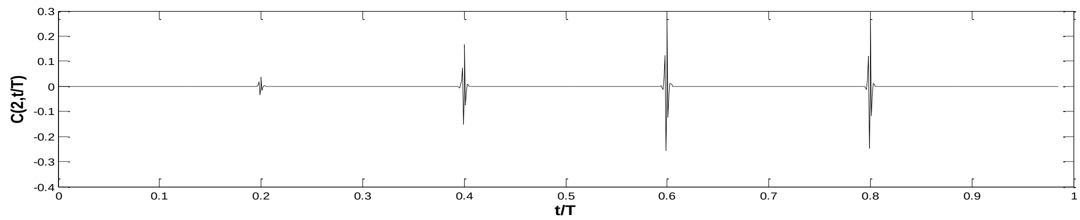
6.3 Cracks identification

The use of moving harmonic excitation aims the detection of small defects by analyzing the amplified response. The maximum of continuous wavelet transform coefficients at a given scale '*a*' is an indicator about the presence of the cracks. For sake of clarity, a numerical simulation has been performed for the aforementioned beam with a single crack at the mid span for β=0.1 and γ=0.5. Both of the first mode shape and the associated forced response have been wavelet transformed. A coefficient line plot at scale a=2 is plotted with respect to relative depths of the crack and either relative position x/L, or position of the moving load Vt/L.

It is clearly demonstrated from Fig. 5 that the coefficients of the forced response wavelet transformed, are 10 times larger than those of the mode shape. On the other hand, cracks with small relative depths (around a/h=0.01) are detected by analyzing the forced response time signal, which is not possible if the classical wavelet procedure based on the mode shapes is used. This allows one to perform a crack identification procedure based on an amplified forced response, by a judicious selection of the moving harmonic excitation parameters F_0, Ω, V.



(a) based on the first eigenmode



(b) based on forced response $q_1(t)$

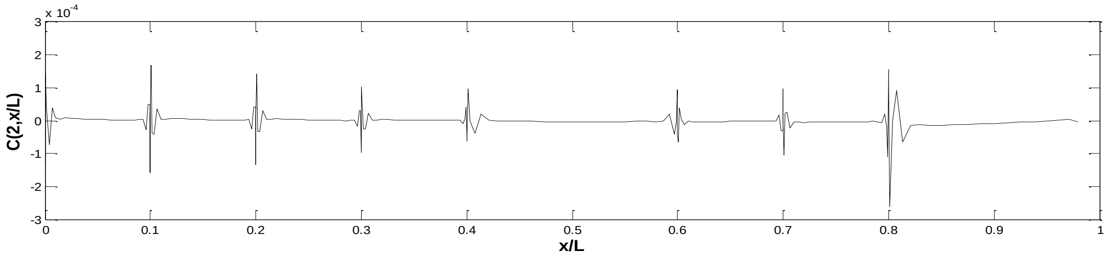
Fig. 7 Wavelet based multi-cracks identification

For more detailed study, impact of the moving harmonic excitation on the detection of damages in beams is investigated in order to find combinations (β_i, γ_i) that indicate better positions of cracks. For a simply supported beam with one crack at mid span, the maximum values of the continuous wavelet transform coefficient of $q_1(t)$ with respect to β and γ, are presented in Fig. 6. It is demonstrated that a better detection can be obtained for $0 \prec \beta \prec 0.2$ and $0.8 \prec \gamma \prec 0.9$. In this zone of ($\beta$, γ), the CWT ($q_1(t)$) is amplified. A deep analysis of can be conducted for optimized values if required.

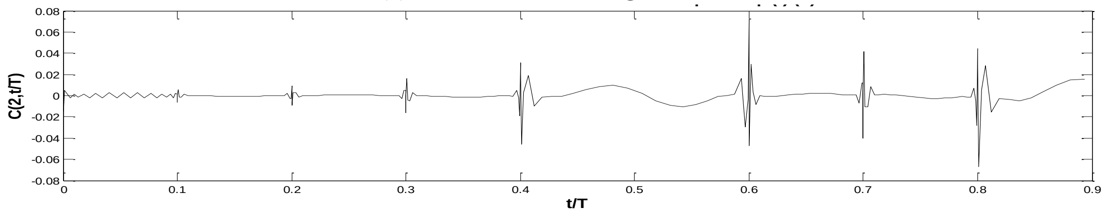
To demonstrate the effectiveness of this procedure, the concept of using moving harmonic excitation force cracks detection is applied to multi-cracked beams. Numerical simulations have been performed for a simply supported beam containing four cracks which positions and relative depths are listed in Table 2. Free and forced responses are obtained based on the DQM.

Fig. 7(a) displays the wavelet transform identification based on the first eigenmode. Note that the four cracks locations are identified, but the amplitude of the signal is too small, Fig. 7(b) presents the wavelet transform of the forced response $q_1(t)$ associated to a moving excitation with β=0.1, γ=0.85, $F_0=10^2$ kN. These results clearly demonstrate that the crack locations are better identified. The signal amplitude is around 10 times the previous one. This leads to a more practical crack identification procedure.

In order to show the robustness and the efficiency of proposed methodological approaches, the number of cracks was increased to 7 for which positions and relative depth ratios are shown in Table 4. The obtained signals were wavelet transformed based on first eigenmode (Fig. 8(a)) and on forced response (Fig. 8(b)) for $F_0=10^2$ kN, β=0.1 and γ=0.85. It is clearly demonstrated that the seven locations were accurately predicted by both procedures and the amplitude of the wavelet coefficient based on the eigenmode is around 1.5 10^{-4}. Furthermore, signals resulted from forced response give largely better detections of cracks locations. The obtained signal amplitude is increased and is around 100 times the eigenmode based one, as clearly presented in Fig. 8(b), by adjusting the harmonic moving excitation parameters (**Ω**, *V*), the wavelet coefficient amplitude, based on the time response, can be amplified.



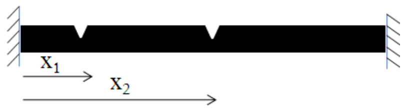
(a) based on the first eigenmode



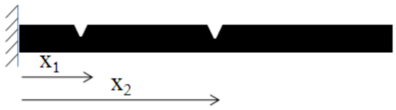
(b) based on forced response $q_1(t)$ for β=0.1 and γ=0.85

Fig. 8 Wavelet based multi-cracks identification

(a) Simply supported



(b) Clamped-Clamped



(c) Clamped-Free

Fig. 9 Double-cracked beam under considered boundary conditions

7. Conclusions

Vibration analysis of multi-cracked beams is investigated using analytical and numerical approaches. The presented analytical approaches are forwards but limited to few numbers of cracks. For an arbitrary number of cracks, the differential quadrature method was adapted. Accuracy, convergence and efficiency of the presented method were proved. Based on the modal expansion, forced responses due to a moving harmonic excitation are obtained for various boundary conditions. A new identification procedure based on the wavelet transform of the forced time response due to a moving harmonic excitation is proposed. This excitation is used as a moving actuator leading to an amplified response. It was demonstrated that the proposed procedure leads to a better cracks detection than the classical ones based on eigenfrequencies and the wavelet transforms of eigenmodes. The moving harmonic excitation parameters β and γ can be used as adjusters for the cracks detection amplitude. This new procedure can be adopted for health monitoring of structures.

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References

Abolbashari, M.H., Nazari, F. and Rad, J.S. (2014), "A multi-crack effects analysis and crack identification in functionally graded beams using particle swarm optimization algorithm and artificial neural network", *Struct. Eng. Mech*., **51**(2), 299-313

Al-Waily, M. (2013), "Theoretical and numerical vibration study of continuous beam with crack size and location effect", *Int. J. Innov. Res. Sci. Eng. Tech.*, **2**(9), 4166-4177.

Andreaus, U., Casini, P. and Vestroni, F. (2007), "Non-linear dynamics of a cracked cantilever beam under harmonic excitation", *Int. J. Nonlin. Mech.*, **42**(3), 566-575.

Behzad, M., Ebrahimi, A. and Meghdari, A. (2010), "A continuous vibration theory for beams with a vertical edge crack", *Scientia Iranica*, **17**(3), 194-204.

Christides, S. and Barr, A. (1984), "One dimensional theory of cracked Bernoulli-Euler beams", *Int. J. Mech. Sci.*, **26**(11-12), 639-648.

Dimarogonas, A.D. (1996), "Vibration of cracked structures: a state of the art review", *Eng. Fract. Mech.*, **55**(5), 831-857.

Doebling, S., Farrar, C., Prime, M. and Shevitz, D. (1996), "Damage identification and health monitoring of structural and mechanical systems from changes in their vibration characteristics: a literature review", Los Alamos National Laboratory Report.

Friswell, M. and Penny, J. (2002), "Crack modeling for structural health monitoring", *Struct. Hlth. Monit.*, **1**(2), 139-148.

Khorram, A., Bakhtiari-Nejad, F. and Rezaein, M. (2012), "Comparison studies between two wavelet based cracked detection methods of a beam subjected to a moving load", *Int. J. Eng. Sci.*, **51**, 204-215.

Khorram, A., Rezaien, M. and Bakhtiari-Nejad, F. (2013), "Multiple cracks detection in a beam subjected to a moving load using wavelet analysis with factorial design", *Eur. J. Mech. A/Solid.*, **40**, 97-113.

Li, Q.S. (2002), "Free vibration analysis of non uniform beams with an arbitrary number of cracks and concentrated masses", *J. Sound Vib.*, **252**(3), 509-525.

Loutridis, S., Douka, E. and Trochidis, A. (2004), "Crack identification in double-cracked beams using wavelet analysis", *J. Sound Vib.*, **277**(4), 1025-1039.

Nassar, M., Matbuly, M.S. and Ragb, O. (2012), "Vibration analysis of structural elements using differential quadrature method", *J. Adv. Res.*, **4**, 93-102.

Orhan, S. (2007), "Analysis of free and forced vibration of a cracked cantilever beam", *NDT&E Int.*, **40**, 443-450.

Ovanesova, A. and Suàrez, L. (2004), "Applications of wavelet transforms to damage detection in frame structures", *Eng. Struct.*, **26**, 39-49.

Papadopoulos, C.A. (2008), "The strain energy release approach for modeling cracks in rotors: a state of the art review", *Mech. Syst. Signal Pr.*, **22**, 763-789.

Ramon, S., Luciano, M. and Brito, M. (2012), "Determination of damages in beams using wavelet transforms", *Proceedings of the World Congress on Engineering*, 1211-1213.

Salawu, O. (1997), "Detection of structural damage through changes in frequency: a review", *Eng. Struct.*, **19**(9), 718-723.

Shifrin, E. and Ruotolo, R. (1999), "Natural frequencies of a beam with an arbitrary number of cracks", *J. Sound Vib.*, **222**(3), 409-423.

Tada, H., Paris, P.C. and R.Irwin, G. (2000), *The Stress Analysis of Cracks Handbook*, ASME Press, New

York.
Torabi, K., Afshari, H. and Haji Aboutalebi, F. (2014), "A DQEM for transverse vibration analysis of multiple cracked non-uniform Timoshenko beams with general boundary conditions", *Comput. Math. Appl.*, **67**(3), 527-541.
Vosoughi, A.R. (2015), "A developed hybrid method for crack identification of beams", *Smart Struct. Syst.*, **16**(3), 401-414.
Zhi, Z. and Yingyan, Z. (2008), *Advanced Differential Quadrature Methods*, CRC Press.
Zhu, X. and Law, S. (2006), "Wavelet-based crack identification of bridge beam from operational deflection time history", *Int. J. Solid. Struct.*, **43**, 2299-2317.

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Appendix

The commonly used approach for cracked beam vibration, called here classical approach, is first used to compute analytically $w_{ij}(x)$, The following analytical expression is used.

$$w_{ij}(x) = A_{ij}\sin(\mu_j x) + B_{ij}\cos(\mu_j x) + C_{ij}\sinh(\mu_j x) + D_{ij}\cos(\mu_j x) \tag{A.1}$$

where A_{ij}, B_{ij}, C_{ij}, and D_{ij} are constants to be determined from the considered boundary conditions and the sub-beams connections. For the sake of clarity, a double cracked beam (r=2) is considered and the mathematical developments are explicitly given for simply supported, clamped and clamped-free cases. The boundary conditions are:

For Simply Supported: $\begin{cases} w_{1j}(0) = 0\,;\, w''_{1j}(0) = 0 \\ w_{3j}(L) = 0\,;\, w''_{3j}(L) = 0 \end{cases}$

For Clamped-Clamped: $\begin{cases} w_{1j}(0) = 0\,;\, w'_{1j}(0) = 0 \\ w_{3j}(L) = 0\,;\, w'_{3j}(L) = 0 \end{cases}$

For Clamped -Free: $\begin{cases} w_{1j}(0) = 0\,;\, w'_{1j}(0) = 0 \\ w''_{3j}(L) = 0\,;\, w'''_{3j}(L) = 0 \end{cases}$

The internal nodes compatibility conditions Eq. (12) are used for i=1, 2 (r=2), the eigen mode is thus given by

$$w(x) = \sum_{i=1}^{3} w_{ij}(x)(H(x - x_{i-1}) - H(x - x_i)) \tag{A.2}$$

in which

$$\begin{cases} w_1(x) = A_1\sin(\mu x) + B_1\cos(\mu x) + C_1\sinh(\mu x) + D_1\cosh(\mu x) \\ w_2(x) = A_2\sin(\mu x) + B_2\cos(\mu x) + C_2\sinh(\mu x) + D_2\cosh(\mu x) \\ w_3(x) = A_3\sin(\mu x) + B_3\cos(\mu x) + C_3\sinh(\mu x) + D_3\cosh(\mu x) \end{cases} \tag{A.3}$$

The insertion of Eq. (A.3) in Eq. (12-a) and Eq. (12-b) leads to a (12×12) homogeneous algebraic linear system of matrix M. The eigenfrequencies, $\mu_j = \frac{\rho A L^4}{EI}\omega_j^2$ are obtained by solving the resulting nonlinear transcendental equation.

$$F(\mu_j) = \det(M) = 0 \tag{A.4}$$