Effect of higher order terms of Maclaurin expansion in non-linear analysis of the Bernoulli beam by single finite element

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Abstract. The second order analysis taking place due to non-linear behavior of the structures under the mechanical and geometric factors through implementing exact and approximate methods is an indispensible issue in the analysis of such structures. Among the exact methods is the slope-deflection method that due to its simplicity and efficiency of its relationships has always been in consideration. By solving the differential equations of the modified slope–deflection method in which the effect of axial compressive force is considered, the stiffness matrix including trigonometric entries would be obtained. The complexity of computations with trigonometric functions causes replacement with their Maclaurin expansion. In most cases only the first two terms of this expansion are used but to obtain more accurate results, more elements are needed. In this paper, the effect of utilizing higher order terms of Maclaurin expansion on reducing the number of required elements and attaining more rapid convergence with less error is investigated for the Bernoulli beam with various boundary conditions. The results indicate that when using only one element along the beam length, utilizing higher order terms in Maclaurin expansion would reduce the relative error in determining the critical buckling load and kinematic parameters in the second order analysis.

Keywords: non-linear behavior; slope-deflection method; axial compressive force; Maclaurin expansion; critical load; kinematic parameters

1. Introduction

Study of structural stability and related constituent parts has always been among important issues discussed in engineering science. Among various explanations for the stability, one is the concept of buckling. If a structure in the equilibrium condition is subjected to axial compressive loading or temperature variation, there is a possibility of buckling and related instability. Hence the problem of buckling has turned into an influencing factor in the analysis and design of structures like beam-columns, slender columns, trusses and shells. This is very important not only in dealing with macroscopic-scale structures, but also in design of structures with micro and nano scales

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(Eltaher *et al*. 2016), Structural analysis is performed linearly or nonlinearly. In linear analysis, the equilibrium equations are written based on the undeformed state of the member while the large deformations are ignored. In many cases this kind of analysis yields acceptable results but in reality the non-linear behavior of structural members under two general mechanical (such as materials characteristics, presence of crack, and time-variant characteristics of the material) and geometric categories, makes clear the necessity of non-linear analysis. When displacements in structural members due to their exerted loads are large so that the positions of the exerted loads change and new conditions in the equilibrium equations are established, then the geometric nonlinear analysis would be necessary. In this kind of analysis the stability, consistency and material criteria in the deformed state are established. These factors influence the stiffness and stability of each element in addition to that of the total structure (Aristizabal-Ochoa 2012), Investigation of pre and post buckling phases in addition to the moment at which buckling occurs is among advantages of non-linear analysis of structures.

Analysis of structures is basically conducted using stiffness and force methods in which the displacements and the forces are unknown, respectively (Samuelsson and Zienkiewicz 2006), Exact analysis requires direct integration of the equilibrium equations, also complete satisfaction of the boundary conditions that in cases such as non-linear analysis produces enormous complexities. Hence using numerical and approximate methods is widely spread in the first and second order analyses. The non-linear geometric elastic behavior in connection with structures comprises of Bernoulli beam and beam-columns are generally examined by two second-order Finite Element Method (Zienkiewicz 1971, Bathe 1996) and the classical stability functions method (Aristizabal-Ochoa 1997), In analysis of structures, issues such as decreasing the volume and time of the calculations in addition to maintaining accuracy of the results are of vital importance. This issue has urged the scientists to develop methods and to implement innovation in deriving stiffness matrix of the structure, so that one could reach the desired results by applying the methods with least number of the elements resulting in reducing degrees of freedom and finally easing the calculations,. For the first time Al-Bermani and Kitipornchai (1990) proposed a matrix which included the effect between axial strength and lateral-torsional deformations as functions of element deformation, so that in most cases using just one element one could find the response for the problems. Chan and Zhou (1994, 1995) presented a new formulation for non-linear analysis, based on the fifth order finite element method with only one element for deriving proper results and considering member imperfection.

IU and Bradford (2010), using higher order finite element including geometric non-linearity, with an updated Lagrangian formulation introduced a technique which was suitable for second order elastic analysis of plane and spatial steel structures using only one element per member. The rigid body movements are eliminated from the local load-displacement relationships and the total secant stiffness is formulated for evaluating the large member deformations. Second order analysis of plane frame using an element was performed by Balling and Lyon (2010), by concentration on non-linear geometry (large deformations) and considering the axial strains due to displacement and rotation of two ends of the member. Also, the sum of elastic and geometric stiffness matrices was converted to a tangent symmetric stiffness matrix which is in fact the differentiation of the overall resistance vector with respect to the overall displacements vector. The other approach is reducing the number of the elements, simplifying the geometric stiffness matrix to constant and compacted geometric matrices (Senjanović *et al*. 2012), the beam element with simply support and triangular and square plate elements is evaluated. The results indicate suitability of using simplified and compacted geometric stiffness matrix, simultaneously. Ibearugbulem *et al.* (2013) presented a new

matrix with less number of elements to analyze the structure. In addition to two ends of the beam element (two degrees of freedom), they considered a degree of freedom corresponding to the mid span displacement of the beam. Applying higher-order beam theory, Nguyen and Lee (2015) obtained appropriate results by analyzing deflections and stresses of cantilever beams considering shear deformation and the warping effects in bending.

In this article, to develop the research work and with respect to the advantages of slope-deflection such as its easy application in design, attempt is made to use less numbers of elements but increase the accuracy of the results. The slope-deflection method which is among stiffness methods was introduced by Wilson and Maney (1915) in 1915 by considering flexural deformation and ignoring axial and shear forces in investigation of beams with rigid connections. The slope-deflection equations for the Timoshenko beam including shear deformations and transverse loads were developed by Bryant and Baile (1977), In continuation researchers developed, modified and added further capabilities for this method such as variation in the cross section along the member length and accounting for connection flexibility (Ermopoulos 1988, Saffari *et al*. 2008, Aristizabal-Ochoa 2008), After that, the relationships were modified by accounting the axial force and the beam stiffness matrix entries were obtained using exact solution of the differential equation. The obtained entries were trigonometric functions which regarding their cumbersome computations they were replaced by Maclaurin expansions. The slope-deflection method, because of its simplicity and its high capability in the structural analysis among the stiffness methods known today, is taught in elementary analysis courses (Norris and Wilbur 1960, Kassimali 1998) and is also widely used in practice (Salmon and Johnson 1996) because it gives sufficient insight concerning internal moments and corresponding displacements. In common beam stiffness matrix derived from the slope-deflection relationships considering the effect of the axial force in reducing the stiffness, only the first two terms of Maclaurin expansion are taken into account and to attain a more exact response with less error there is a need to increase the number of elements which consequently increases the number of degrees of freedom and computation process.

Therefore in the present paper, the effect of higher order terms of the Maclaurin expansion to reduce the relative error in determination of critical buckling load and kinematic parameters in second order analysis of beams with varying boundary conditions is investigated. First the constituent functions of the stiffness matrix entries with varying number of Maclaurin expansion terms are extracted. To ensure the usefulness of increasing the number of the terms employed in increasing accuracy of the stiffness matrix, the convergence of the mentioned functions in both exact and approximate cases (replacement of trigonometric terms with the Maclaurin expansion) are evaluated using a few examples.

2. Modified slope-deflection relationships considering the axial force

Considering the axial force in slope-deflection equation, elastic relationship between force and nodal displacement is as follows

$$\mathbf{F} = \mathbf{KD} \tag{1}$$

in which the vector of degrees of freedom (**D**) and forces (**F**) are defined as

$$\mathbf{D} = \begin{bmatrix} u_1 & v_1 & \theta_1 & u_2 & v_2 & \theta_2 \end{bmatrix}^{\mathrm{T}} \qquad \mathbf{F} = \begin{bmatrix} p_1 & s_1 & m_1 & p_2 & s_2 & m_2 \end{bmatrix}^{\mathrm{T}} \tag{2}$$

where p_i, s_i, m_i, u_i, v_i, θ_i ($i=1,2$) are axial force, shear force, flexural moment, horizontal

displacement, vertical displacement and rotation, respectively. Considering the axial force in slope-deflection relationships, the stiffness matrix ([K]) for the beam element would be as follows (Lui and Chen 1987)

$$
[\mathbf{K}] = \begin{bmatrix}
\dfrac{EA}{L} & 0 & 0 & -\dfrac{EA}{L} & 0 & 0 \\[2mm]
 & q^* & \dfrac{k^*(1+c^*)}{L} & 0 & -q^* & \dfrac{k^*(1+c^*)}{L} \\[2mm]
 & & k^* & 0 & -\dfrac{k^*(1+c^*)}{L} & c^* k^* \\[2mm]
 & & & -\dfrac{EA}{L} & 0 & 0 \\[2mm]
 & syms. & & & q^* & -\dfrac{k^*(1+c^*)}{L} \\[2mm]
 & & & & & k^*
\end{bmatrix}
\tag{3}
$$

In which *EA/L*, k^* and c^* are the axial stiffness, flexural stiffness and modified transfer coefficient because of presence of axial compressive load, respectively. The parameter q^* is also defined by Eq. (4)

$$
q^* = \frac{2k^*(1+c^*)}{L^2} - \frac{p}{L}
\tag{4}
$$

in which *p* is axial compressive load of the beam. By assuming $\alpha = \sqrt{p/EI}\, L$, the stiffness matrix entries would be

$$
k^* = \frac{EI}{L} \cdot \frac{\dfrac{\alpha}{2}\left(1 - \alpha\cot(\alpha)\right)}{\tan\left(\dfrac{\alpha}{2}\right) - \dfrac{\alpha}{2}}
\tag{5-a}
$$

$$
c^* k^* = \frac{EI}{L} \cdot \frac{\dfrac{\alpha}{2}\left(\alpha\csc(\alpha) - 1\right)}{\tan\left(\dfrac{\alpha}{2}\right) - \dfrac{\alpha}{2}}
\tag{5-b}
$$

$$
\frac{k^*(1+c^*)}{L} = \frac{EI}{L^2} \cdot \frac{\dfrac{\alpha^2}{2}\tan\left(\dfrac{\alpha}{2}\right)}{\tan\left(\dfrac{\alpha}{2}\right) - \dfrac{\alpha}{2}}
\tag{5-c}
$$

$$
q^* = \frac{EI}{L^3} \cdot \frac{\dfrac{\alpha^3}{2}}{\tan\left(\dfrac{\alpha}{2}\right) - \dfrac{\alpha}{2}}
\tag{5-d}
$$

Using the above matrix in the analysis of structures would yield exact response. But matrix operations with the above mentioned trigonometric terms are complex. Therefore, attempt is made here to reach out results with acceptable level of accuracy implementing a more simple solution. One useful way is to use the Maclaurin series. Implementing the Maclaurin expansion terms would yield a good approximation of a function around the point $\alpha=0$ up to the radius of convergence. To attain an acceptable accuracy of a function by the Maclaurin expansion up to its radius of convergence, there is a need for implementing more terms. Each of the above functions based on the Maclaurin expansion could be presented in two forms. In the first form the Maclaurin expansion is written separately for the numerator and denominator of each function (Eq. (6)) (Lui and Chen 1987)

$$k^* = \frac{EI}{L} \cdot \frac{\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{(2n+1)!}\alpha^{2n-2}}{\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{(2n+2)!}\alpha^{2n-2}} \tag{6-a}$$

$$c^*k^* = \frac{EI}{L} \cdot \frac{\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(2n+1)!}\alpha^{2n-2}}{\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{(2n+2)!}\alpha^{2n-2}} \tag{6-b}$$

$$\frac{k^*(1+c^*)}{L} = \frac{EI}{L^2} \cdot \frac{\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(2n)!}\alpha^{2n-2}}{\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{(2n+2)!}\alpha^{2n-2}} \tag{6-c}$$

$$q^* = \frac{EI}{L^3} \cdot \frac{\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(2n-1)!}\alpha^{2n-2}}{\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{(2n+2)!}\alpha^{2n-2}} \tag{6-d}$$
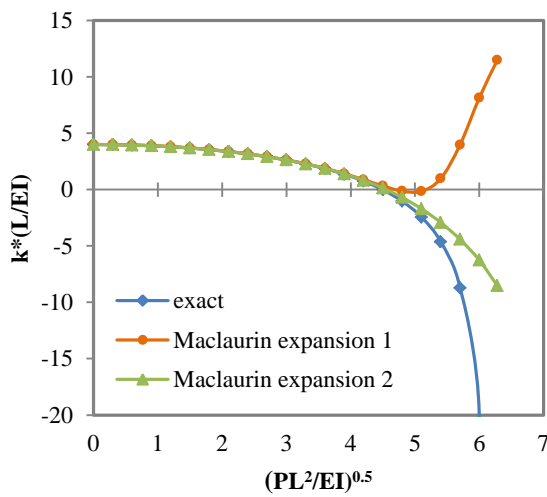
The other form of representing these functions is by implementing the general form of the Maclaurin expansion. The first six terms of the Maclaurin expansion with the radius of convergence of the above four functions are as following

$$k^* = \frac{4EI}{L} - \frac{2EI}{15L}\alpha^2 - \frac{11EI}{63\times10^2 L}\alpha^4 - \frac{EI}{27\times10^3 L}\alpha^6 - \frac{509EI}{58212\times10^4 L}\alpha^8$$
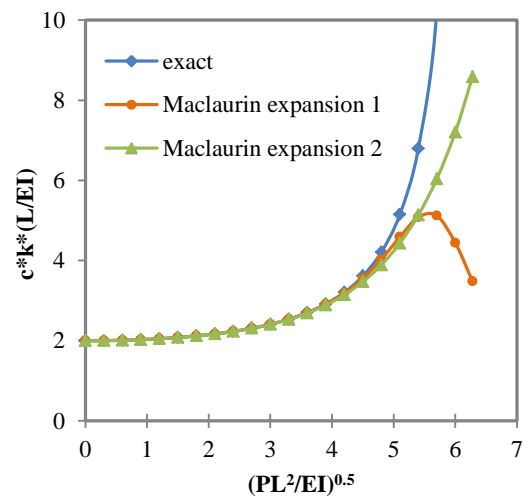$$- \frac{14617EI}{6810804\times10^5 L}\alpha^{10}... \qquad |\alpha| < 2\pi \tag{7-a}$$

$$c^*k^* = \frac{2EI}{L} + \frac{EI}{30L}\alpha^2 + \frac{13EI}{126\times10^2 L}\alpha^4 + \frac{11EI}{378\times10^3 L}\alpha^6 + \frac{907EI}{116424\times10^4 L}\alpha^8$$
$$+ \frac{27641EI}{13621608\times10^5 L}\alpha^{10}... \qquad |\alpha| < 2\pi \tag{7-b}$$

$$\frac{k^*\left(1+c^*\right)}{L} = \frac{6EI}{L^2} - \frac{EI}{10L^2}\alpha^2 - \frac{EI}{14\times10^2 L^2}\alpha^4 - \frac{EI}{126\times10^3 L^2}\alpha^6 - \frac{37EI}{38808\times10^4 L^2}\alpha^8$$

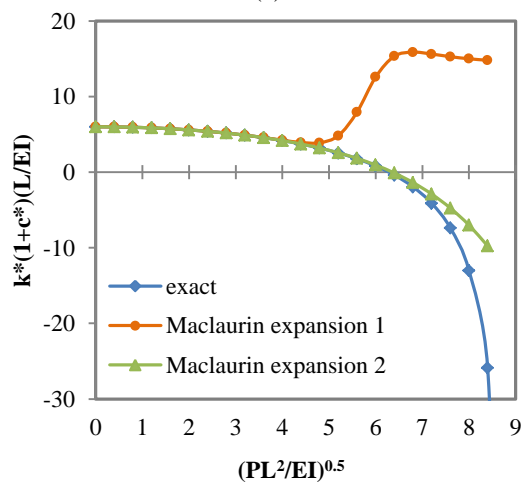$$- \frac{59EI}{504504\times10^5 L^2}\alpha^{10}... \qquad |\alpha| < R = 2.86\pi \tag{7-c}$$

$$q^* = \frac{12EI}{L^3} - \frac{6EI}{5L^3}\alpha^2 - \frac{EI}{7\times10^2 L^3}\alpha^4 - \frac{EI}{63\times10^3 L^3}\alpha^6 - \frac{37EI}{19404\times10^4 L^3}\alpha^8$$

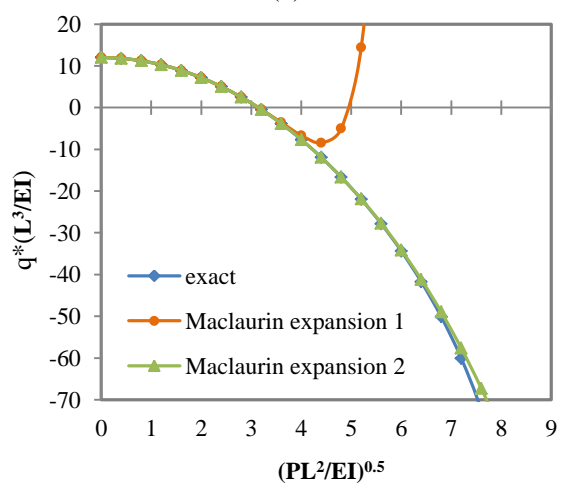$$- \frac{59EI}{252252\times10^5 L^3}\alpha^{10}... \qquad |\alpha| < R = 2.86\pi \tag{7-d}$$

Fig. 1 Comparison of convergence of two forms of Maclaurin expansion values per first 5 terms
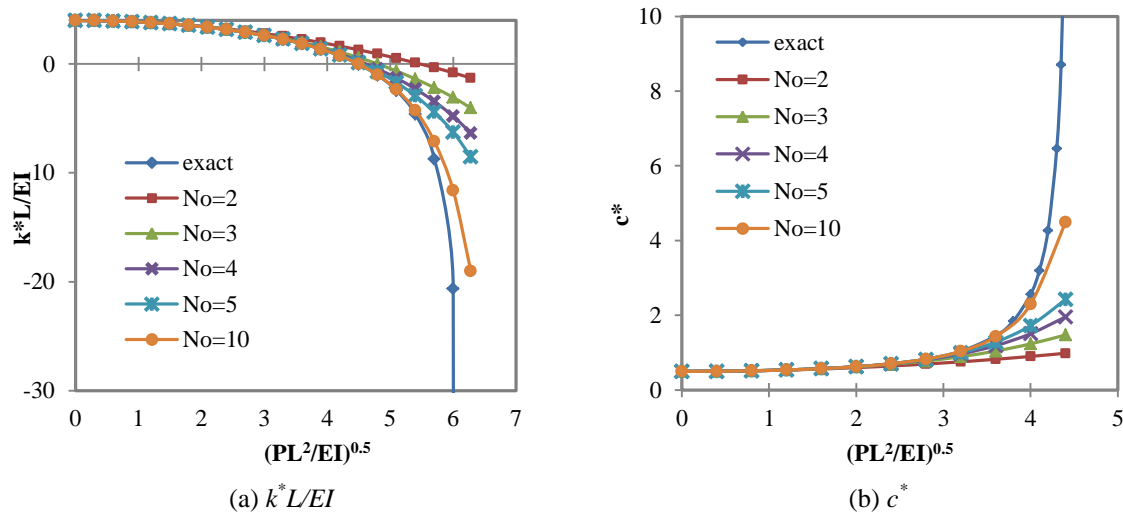
(a) k^*L/EI (b) c^*

Fig. 2 Comparison of exact diagram of functions with diagrams obtained from different number of terms of the second form of Maclaurin expansion; i.e., No

R, is the first root of the equation $\tan(\alpha/2)=\alpha/2$ which is approximately equal to 2.86π. In order to investigate the convergence of those two represented forms of the Maclaurin expansion of the mentioned functions, their differences with respect to the exact values are shown in Fig. 1. Dimensionless forms of the functions are used in these graphs.

In addition to the simplicity of the second form of the polynomial, as observed from the above figure in all functions, compatibility with the exact value in the second form is better achieved than that in the first form (per first 5 terms of the Maclaurin expansion), With respect to the advantages of implementing the second form of the Maclaurin expansion, in continuation its application effect on k^*L/EI and c^* function values would be examined.

Fig. 2, which is a comparison between exact diagram of k^*L/EI and c^* functions and diagrams obtained from the terms of the second form of the Maclaurin expansion, indicates that in case of using the Maclaurin expansion, with increasing the value of α, difference between the real and approximate values increases so that in proximity of the upper bound of the radius of convergence, to attain lower levels of error there is a need for higher order terms of the Maclaurin expansion. Therefore, the more the α value is close to zero, to attain better accuracy less number of terms in the Maclaurin expansion is needed.

In Tables 1 and 2 also the exact and approximate values of k^*L/EI and c^* for different number of terms of Maclaurin expansion, in the range of zero to vertical asymptote of them, are given.

As obtained from the results of the table, for a small α value, convergence to the exact response is attained rapidly and the more the α value increases, to attain higher accuracy more Maclaurin expansion terms are needed such that in regions close to the vertical asymptotes, this convergence is hardly attainable. Calculations have shown that in case that $p/p_E<0.4((\alpha/\pi))^2<0.4$ (p_E is the Euler buckling load equal to $\pi^2 EI/L^2$), the calculations results using two terms of the Maclaurin expansion would differ less than one percent from the real value. In case that $p/p_E>0.4$ the error would be more than one percent and when using two terms of the Maclaurin expansion to reduce this error, the beam is divided into several elements and the stiffness matrix is calculated for each member. The most critical case for determining the first mode of buckling load belongs to the

Table 1 Value of k^*L/EI with respect to the number of terms in the Maclaurin expansion (No)

k^*L/EI		α		
No	0.5π	π	1.5π	1.99999π
2	3.6710	2.6841	1.0391	-3.9850
3	3.6604	2.5140	0.1781	-3.9850
4	3.6598	2.4784	-0.2275	-6.2637
5	3.6598	2.4701	-0.4401	-8.3876
6	3.6598	2.4681	-0.5560	-10.4456
Exact	3.6598	2.4674	-0.7020	-199999

Table 2 Value of c^* with respect to the number of terms in the Maclaurin expansion (No)

c^*		α	
No	0.5π	π	1.43029π
2	0.5617	0.7647	1.0048
3	0.5697	0.8743	1.5386
4	0.5707	0.9382	2.0857
5	0.5708	0.9697	2.6379
6	0.5708	0.9852	3.1919
Exact	0.5708	1.0000	59662

beam fixed at both ends with the value of p/p_E equal to 4. In case of using 3 elements for the beam fixed at both ends, in stiffness matrix the length L is turned into $L/3$ and causes reduction of p/p_E from 4 to 0.44. In this case the error corresponding to the first two terms of the Maclaurin expansion would amount to less than one percent and yields acceptable results.

3. Elastic and geometric stiffness matrices

When implementing more elements to attain better accuracy, due to increasing degrees of freedom, the order of the matrix is increased which results in increased volume and time of calculations. Therefore to prevent such problem, in this research instead of using several elements, more terms of the Maclaurin expansion are used. By considering the first two terms of Maclaurin expansion for each matrix entry and substituting $\alpha = \sqrt{p/EI}\,L$, the stiffness matrix is separated into two elastic stiffness and geometric stiffness matrices (Eq. (8-a))

$$[\mathbf{K}] = [\mathbf{K}_E] + [\mathbf{K}_G] \tag{8-a}$$

$$k^* = \frac{4EI}{L} - \frac{2pL}{15} - \ldots \tag{8-b}$$

$$c^*k^* = \frac{2EI}{L} + \frac{pL}{30} + \ldots \tag{8-c}$$

$$\frac{k^*\left(1+c^*\right)}{L}=\frac{6EI}{L^2}-\frac{p}{10}-\cdots \tag{8-d}$$

$$q^*=\frac{12EI}{L^3}-\frac{6p}{5L}-\cdots \tag{8-e}$$

The first terms of Maclaurin expansion make the elastic stiffness matrix ($[\mathbf{K}_E]$) which is in fact the same stiffness matrix without presence of the axial load.

$$[\mathbf{K}_E]=\frac{EI}{L^3}\begin{bmatrix} \dfrac{AL^2}{I} & 0 & 0 & -\dfrac{AL^2}{I} & 0 & 0 \\[2mm] & 12 & 6L & 0 & -12 & 6L \\[2mm] & & 4L^2 & 0 & -6L & 2L^2 \\[2mm] & & & \dfrac{AL^2}{I} & 0 & 0 \\[2mm] & syms. & & & 12 & -6L \\[2mm] & & & & & 4L^2 \end{bmatrix} \tag{9}$$

The second terms in the Maclaurin expansion also are the matrix entries of the geometric stiffness matrix in which the effects of stiffness reduction due to the axial compressive load are considered in the elastic stiffness matrix. Zero values of the 1st and 4th row and column in the geometric stiffness matrix indicate that the axial load induces no change in the axial stiffness of the members and only causes stiffness change at axes perpendicular to the member axis.

$$[\mathbf{K}_G]=\frac{p}{L}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\[2mm] & -\dfrac{6}{5} & -\dfrac{L}{10} & 0 & \dfrac{6}{5} & -\dfrac{L}{10} \\[2mm] & & -\dfrac{2L^2}{15} & 0 & \dfrac{L}{10} & \dfrac{L^2}{30} \\[2mm] & & & 0 & 0 & 0 \\[2mm] & syms. & & & -\dfrac{6}{5} & \dfrac{L}{10} \\[2mm] & & & & & -\dfrac{2L^2}{15} \end{bmatrix} \tag{10}$$

Implementing more terms of the Maclaurin expansion, results in stiffness matrices with more complete terms. Using the 3rd term of the Maclaurin expansion, the stiffness matrix is separated as shown in Eq. (11) in which [$\mathbf{K}'_G$] matrix corresponds to 3rd terms of the Maclaurin expansion. By application of higher order terms of Maclaurin expansion, the number of geometric stiffness matrices would increase.

$$[\mathbf{K}]=[\mathbf{K}_E]+[\mathbf{K}_G]+[\mathbf{K}'_G] \tag{11-a}$$

$$[\mathbf{K}_G'] = \frac{p^2 L}{EI}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ & -\dfrac{1}{700} & -\dfrac{L}{1400} & 0 & \dfrac{1}{700} & -\dfrac{L}{1400} \\ & & -\dfrac{11L^2}{6300} & 0 & \dfrac{L}{1400} & \dfrac{13L^2}{12600} \\ & & & 0 & 0 & 0 \\ & syms. & & & -\dfrac{1}{700} & \dfrac{L}{1400} \\ & & & & & -\dfrac{11L^2}{6300} \end{bmatrix} \quad (11\text{-}b)$$

4. P-Delta effects

The P-delta effects include two parts of $p{-}\Delta$ and $p{-}\delta$. The effect of $p{-}\Delta$ induces an extra moment due to eccentricity of axial load p with respect to its lateral displacement. On the other hand, presence of the axial load reduces the stiffness of the beam which is called the $p{-}\delta$ effect. By eliminating the row and column related to axial force and displacement of stiffness matrix (1) and separating the geometric stiffness matrix into three following matrices, the effects of $p{-}\Delta$ and $p{-}\delta$ are shown in separate matrices (Eq. (12))

$$\begin{bmatrix} s_1 & m_1 & s_2 & m_2 \end{bmatrix}^T = \begin{bmatrix} K_E + K_G \end{bmatrix}\begin{bmatrix} v_1 & \theta_1 & v_2 & \theta_2 \end{bmatrix}^T$$
$$[\mathbf{K}_G] = \begin{bmatrix} \mathbf{K}_{G_1} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{G_2} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{G_3} \end{bmatrix} \quad (12\text{-}a)$$

$$\begin{bmatrix} \mathbf{K}_{G_1} \end{bmatrix} = \begin{bmatrix} -\dfrac{p}{5L} & -\dfrac{p}{10} & \dfrac{p}{5L} & -\dfrac{p}{10} \\ & -\dfrac{pL}{15} & \dfrac{p}{10} & -\dfrac{pL}{30} \\ & & -\dfrac{p}{5L} & \dfrac{p}{10} \\ syms. & & & -\dfrac{pL}{15} \end{bmatrix} \quad (12\text{-}b)$$

$$\begin{bmatrix} \mathbf{K}_{G_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & -\dfrac{pL}{15} & 0 & \dfrac{pL}{15} \\ & & 0 & 0 \\ syms. & & & -\dfrac{pL}{15} \end{bmatrix} \quad (12\text{-}c)$$

$$[\mathbf{K}_{G_3}] = \begin{bmatrix} -\dfrac{p}{L} & 0 & \dfrac{p}{L} & 0 \\ & 0 & 0 & 0 \\ & & -\dfrac{p}{L} & 0 \\ \text{syms.} & & & 0 \end{bmatrix} \tag{12-d}$$

Combination of $[K_E]$ and $[K_{G_1}]$ matrices indicates the $p-\delta$ effect (Eq. (13))

$$[\mathbf{K}_E] + [\mathbf{K}_{G_1}] = \begin{bmatrix} \dfrac{12EI}{L^3} - \dfrac{p}{5L} & \dfrac{6EI}{L^2} - \dfrac{p}{10} & -\dfrac{12EI}{L^3} + \dfrac{p}{5L} & \dfrac{6EI}{L^2} - \dfrac{p}{10} \\ & \dfrac{4EI}{L} - \dfrac{pL}{15} & -\dfrac{6EI}{L^2} + \dfrac{p}{10} & \dfrac{2EI}{L} - \dfrac{pL}{30} \\ & & \dfrac{12EI}{L^3} - \dfrac{p}{5L} & -\dfrac{6EI}{L^2} + \dfrac{p}{10} \\ \text{syms.} & & & \dfrac{4EI}{L} - \dfrac{pL}{15} \end{bmatrix} \Rightarrow$$

$$[\mathbf{K}_E] + [\mathbf{K}_{G_1}] = \left(1 - \dfrac{pL^2}{60EI}\right) \begin{bmatrix} \dfrac{12EI}{L^3} & \dfrac{6EI}{L^2} & -\dfrac{12EI}{L^3} & \dfrac{6EI}{L^2} \\ & \dfrac{4EI}{L} & -\dfrac{6EI}{L^2} & \dfrac{2EI}{L} \\ & & \dfrac{12EI}{L^3} & -\dfrac{6EI}{L^2} \\ \text{syms.} & & & \dfrac{4EI}{L} \end{bmatrix} = \left(1 - \dfrac{pL^2}{60EI}\right)\mathbf{K}_E \tag{13}$$

Reduction in the elastic stiffness due to the presence of the axial compressive load is obtained through the $(1-(pL^2/60EI))$ coefficient. The stiffness matrix $[K_{G_2}]$ which corresponds to θ_1 and θ_2 degrees of freedom has negligible effect in the analysis results of the frames, because in frames usually $\theta_1 \approx \theta_2$, consequently

$$m_1 = \dfrac{pL}{15}\theta_1 - \dfrac{pL}{15}\theta_2 \approx 0$$

$$m_2 = -\dfrac{pL}{15}\theta_1 + \dfrac{pL}{15}\theta_2 \approx 0 \tag{14}$$

And finally the $[K_{G_3}]$ matrix is related to the $p-\Delta$ effect. Utilizing more terms of the Maclaurin expansion in the geometric stiffness matrix only causes changes in the entries of $[K_{G_1}]$ and $[K_{G_2}]$ matrices increasing the accuracy of the stiffness reduction coefficient due to presence of the axial load. For example, using 3rd term of the Maclaurin expansion modifies the stiffness reduction coefficient and concerning the $[K_{G_2}]$ matrix causes addition of another term to its non-

zero entries.

$$\left[1 - \frac{pL^2}{60EI} - \frac{p^2 L^4}{8400 (EI)^2} \right] \tag{15}$$

$$\left[\mathbf{K}_{G_2} \right] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & -\frac{pL}{15} - \frac{2p^2 L^3}{1575EI} & 0 & \frac{pL}{15} + \frac{2p^2 L^3}{1575EI} \\ & & 0 & 0 \\ syms. & & & -\frac{pL}{15} - \frac{2p^2 L^3}{1575EI} \end{bmatrix} \tag{16}$$

5. Numerical study

5.1 Determining the critical load

The critical buckling load for a two-end simply supported beam is equal to $n^2\pi^2 EI/L^2$ (Timoshenko and Gere 1961) that for $n=1,2$ its critical buckling loads in the first and second modes would be obtained. The critical buckling load value is determined by solving the eigenvalue problem $[K]\{D\}=0$. If determinant of the stiffness matrix is zero $|\mathbf{K}|=0$, the equation has non-trivial answer. By assuming a single element and using two terms of the Maclaurin expansion, the critical load value is determined as follows

$$\begin{bmatrix} \frac{EA}{L} & 0 & 0 & -\frac{EA}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} - \frac{6p}{5L} & \frac{6EI}{L^2} - \frac{p}{10} & 0 & -\frac{12EI}{L^3} + \frac{6p}{5L} & \frac{6EI}{L^2} - \frac{p}{10} \\ 0 & \frac{6EI}{L^2} - \frac{p}{10} & \frac{4EI}{L} - \frac{2pL}{15} & 0 & -\frac{6EI}{L^2} + \frac{p}{10} & \frac{2EI}{L} + \frac{pL}{30} \\ -\frac{EA}{L} & 0 & 0 & \frac{EA}{L} & 0 & 0 \\ 0 & -\frac{12EI}{L^3} + \frac{6p}{5L} & -\frac{6EI}{L^2} + \frac{p}{10} & 0 & \frac{12EI}{L^3} - \frac{6p}{5L} & -\frac{6EI}{L^2} + \frac{p}{10} \\ 0 & \frac{6EI}{L^2} - \frac{p}{10} & \frac{2EI}{L} + \frac{pL}{30} & 0 & -\frac{6EI}{L^2} + \frac{p}{10} & \frac{4EI}{L} - \frac{2pL}{15} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \theta_1 \\ u_2 \\ 0 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{17}$$

By selecting the 3rd and 6th rows

$$\begin{bmatrix} \frac{4EI}{L} - \frac{2pL}{15} & \frac{2EI}{L} + \frac{pL}{30} \\ \frac{2EI}{L} + \frac{pL}{30} & \frac{4EI}{L} - \frac{2pL}{15} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{18}$$

The critical load is obtained when determinant of the above matrix is equal to zero

$$|\mathbf{K}| = 0 \quad \Rightarrow \quad \left(\frac{4EI}{L} - \frac{2pL}{15}\right)^2 - \left(\frac{2EI}{L} + \frac{pL}{30}\right)^2 = 0 \quad \Rightarrow \quad \begin{cases} \dfrac{2EI}{L} = \dfrac{5pL}{30} \quad \Rightarrow \quad p_{cr_1} = \dfrac{12EI}{L^2} \\ \dfrac{6EI}{L} = \dfrac{pL}{10} \quad \Rightarrow \quad p_{cr_2} = \dfrac{60EI}{L^2} \end{cases} \tag{19}$$

The error percentage for the first and second modes would be

$$e = \frac{\dfrac{12EI}{L^2} - \dfrac{\pi^2 EI}{L^2}}{\dfrac{\pi^2 EI}{L^2}} \times 100 = 21.59\% \qquad , \qquad e = \frac{\dfrac{60EI}{L^2} - \dfrac{4\pi^2 EI}{L^2}}{\dfrac{4\pi^2 EI}{L^2}} \times 100 = 51.98\% \tag{20}$$

In case of adding the third term of the Maclaurin expansion, the first two modes of the critical buckling load would be

$$\begin{bmatrix} \dfrac{4EI}{L} - \dfrac{2pL}{15} - \dfrac{11p^2 L^3}{6300EI} & \dfrac{2EI}{L} + \dfrac{pL}{30} + \dfrac{13p^2 L^3}{12600EI} \\ \dfrac{2EI}{L} + \dfrac{pL}{30} + \dfrac{13p^2 L^3}{12600EI} & \dfrac{4EI}{L} - \dfrac{2pL}{15} - \dfrac{11p^2 L^3}{6300EI} \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{21-a}$$

$$|\mathbf{K}| = 0 \quad \Rightarrow \quad \left(\frac{4EI}{L} - \frac{2pL}{15} - \frac{11p^2 L^3}{6300EI}\right)^2 - \left(\frac{2EI}{L} + \frac{pL}{30} + \frac{13p^2 L^3}{12600EI}\right)^2 = 0 \quad \Rightarrow$$

$$\begin{cases} \dfrac{2EI}{L} - \dfrac{pL}{6} - \dfrac{p^2 L^3}{360EI} = 0 \\ \dfrac{6EI}{L} - \dfrac{pL}{10} - \dfrac{p^2 L^3}{1400EI} = 0 \end{cases} \tag{21-b}$$

Assuming $\dfrac{pL^2}{EI} = \beta$

$$\begin{cases} 2 - \dfrac{\beta}{6} - \dfrac{\beta^2}{360} = 0 \quad \Rightarrow \quad \beta = 10.2492 \quad \Rightarrow \quad p_{cr_1} = \dfrac{10.2492EI}{L^2} \\ 6 - \dfrac{\beta}{10} - \dfrac{\beta^2}{1400} = 0 \quad \Rightarrow \quad \beta = 45.3256 \quad \Rightarrow \quad p_{cr_2} = \dfrac{45.3256EI}{L^2} \end{cases} \tag{22}$$

And the error percentage is equal to

$$e = \frac{\dfrac{10.249EI}{L^2} - \dfrac{\pi^2 EI}{L^2}}{\dfrac{\pi^2 EI}{L^2}} \times 100 = 3.84\% \qquad \Rightarrow \qquad e = \frac{\dfrac{45.3256EI}{L^2} - \dfrac{4\pi^2 EI}{L^2}}{\dfrac{4\pi^2 EI}{L^2}} \times 100 = 14.81\% \tag{23}$$

Using 4 terms of the mentioned expansion

$$
\begin{bmatrix}
\dfrac{4EI}{L} - \dfrac{2pL}{15} - \dfrac{11p^2L^3}{6300EI} - \dfrac{p^3L^5}{27000(EI)^2} & \dfrac{2EI}{L} + \dfrac{pL}{30} + \dfrac{13p^2L^3}{12600EI} + \dfrac{11p^3L^5}{378000(EI)^2} \\[4mm]
\dfrac{2EI}{L} + \dfrac{pL}{30} + \dfrac{13p^2L^3}{12600EI} + \dfrac{11p^3L^5}{378000(EI)^2} & \dfrac{4EI}{L} - \dfrac{2pL}{15} - \dfrac{11p^2L^3}{6300EI} - \dfrac{p^3L^5}{27000(EI)^2}
\end{bmatrix}
\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\tag{24-a}
$$

$$
|\mathbf{K}| = 0 \;\Rightarrow\; \left(\dfrac{4EI}{L} - \dfrac{2pL}{15} - \dfrac{11p^2L^3}{6300EI} - \dfrac{p^3L^5}{27000(EI)^2} \right)^2 - \left(\dfrac{2EI}{L} + \dfrac{pL}{30} + \dfrac{13p^2L^3}{12600EI} + \dfrac{11p^3L^5}{378000(EI)^2} \right)^2 = 0
\tag{24-b}
$$

$$
\Rightarrow
\begin{cases}
\dfrac{2EI}{L} - \dfrac{pL}{6} - \dfrac{p^2L^3}{360EI} - \dfrac{p^3L^5}{15120(EI)^2} = 0 & \Rightarrow & 2 - \dfrac{\beta}{6} - \dfrac{\beta^2}{360} - \dfrac{\beta^3}{15120} = 0 \\[4mm]
\dfrac{6EI}{L} - \dfrac{pL}{10} - \dfrac{p^2L^3}{1400EI} - \dfrac{p^3L^5}{126000(EI)^2} = 0 & \Rightarrow & 6 - \dfrac{\beta}{10} - \dfrac{\beta^2}{1400} - \dfrac{\beta^3}{126000} = 0
\end{cases}
\tag{24-c}
$$

$$
\Rightarrow \quad \beta = 9.9562 \qquad \Rightarrow \qquad p_{cr_1} = \dfrac{9.9562EI}{L^2}
$$

$$
\Rightarrow \quad \beta = 41.7619 \qquad \Rightarrow \qquad p_{cr_1} = \dfrac{41.7619EI}{L^2}
\tag{24-d}
$$

And the error percentage in this case is as follows

$$
e = \dfrac{\dfrac{9.9562EI}{L^2} - \dfrac{\pi^2EI}{L^2}}{\dfrac{\pi^2EI}{L^2}} \times 100 = 0.88\% \qquad \Rightarrow \qquad e = \dfrac{\dfrac{41.7619EI}{L^2} - \dfrac{4\pi^2EI}{L^2}}{\dfrac{4\pi^2EI}{L^2}} \times 100 = 5.78\%
\tag{25}
$$

Using 5 terms, the error for the first mode is 0.21% and for the second mode is 2.52%. The relative error for 3 different boundary conditions: S-S(simple-simple), C-F(clamped-free) and C-S(clamped-simple) per a single element and the number of the Maclaurin expansion terms are shown in Table 3.

Noticing Table 3, the following results are deducted:

(1) For all boundary conditions, an increase in the number of Maclaurin expansion terms reduces the relative error. Meanwhile the relative error per initial terms compared with the higher order terms, decreases with a sharper slope.

(2) With higher number of the degrees of freedom or lower critical load, convergence to the exact response is faster attained. For example in the C-F beam, per 4 expansion terms, the first mode critical load up to 4th decimal place is compatible with the exact solution. While with larger critical load (larger α value) more Maclaurin expansion terms are needed for convergence. The largest relative error per 7 terms for the second mode critical load of the C-F beam which has the largest α value with respect to the other cases has been obtained.

(3) The number of buckling modes is obtained based on the existing degrees of freedom of the beam. For the case C-S, due to the only rotational degree of freedom of the simply support per

Table 3 Variation in the relative error with increasing the number of the Maclaurin expansion terms (No) for the beam with different boundary conditions

Boundary condition	Mode number	$(p_{cr}L^2/EI)_{\text{Exact}}$	Errors of $(p_{cr}L^2/EI)(\%)$					
			No.					
			2	3	4	5	6	7
S-S	1	9.8696	21.5854	3.8461	0.8774	0.2127	0.0526	0.0131
	2	39.4784	51.9818	14.8111	5.7839	2.5267	1.1631	0.5501
C-F	1	2.4674	0.7497	0.0405	0.0000	0.0000	0.0000	0.0000
	2	22.2066	44.9150	13.6247	5.8212	2.8324	1.4680	0.7876
C-S	1	20.1907	48.5830	14.1380	5.6841	2.5708	1.2296	0.6056

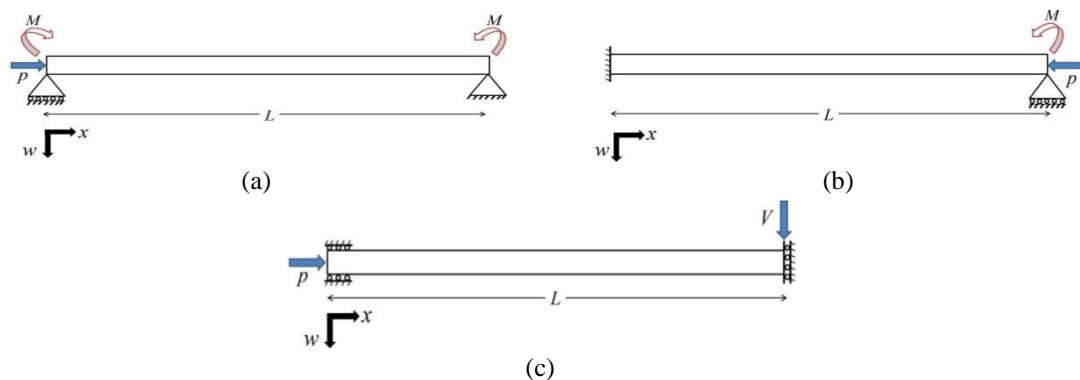
(a)   (b)

(c)

Fig. 3 Schematic views of the beams being studied

one single element, only its first mode is attainable. Therefore to attain other modes, more elements should be utilized. It should be noted that in the first step of matrix analysis of the beam with two-end-clamped condition (C-C), at least two elements are needed.

5.2 Second order analysis of the beam

Using higher order Maclaurin expansion terms for determining the kinematic parameters in a second order analysis could result in increased accuracy. In continuation, to investigate the effect of the number of Maclaurin expansion terms in the non-linear analysis, the kinematic parameters of 3 types of beams with various boundary conditions subjected to the loadings according to Fig. 3 would be compared in two exact and approximate cases.

The beam displacement relationship is obtained by solution of moment equilibrium differential equation and applying the boundary conditions (for determining response parameters of the differential equation), The moment equilibrium equation for the general case regarding each beam being studied is according to Eq. (26)

$$EI \frac{d^2w(x)}{dx^2} + p\,w(x) = f(x) \tag{26}$$

$f(x)$ is the internal moment due to the transverse loading. Assuming $p/EI=\lambda^2$ and solving the

differential equation and replacing the boundary conditions, the vertical displacement functions for each model are obtained as follows

$$w(x) = \frac{ML^2}{EI}\left[\frac{\cos(\lambda x)-1}{(\lambda L)^2}+\frac{1-\cos(\lambda L)}{(\lambda L)^2\sin(\lambda L)}\sin(\lambda x)\right] \tag{27-a}$$

$$w(x) = \frac{ML^2}{EI}\left\{\frac{\left[\lambda L-\sin(\lambda L)\right]\left[\cos(\lambda x)-1\right]+\left[\cos(\lambda L)-1\right]\left[\sin(\lambda x)-\lambda x\right]}{(\lambda L)^2\sin(\lambda L)-(\lambda L)^3\cos(\lambda L)}\right\} \tag{27-b}$$

$$w(x) = \frac{VL^3}{EI}\left\{\frac{\sin(\lambda x)}{(\lambda L)^3}-\frac{1}{(\lambda L)^2}\left(\frac{x}{L}\right)+\frac{1-\cos(\lambda L)}{(\lambda L)^3\sin(\lambda L)}\left[1-\cos(\lambda x)\right]\right\} \tag{27-c}$$

The angle change for end beams no. 1 and 2 and displacement of the end beam no. 3, respectively, are in the following order

$$w'(L) = \frac{ML}{EI}\left[\frac{\cos(\lambda L)-1}{\lambda L\sin(\lambda L)}\right] \tag{28-a}$$

$$w'(L) = \frac{ML}{EI}\left[\frac{2-2\cos(\lambda L)-\lambda L\sin(\lambda L)}{\lambda L\sin(\lambda L)-(\lambda L)^2\cos(\lambda L)}\right] \tag{28-b}$$

$$w(L) = \frac{VL^3}{EI}\left[\frac{2-2\cos(\lambda L)-\lambda L\sin(\lambda L)}{(\lambda L)^3\sin(\lambda L)}\right] \tag{28-c}$$

Another method for exact determination of kinematic parameters in nodal points of a beam is to utilize the stiffness matrix obtained from the slope-deflection method. Implementing fewer number of the Maclaurin expansion terms in entries of this matrix, instead of the original functions, causes distancing of kinematic parameters from the exact values. Assuming the material and geometric properties also the loading values as follows, the relative errors for the number of the Maclaurin expansion terms for beams of Fig. 3 are given in Table 4.

$L=200$ cm $\qquad\qquad A=25\pi$ cm^2 $\qquad\qquad EI=312.5\pi\times10^6$ kg/cm^2

$p=2\times10^5$ kg $\qquad\qquad V=2000$ kg $\qquad\qquad M=5\times10^4$ kg.cm

Table 4 The relative errors of kinematic parameters with respect to the number of Maclaurin expansion terms (No)

	Model	Exact	Errors of kinematic parameters (%)			
			No. of Maclaurin expansion terms			
			2	3	4	5
w' (rad)	1	0.0246	28.74	9.72	1.13	0.40
	2	0.0036	4.84	0.90	0.20	0.05
w (rad)	3	7.6979	4.70	0.4482	0.04	0.01

As the results in Table 4 show, implementing larger number of the Maclaurin expansion terms has reduced the error. In case of using 5 terms, with one element, the error is reduced to less than 0.5%. On the other hand, the initial terms were more effective in reducing the error compared to the higher order terms.

6. Conclusions

Non-linear analysis of structures to provide greater compatibility with reality has always been in consideration studied using exact and approximate methods. One of these exact methods is the slope-deflection method which has the advantage of being straightforward and deriving its relationships is simple. Considering the effect of axial compressive load on stiffness reduction and using the mentioned method, the stiffness matrix of the beam with trigonometric entries is obtained and to reduce computational complexity they are replaced by limited number of the Maclaurin expansion terms. In most problems only the first two terms are used and to attain a suitable response, more elements are utilized.

In this paper, in order to reduce the number of elements which results in reduced volume and time of computations, the effect of higher order terms of the Maclaurin expansion is investigated to attain rapid convergence and meanwhile obtain results with less relative error. The numerical results indicated that using higher order terms of Maclaurin expansion has caused error reduction in determining the critical buckling load and kinematic parameters of the non-linear analysis. Also with larger axial loads, to attain a constant accuracy, there is a need for larger number of the Maclaurin expansion terms.

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