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Description of reversed yielding in thin hollow discs subject to external pressure

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Abstract. This paper presents an elastic/plastic model that neglects strain hardening during loading, but accounts for the Bauschinger effect. These mathematical features of the model represent reasonably well the actual behavior of several materials such as high strength steels. Previous attempts to describe the behavior of this kind of materials have been restricted to a class of boundary value problems in which the state of stress in the plastic region is completely controlled by the yield stress in tension or torsion. In particular, the yield stress is supposed to be constant during loading and the forward plastic strain reduces the yield stress to be used to describe reversed yielding. The new model generalizes this approach on plane stress problems assuming that the material obeys the von Mises yield criterion during loading. Then, the model is adopted to describe reversed yielding in thin hollow discs subject to external pressure.

Keywords: bauschinger effect; Mises yield criterion; thin disc; new material model

1. Introduction

In the case of loading-unloading processes the elastic range is often reduced with an accompanying Bauschinger effect. A comprehensive overview of theories that accounts for the Bauschinger effect has been given in Rees (1981). The present paper is devoted to materials which show little or no forward hardening (Franklin and Morrison 1960, Milligan *et al.* 1966, Findley and Reed 1983, Rees 2006). Most previous solutions for such materials are restricted to boundary value problems in which the state of stress in the plastic region is completely controlled by the yield stress in tension or torsion (Rees 2007, 2009, Alexandrov and Hwang, 2011, Gui *et al.* 2015). In this case, there is no need to account for the effect of forward plastic strain on the yield criterion. For a class of boundary value problems, Tresca's yield criterion can be modified to take into account both perfectly plastic material behavior at loading and the Bauschinger effect (Chen, 1986, Alexandrov *et al.* 2016). The state of stress in such boundary value problems corresponds to

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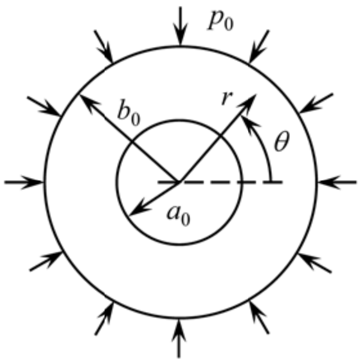

Fig. 1 Disc under external pressure-notation

one face of the yield surface at loading and the opposite face in reversed yielding. Therefore, in fact, the state of stress in the plastic region is completely controlled by the yield stress as in the aforementioned works. The approach developed in Chen (1986) and Alexandrov *et al.* (2016) is not applicable to smooth yield surfaces. The model proposed in the present paper generalizes this approach. In particular, it combines the von Mises yield criterion with a constant yield stress at the stage of loading and Prager's law (Prager 1956) at the stage of reversed yielding. The model is restricted to plane stress problems.

The model proposed is used to describe reversed yielding in thin hollow discs subject to external pressure. This is one of the classical problems of plasticity. A comprehensive overview of available solutions of this problem is provided in Cohen *et al.* (2009). A mathematical advantage of the problem under consideration, as compared to somehow a similar boundary value problem for thin hollow discs subject to internal pressure, is that the elastic strain rates vanish in the plastic region. Therefore, the solutions based on flow and deformation theories of plasticity coincide. The solution is semi-analytic. A numerical technique is only necessary to solve transcendental equations.

2. Statement of the problem

Consider a thin hollow disc of initial yield stress σ_0, Poisson's ratio v, Young's modulus E, outer radius b_0, and inner radius a_0 subject to uniform pressure p_0 over its outer radius (Fig. 1).

The disc has no stress at p_0=0. Strains are supposed to be infinitesimal. The state of stress is two dimensional (σ_z=0) in a cylindrical coordinate system (r,θ,z) with its z axis coinciding with the axis of symmetry of the disc. Here σ_z is the axial stress (σ_r and σ_θ will stand for the radial and circumferential stresses, respectively). The boundary value problem is axisymmetric and its solution is independent of θ. The circumferential displacement vanishes everywhere. The normal stresses in the cylindrical system of coordinates are the principal stresses. The boundary conditions at loading are

$$\sigma_r = 0 \tag{1}$$

for r=a_0 and

$$\sigma_r = -p_0 \tag{2}$$

for $r=b_0$. It is assumed that p_0 is large enough to initiate plastic yielding in the disc. Therefore, the disc consists of two regions, elastic and plastic. The constitutive equations in the elastic region are

$$\varepsilon_r^e = \frac{\sigma_r - \nu\sigma_\theta}{E}, \quad \varepsilon_\theta^e = \frac{\sigma_\theta - \nu\sigma_r}{E}, \quad \varepsilon_z^e = -\frac{\nu\left(\sigma_r + \sigma_\theta\right)}{E}. \tag{3}$$

The superscript e denotes the elastic part of the strain. In the elastic region the whole strain is elastic. The superscript e is employed in (3) as the same equations are satisfied by the elastic part of the strain in the plastic region. The superscript can be dropped in the elastic region. Many materials show little or no forward hardening (Rees 2006). However, the yield stress significantly decreases with a stress reversal (Bauschinger effect). There are boundary value problems in which it is sufficient to assume that the yield stress is constant at loading and depends on the forward equivalent strain at reversed yielding to represent such behavior of materials (Rees 2007, 2009, Alexandrov and Hwang 2011, Gui *et al.* 2015). However, in most cases it is necessary to propose constitutive equations that predict that the material behaves as perfectly plastic at loading and its behaviour at reversed loading accounts for the Bauschinger effect. It is evident that the classical constitutive equations, such as Prager's law (Prager 1956), are not capable of predicting this kind of material behavior. In Chen (1986) a generalization of Tresca's yield criterion has been proposed to satisfy the aforementioned requirement. However, this model is valid if the regime of loading corresponds to one face of the yield surface and the regime of reversed loading to the opposite face. Therefore, this approach is restricted to a certain class of boundary value problems and is not applicable for smooth yield surfaces. In the present paper, it is assumed that the original material obeys the von Mises yield criterion. The corresponding yield surface does not change its shape and position as the deformation progresses in the regime of loading. However, it is assumed that there is another yield surface whose position is controlled by Prager's law (Prager 1956). Therefore, the position of this surface depends on the forward plastic strain. This surface controls reversed yielding. Both surfaces are shown in Fig. 2 (the solid line corresponds to the original von Mises yield surface and the broken line to the yield surface that controls reversed yielding). A possible strain path is also illustrated in this figure. The associated flow rule is used in conjunction with each of the yield criteria. In the case under consideration the original von Mises yield criterion is

$$\sigma_r^2 + \sigma_\theta^2 - \sigma_\theta\sigma_r = \sigma_0^2. \tag{4}$$

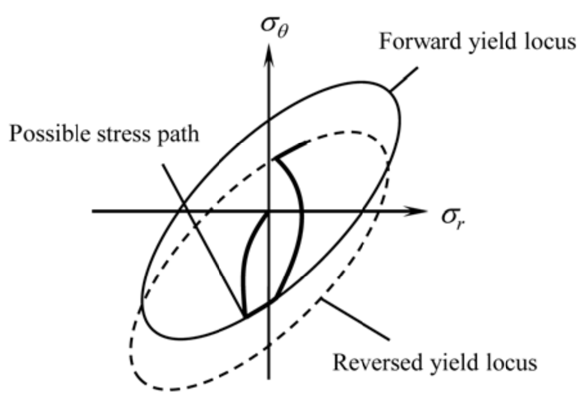

Fig. 2 Forward and reversed yield loci

Here σ_0 is the tensile yield stress, a material constant. The solid line in Fig. 2 corresponds to Eq. (4). According to Prager law (Prager 1956) the yield criterion is

$$\frac{3}{2}\left[\left(s_r-\alpha_r\right)^2+\left(s_\theta-\alpha_\theta\right)^2+\left(s_z-\alpha_z\right)^2\right]=\sigma_0^2 \tag{5}$$

where

$$s_r=\sigma_r-\sigma\,,\; s_\theta=\sigma_\theta-\sigma\,,\; s_z=\sigma_z-\sigma=-\sigma\,,\; \sigma=\left(\sigma_r+\sigma_\theta+\sigma_z\right)/3=\left(\sigma_r+\sigma_\theta\right)/3 \tag{6}$$

and

$$\alpha_r=C\varepsilon_r^p,\quad \alpha_\theta=C\varepsilon_\theta^p,\quad \alpha_z=C\varepsilon_z^p \tag{7}$$

where $\varepsilon_r^p, \varepsilon_\theta^p, \varepsilon_z^p$ are the plastic strains and C is a material constant. Substituting Eqs. (6) and (7) into Eq. (5) leads to

$$\sigma_r^2+\sigma_\theta^2-\sigma_r\sigma_\theta+\sigma_r\left(\alpha_z+\alpha_\theta-2\alpha_r\right)+\sigma_\theta\left(\alpha_z+\alpha_r-2\alpha_\theta\right)+\frac{3}{2}\left(\alpha_r^2+\alpha_\theta^2+\alpha_z^2\right)=\sigma_0^2\,. \tag{8}$$

Since the material is plastically incompressible, $\varepsilon_r^p+\varepsilon_\theta^p+\varepsilon_z^p=0$. Then, it follows from Eq. (7) that $\alpha_r+\alpha_\theta+\alpha_z=0$. Using this equation it is possible to transform Eq. (8) to

$$T_r^2+T_\theta^2-T_rT_\theta=\sigma_0^2 \tag{9}$$

where

$$T_r=\sigma_r-\alpha_r+\alpha_z,\quad T_\theta=\sigma_\theta-\alpha_\theta+\alpha_z\,. \tag{10}$$

The flow rule associated with the von Mises yield criterion is

$$\dot{\varepsilon}_r^p=\lambda s_r,\quad \dot{\varepsilon}_\theta^p=\lambda s_\theta,\quad \dot{\varepsilon}_z^p=\lambda s_z. \tag{11}$$

Here $\dot{\varepsilon}_r^p$, $\dot{\varepsilon}_\theta^p$ and $\dot{\varepsilon}_z^p$ are the plastic strain rates and λ is a non-negative multiplier. The flow rule associated with the yield criterion (5) is

$$\dot{\varepsilon}_r^p=\lambda_1\left(s_r-\alpha_r\right),\quad \dot{\varepsilon}_\theta^p=\lambda_1\left(s_\theta-\alpha_\theta\right),\quad \dot{\varepsilon}_z^p=\lambda_1\left(s_z-\alpha_z\right) \tag{12}$$

where λ_1 is a non-negative multiplier.

The only nontrivial equilibrium equation reduces to

$$\frac{\partial\sigma_r}{\partial r}+\frac{\sigma_r-\sigma_\theta}{r}=0. \tag{13}$$

It is convenient to introduce the following dimensionless quantities

$$a=\frac{a_0}{b_0},\quad \rho=\frac{r}{b_0},\quad k=\frac{\sigma_0}{E},\quad c=\frac{C}{E},\quad p=\frac{p_0}{\sigma_0}. \tag{14}$$

Also, the material model is rate-independent. Therefore, it is possible to replace the strain rates with strain derivatives with respect to any time-like parameter, q. In particular, denote

$$\xi_r^p=\frac{\partial\varepsilon_r^p}{\partial q},\quad \xi_\theta^p=\frac{\partial\varepsilon_\theta^p}{\partial q},\quad \xi_z^p=\frac{\partial\varepsilon_z^p}{\partial q}. \tag{15}$$

Then, Eqs. (11) and (12) can be rewritten as

$$\xi_r^p = \lambda_2 s_r, \quad \xi_\theta^p = \lambda_2 s_\theta, \quad \xi_z^p = \lambda_2 s_z \tag{16}$$

and

$$\xi_r^p = \lambda_3 \left(s_r - \alpha_r\right), \quad \xi_\theta^p = \lambda_3 \left(s_\theta - \alpha_\theta\right), \quad \xi_z^p = \lambda_3 \left(s_z - \alpha_z\right), \tag{17}$$

respectively. Here $\lambda = \lambda_2 \, dq/dt$ and $\lambda_1 = \lambda_3 \, dq/dt$ where t is the time.

3. Loading

The solution at loading is purely analytic (Alexandrov 2015). For completeness, this solution is briefly discussed in this section.

3.1 Elastic solution

The general elastic solution is well known (see, for example, Hill 1950). Using Eq. (14) this solution is represented as

$$\begin{gathered} \frac{\sigma_r}{\sigma_0} = \frac{A}{\rho^2} + B, \quad \frac{\sigma_\theta}{\sigma_0} = -\frac{A}{\rho^2} + B, \\ \frac{\varepsilon_r}{k} = \frac{A(1+\nu)}{\rho^2} + B(1-\nu), \quad \frac{\varepsilon_\theta}{k} = -\frac{A(1+\nu)}{\rho^2} + B(1-\nu), \quad \frac{\varepsilon_z}{k} = -2\nu B. \end{gathered} \tag{18}$$

Here A and B are constants of integration. In the case of purely elastic solution these constants are found from the boundary conditions (1) and (2). Substituting these boundary conditions into Eq. (18) and using Eq. (14) lead to $A=A_e=pa^2(1-a^2)^{-1}$ and $B=B_e=-p(1-a^2)^{-1}$. The distribution of stresses and strains in the purely elastic disc follows from Eq. (18) in which A and B should be replaced with A_e and B_e, respectively. Substituting this stress solution into Eq. (4) shows that the plastic region starts to develop from the hole at $p=p_e=(1-a^2)/2$. In what follows, it is assumed that $p>p_e$.

3.2 Elastic/plastic solution for stress

At $p_p>p>p_e$ the disc consists of two regions, elastic and plastic. Here p_p is the value of p at which the entire disc becomes plastic. This value should be found from the elastic/plastic solution. Let ρ_c be the dimensionless radius of the elastic/plastic boundary. Consider the plastic region, $a \le \rho \le \rho_c$. In this region, the distribution of stresses is found from Eqs. (4) and (13). In particular, Eq. (4) is satisfied by the following substitution

$$\frac{\sigma_r}{\sigma_0} = -\frac{2\sin\psi}{\sqrt{3}}, \quad \frac{\sigma_\theta}{\sigma_0} = -\frac{\sin\psi}{\sqrt{3}} - \cos\psi. \tag{19}$$

Eqs. (13) and (19) combine to give

$$2\rho\cos\psi\frac{\partial\psi}{\partial\rho}=\sqrt{3}\cos\psi-\sin\psi. \tag{20}$$

The boundary condition to this equation follows from Eqs. (1) and (19). In particular, $\psi=0$ or $\psi=\pi$ at $r=a_0$. It is evident that $\sigma_\theta<0$ at $r=a_0$. Therefore, using Eq. (14) the boundary condition to Eq. (20) can be represented as $\psi=0$ for $\rho=a$. The solution of Eq. (20) satisfying this boundary condition is

$$\rho=\frac{\sqrt{\sqrt{3}a}}{\sqrt{2}\sqrt{\sin(\pi/3-\psi)}}\exp\left(\frac{\sqrt{3}}{2}\psi\right). \tag{21}$$

Eqs. (19) and (21) supply the radial distribution of the radial and circumferential stresses in the plastic region in parametric form with ψ being the parameter. Let ψ_c be the value of ψ at the elastic/plastic boundary. Then, it follows from Eq. (21) that

$$\rho_c=\frac{\sqrt{\sqrt{3}a}}{\sqrt{2}\sqrt{\sin(\pi/3-\psi_c)}}\exp\left(\frac{\sqrt{3}}{2}\psi_c\right). \tag{22}$$

The stress solution given in Eq. (18) is valid in the elastic region, $\rho_c\leq\rho\leq 1$. However, $A\neq A_e$ and $B\neq B_e$. This stress solution must satisfy the boundary condition (2). Therefore, it follows from Eqs. (14) and (18) that

$$A+B=-p \tag{23}$$

and

$$\frac{\sigma_r}{\sigma_0}=A\left(\frac{1}{\rho^2}-1\right)-p,\quad \frac{\sigma_\theta}{\sigma_0}=-A\left(\frac{1}{\rho^2}+1\right)-p \tag{24}$$

in the elastic region. Both the radial and circumferential stresses should be continuous across the elastic/plastic boundary (Hill 1950). Therefore, it follows from Eqs. (19) and (24) that

$$\frac{2\sin\psi_c}{\sqrt{3}}=A\left(1-\frac{1}{\rho_c^2}\right)+p,\quad \frac{\sin\psi_c}{\sqrt{3}}+\cos\psi_c=A\left(1+\frac{1}{\rho_c^2}\right)+p. \tag{25}$$

It is convenient to put $q=\psi_c$ in Eq. (15). Eqs. (22) and (25) determine p and A as functions of ψ_c. In particular

$$A=\frac{a^2}{2}\exp\left(\sqrt{3}\psi_c\right),\quad p=\sin\left(\frac{\pi}{6}+\psi_c\right)-\frac{a^2}{2}\exp\left(\sqrt{3}\psi_c\right). \tag{26}$$

Substituting Eq. (26) into Eq. (24) gives

$$\frac{\sigma_r}{\sigma_0}=\frac{a^2}{2\rho^2}\exp\left(\sqrt{3}\psi_c\right)-\sin\left(\frac{\pi}{6}+\psi_c\right),\quad \frac{\sigma_\theta}{\sigma_0}=-\frac{a^2}{2\rho^2}\exp\left(\sqrt{3}\psi_c\right)-\sin\left(\frac{\pi}{6}+\psi_c\right) \tag{27}$$

in the elastic region. The entire disc is plastic when $\rho_c=1$. Let ψ_p be the value of ψ_c when $\rho_c=1$. The equation for ψ_p follows from Eq. (22) in the form

$$\frac{\sqrt{3}a^2}{2\sin\left(\pi/3-\psi_p\right)}\exp\left(\sqrt{3}\psi_p\right)=1\,. \tag{28}$$

This equation should be solved numerically. Then, the value of p_p is found from Eq. (26) as

$$p_p=\sin\left(\frac{\pi}{6}+\psi_p\right)-\frac{a^2}{2}\exp\left(\sqrt{3}\psi_p\right)\,. \tag{29}$$

In what follows, it is assumed that $p_p>p>p_e$.

3.3 Elastic/plastic solution for strain

The distribution of strains in the elastic region is determined from Eqs. (3), (14) and (27) as

$$\begin{gathered}\frac{\varepsilon_r}{k}=\frac{a^2(1+\nu)}{2\rho^2}\exp\left(\sqrt{3}\psi_c\right)-(1-\nu)\sin\left(\frac{\pi}{6}+\psi_c\right),\\ \frac{\varepsilon_\theta}{k}=-\frac{a^2(1+\nu)}{2\rho^2}\exp\left(\sqrt{3}\psi_c\right)-(1-\nu)\sin\left(\frac{\pi}{6}+\psi_c\right),\\ \frac{\varepsilon_z}{k}=2\nu\sin\left(\frac{\pi}{6}+\psi_c\right).\end{gathered} \tag{30}$$

Substituting Eq. (19) into Eq. (3) and using Eq. (14) yield

$$\frac{\varepsilon_r^e}{k}=\frac{(\nu-2)\sin\psi}{\sqrt{3}}+\nu\cos\psi,\quad \frac{\varepsilon_\theta^e}{k}=-\frac{(1-2\nu)}{\sqrt{3}}\sin\psi-\cos\psi,\quad \frac{\varepsilon_z^e}{k}=2\nu\sin\left(\frac{\pi}{6}+\psi\right) \tag{31}$$

in the plastic region. Eqs. (31) and (21) supply the radial distribution of the elastic strains in the plastic region in parametric form with ψ being the parameter. Eliminating s_r, s_θ and s_z in Eq. (16) by means of Eqs. (6) and (19) gives

$$\xi_r^p=-\frac{2}{3}\lambda_2\sin\left(\psi-\frac{\pi}{6}\right),\quad \xi_\theta^p=-\frac{2}{3}\lambda_2\cos\psi,\quad \xi_z^p=\frac{2}{3}\lambda_2\sin\left(\psi+\frac{\pi}{6}\right).$$

Eliminating λ_2 between these equations results in

$$\xi_r^p=\frac{\sin(\psi-\pi/6)}{\cos\psi}\xi_\theta^p,\quad \xi_z^p=-\frac{\sin(\psi+\pi/6)}{\cos\psi}\xi_\theta^p. \tag{32}$$

It is seen from Eq. (21) that ψ is a function only ρ. Therefore, Eq. (32) can be immediately integrated with respect to ψ_c to give

$$\varepsilon_r^p=\frac{\sin(\psi-\pi/6)}{\cos\psi}\varepsilon_\theta^p,\quad \varepsilon_z^p=-\frac{\sin(\psi+\pi/6)}{\cos\psi}\varepsilon_\theta^p. \tag{33}$$

Also, since ψ is independent of ψ_c then the elastic strains in the plastic region given by Eq. (31) are independent of the time. Therefore, the elastic strain rates vanish in the plastic region. In this case the equation of strain rate compatibility is equivalent to

$$\rho\frac{\partial\xi_\theta^p}{\partial\rho}=\xi_r^p-\xi_\theta^p.$$

Replacing in this equation differentiation with respect to ρ with differentiation with respect to ψ by means of Eq. (20) and eliminating ξ_r^p by means of Eq. (32) lead to

$$\frac{\partial\xi_\theta^p}{\partial\psi}=-\sqrt{3}\xi_\theta^p. \tag{34}$$

The total circumferential strain rate should be continuous across the elastic/plastic boundary. Therefore

$$\xi_\theta^p=\frac{\partial\varepsilon_\theta^e}{\partial\psi_c}\equiv\xi_\theta^e \tag{35}$$

at $\psi=\psi_c$. It is understood here that ξ_θ^p is calculated on the plastic side of the elastic/plastic boundary and $\xi_\theta^e=\partial\varepsilon_\theta^e/\partial\psi_c$ on the elastic side of the elastic/plastic boundary. It follows from Eq. (30) that

$$\frac{\xi_\theta^e}{k}=-\frac{\sqrt{3}a^2(1+\nu)}{2\rho^2}\exp\left(\sqrt{3}\psi_c\right)-(1-\nu)\cos\left(\frac{\pi}{6}+\psi_c\right).$$

Then, the value of ξ_θ^e on the elastic side of the elastic/plastic boundary is given by

$$\frac{\xi_\theta^e}{k}=-\frac{\sqrt{3}a^2(1+\nu)}{2\rho_c^2}\exp\left(\sqrt{3}\psi_c\right)-(1-\nu)\cos\left(\frac{\pi}{6}+\psi_c\right). \tag{36}$$

Eliminating here ρ_c by means of Eq. (22) and substituting the resulting expression into Eq. (35) supply the boundary condition to Eq. (34) in the form

$$\xi_\theta^p=2k\sin\left(\psi_c-\frac{\pi}{3}\right).$$

for $\psi=\psi_c$. The solution of Eq. (34) satisfying this boundary condition is

$$\frac{\xi_\theta^p}{k}=2\sin\left(\psi_c-\frac{\pi}{3}\right)\exp\left[\sqrt{3}\left(\psi_c-\psi\right)\right]. \tag{37}$$

Let ψ_m be the value of ψ_c at $\rho_c=\rho_m$. Then, it follows from Eq. (22) that the equation for ψ_m is

$$\rho_m^2=\frac{\sqrt{3}a^2}{2\sin\left(\pi/3-\psi_m\right)}\exp\left(\sqrt{3}\psi_m\right). \tag{38}$$

This equation should be solved numerically. In order to find the radial distribution of ε_θ^p at $\rho_c=\rho_m$, it is necessary to integrate Eq. (37) with respect to ψ_c taking into account that $\varepsilon_\theta^p=0$ at $\psi=\psi_c$. As a result

$$\frac{\varepsilon_\theta^p}{k}=\cos\psi-\cos\psi_m\exp\left[\sqrt{3}\left(\psi_m-\psi\right)\right]. \tag{39}$$

Substituting Eq. (39) into Eq. (33) yields

$$\frac{\varepsilon_r^p}{k} = \frac{\sin(\psi - \pi/6)}{\cos\psi}\left\{\cos\psi - \cos\psi_m \exp\left[\sqrt{3}(\psi_m - \psi)\right]\right\},$$
$$\frac{\varepsilon_z^p}{k} = -\frac{\sin(\psi + \pi/6)}{\cos\psi}\left\{\cos\psi - \cos\psi_m \exp\left[\sqrt{3}(\psi_m - \psi)\right]\right\}. \tag{40}$$

Eqs. (39), (40) and (21) supply the radial distribution of the plastic strains in the plastic region in parametric form with ψ being the parameter. The distribution of the total strains can be found by substituting Eqs. (31), (39) and (40) into the equations $\varepsilon_r = \varepsilon_r^e + \varepsilon_r^p$, $\varepsilon_\theta = \varepsilon_\theta^e + \varepsilon_\theta^p$, $\varepsilon_z = \varepsilon_z^e + \varepsilon_z^p$.

4. Unloading

Using Eqs. (7), (10), (39) and (40) it is possible to find

$$T_r = \sigma_r - \sqrt{3}Ck\tan\psi\left\{\cos\psi - \cos\psi_m \exp\left[\sqrt{3}(\psi_m - \psi)\right]\right\},$$
$$T_\theta = \sigma_\theta - \sqrt{3}Ck\frac{\sin(\psi + \pi/3)}{\cos\psi}\left\{\cos\psi - \cos\psi_m \exp\left[\sqrt{3}(\psi_m - \psi)\right]\right\}. \tag{41}$$

Substituting Eq. (41) into Eq. (9) leads to the yield criterion in reversed yielding. It is evident that this criterion depends on ρ and the value of the elastic/plastic radius at the end of loading. The stresses involved in Eq. (41) are determined as

$$\sigma_r = \sigma_r^f + \Delta\sigma_r, \quad \sigma_\theta = \sigma_\theta^f + \Delta\sigma_\theta. \tag{42}$$

Here σ_r^f and σ_θ^f are the radial and circumferential stresses, respectively, at the end of loading. These stresses have been found in the previous section. $\Delta\sigma_r$ and $\Delta\sigma_\theta$ are the increments of the radial and circumferential stresses, respectively, in course of the process of unloading. Analogously, the total strains at the end of unloading are determined as

$$\varepsilon_r = \varepsilon_r^f + \Delta\varepsilon_r, \quad \varepsilon_\theta = \varepsilon_\theta^f + \Delta\varepsilon_\theta, \quad \varepsilon_z = \varepsilon_z^f + \Delta\varepsilon_z. \tag{43}$$

Here ε_r^f, ε_θ^f and ε_z^f are the total radial, circumferential and axial strains, respectively, at the end of loading. These strains have been found in the previous section. $\Delta\varepsilon_r$, $\Delta\varepsilon_\theta$ and $\Delta\varepsilon_z$ are the increments of the radial, circumferential and axial strains, respectively, in course of the process of unloading.

The boundary conditions imposed on the increments of the radial stress are

$$\Delta\sigma_r = 0 \tag{44}$$

for $\rho = a$ and

$$\Delta\sigma_r = \sigma_0 p_m \tag{45}$$

for $\rho = 1$. Here p_m is the value of p at the end of loading. It is seen from Eq. (26) that

$$p_m = \sin\left(\frac{\pi}{6} + \psi_m\right) - \frac{a^2}{2}\exp\left(\sqrt{3}\psi_m\right). \tag{46}$$

Using the solution of Eq. (38) it is possible to connect p_m and ρ_m.

4.1 Elastic solution

In the case of purely elastic unloading the solution for the increment of stresses and strains has the form of Eq. (18). In particular

$$\frac{\Delta\sigma_r}{\sigma_0} = \frac{A_1}{\rho^2} + B_1, \quad \frac{\Delta\sigma_\theta}{\sigma_0} = -\frac{A_1}{\rho^2} + B_1,$$
$$\frac{\Delta\varepsilon_r}{k} = \frac{A_1(1+\nu)}{\rho^2} + B_1(1-\nu), \quad \frac{\Delta\varepsilon_\theta}{k} = -\frac{A_1(1+\nu)}{\rho^2} + B_1(1-\nu), \quad \frac{\Delta\varepsilon_z}{k} = -2\nu B_1. \tag{47}$$

Here A_1 and B_1 are new constants of integration. Using the boundary conditions (44) and (45) these constants are determined as

$$A_1 = A_1^e = \frac{a^2 p_m}{a^2 - 1}, \quad B_1 = B_1^e = \frac{p_m}{1 - a^2}. \tag{48}$$

Substituting Eq. (48) into Eq. (47) gives

$$\frac{\Delta\sigma_r}{\sigma_0} = \frac{p_m}{(1-a^2)}\left(1 - \frac{a^2}{\rho^2}\right), \quad \frac{\Delta\sigma_\theta}{\sigma_0} = \frac{p_m}{(1-a^2)}\left(1 + \frac{a^2}{\rho^2}\right). \tag{49}$$

It is natural to assume that the reversed plastic region starts to develop at $\rho=a$. It is seen from Eqs. (1) and (44) that $\sigma_r=0$ at $\rho=a$ at any stage of the process of deformation and from Eq. (21) that $\psi=0$ at $\rho=a$. Therefore, it follows from Eq. (41) that $T_r=0$ at $\rho=a$. It is seen from Eqs. (1) and (4) that $\sigma_\theta=\sigma_0$ at $\rho=a$ at any stage of the process of elastic/plastic loading. Therefore, $\sigma_\theta^f = \sigma_0$ in Eq. (42). Then, using Eqs. (19), (46) and (49) it is possible to transform Eq. (9) at $\rho=a$ to

$$\sin\left(\frac{\pi}{6} + \psi_{rp}\right) - \frac{a^2}{2}\exp\left(\sqrt{3}\psi_{rp}\right) = \frac{3}{4}\left(1-a^2\right)ck\left[1 - \cos\psi_{rp}\exp\left(\sqrt{3}\psi_{rp}\right)\right]. \tag{50}$$

Here ψ_{rp} is the maximum value of ψ_m at which unloading is purely elastic. Eq. (50) should be solved for ψ_{rp} numerically. Then, the corresponding value of p_{rp} can be found from Eq. (46). No reversed yielding occurs if $p_m \le p_{rp}$.

4.2 Elastic/plastic solution for stress

Assume that $p_m > p_{rp}$ and that reversed plasticity occurs in the region $a \le \rho \le \rho_{rc}$. Eq. (9) is valid in this region. This equation is satisfied by the following substitution

$$\frac{T_r}{\sigma_0} = \frac{2\sin\varphi}{\sqrt{3}}, \quad \frac{T_\theta}{\sigma_0} = \frac{\sin\varphi}{\sqrt{3}} + \cos\varphi. \tag{51}$$

Then, the radial and circumferential stresses are found from Eq. (41) using Eq. (14) as

$$\frac{\sigma_r}{\sigma_0} = \frac{2\sin\varphi}{\sqrt{3}} + \sqrt{3}c\tan\psi\left\{\cos\psi - \cos\psi_m \exp\left[\sqrt{3}\left(\psi_m - \psi\right)\right]\right\},$$
$$\frac{\sigma_\theta}{\sigma_0} = \frac{\sin\varphi}{\sqrt{3}} + \cos\varphi + \sqrt{3}c\frac{\sin\left(\psi + \pi/3\right)}{\cos\psi}\left\{\cos\psi - \cos\psi_m \exp\left[\sqrt{3}\left(\psi_m - \psi\right)\right]\right\}. \tag{52}$$

Using Eqs. (14) and (20) it is possible to rewrite Eq. (13) as

$$\frac{\left(\sqrt{3}\cos\psi - \sin\psi\right)}{2\cos\psi}\frac{\partial\sigma_r}{\partial\psi} + \sigma_r - \sigma_\theta = 0\,. \tag{53}$$

The quantity φ may be regarded as a function of ψ. Then, substituting Eq. (52) into Eq. (53) yields the following equation for φ

$$\cos\varphi\frac{\partial\varphi}{\partial\psi} + \frac{\left(\sin\varphi - \sqrt{3}\cos\varphi\right)}{\left(\sqrt{3} - \tan\psi\right)} + \frac{3c}{2}\cos\psi_m \tan\psi\left(\sqrt{3} - \tan\psi\right)\exp\left[\sqrt{3}\left(\psi_m - \psi\right)\right] = 0. \tag{54}$$

It is seen from Eqs. (1) and (44) that $\sigma_r=0$ at $\rho=a$ at any stage of unloading. Moreover, $\psi=0$ at $\rho=a$. Therefore, it follows from Eq. (52) that the boundary condition to Eq. (54) is

$$\varphi = 0 \tag{55}$$

for $\psi=0$. It is seen from this boundary condition and the structure of Eq. (54) that φ is a function only ψ (or ρ) and is independent of q. Eq. (54) should be solved numerically. Then, the stress field in the region $a\le\rho\le\rho_m$ is determined from Eq. (52). In particular

$$\frac{\sigma_{rc}}{\sigma_0} = \frac{2\sin\varphi_c}{\sqrt{3}} + \sqrt{3}c\tan\psi_{rc}\left\{\cos\psi_{rc} - \cos\psi_m \exp\left[\sqrt{3}\left(\psi_m - \psi_{rc}\right)\right]\right\},$$
$$\frac{\sigma_{\theta c}}{\sigma_0} = \frac{\sin\varphi_c}{\sqrt{3}} + \cos\varphi_c + \sqrt{3}c\frac{\sin\left(\psi_{rc} + \pi/3\right)}{\cos\psi_{rc}}\left\{\cos\psi_{rc} - \cos\psi_m \exp\left[\sqrt{3}\left(\psi_m - \psi_{rc}\right)\right]\right\}. \tag{56}$$

Here σ_{rc} is the value of the radial stress, $\sigma_{\theta c}$ is the value of the circumferential stress and φ_c is the value of φ at $\psi=\psi_{rc}$ (or $\rho=\rho_{rc}$). The following relation is immediate from Eq. (21)

$$\rho_{rc} = \frac{\sqrt{\sqrt{3}a}}{\sqrt{2}\sqrt{\sin\left(\pi/3 - \psi_{rc}\right)}}\exp\left(\frac{\sqrt{3}}{2}\psi_{rc}\right). \tag{57}$$

The increment of the radial and circumferential stresses at $\rho=\rho_{rc}$ is found Eqs. (19) and (56) as

$$\left.\frac{\Delta\sigma_r}{\sigma_0}\right|_{\rho=\rho_{rc}} = \frac{\sigma_{rc}}{\sigma_0} + \frac{2\sin\psi_{rc}}{\sqrt{3}}, \quad \left.\frac{\Delta\sigma_\theta}{\sigma_0}\right|_{\rho=\rho_{rc}} = \frac{\sigma_{\theta c}}{\sigma_0} + \frac{\sin\psi_{rc}}{\sqrt{3}} + \cos\psi_{rc}. \tag{58}$$

Unloading is elastic and Eq. (47) is valid in the region $\rho_{rc}\le\rho\le 1$. Of course, A_1 and B_1 are not given by Eq. (48). Substituting Eq. (47) into Eq. (45) yields $B_1=p_m-A_1$. Then

$$\frac{\Delta\sigma_r}{\sigma_0} = p_m + A_1\left(\frac{1}{\rho^2} - 1\right), \quad \frac{\Delta\sigma_\theta}{\sigma_0} = p_m - A_1\left(\frac{1}{\rho^2} + 1\right). \tag{59}$$

The quantities $\Delta\sigma_r$ and $\Delta\sigma_\theta$ should be continuous across the surface $\rho=\rho_{cr}$. Therefore, it follows from Eqs. (58) and (59) that

$$\frac{\sigma_{rc}}{\sigma_0}+\frac{2\sin\psi_{rc}}{\sqrt{3}}-p_m-A_1\left(\frac{1}{\rho_{rc}^2}-1\right)=0,$$
$$\frac{\sigma_{\theta c}}{\sigma_0}+\frac{\sin\psi_{rc}}{\sqrt{3}}+\cos\psi_{rc}-p_m+A_1\left(\frac{1}{\rho_{rc}^2}+1\right)=0. \tag{60}$$

Using Eqs. (56) and (57) it is possible to eliminate σ_{rc}, $\sigma_{\theta c}$ and ρ_{rc} in Eq. (60). The resulting system of equations involves two unknowns, A_1 and ψ_{rc}. This system should be solved numerically. Then, the distribution of the residual radial and circumferential stresses is determined from Eqs. (19), (21), (42), and (59) in the region $\rho_{rc}\le\rho\le\rho_m$ and from Eqs. (27), (42), and (59) in the region $\rho_m\le\rho\le1$. The distribution in the region $\rho_{rc}\le\rho\le\rho_m$ is in parametric form with ψ being the parameter.

4.3 Elastic/plastic solution for strain

Since A_1 and B_1 have been determined in Section 4.2, the increment of strains in the region $\rho_{rc}\le\rho\le1$ is immediately found from Eq. (47). The distribution of the residual strains follows from Eqs. (30), (43) and (47) in the region $\rho_m\le\rho\le1$ and from Eqs. (21), (31), (39), (40), (43), and (47) in the region $\rho_{rc}\le\rho\le\rho_m$. The distribution in the region $\rho_{rc}\le\rho\le\rho_m$ is in parametric form with ψ being the parameter. In Eq. (30) it is necessary to replace ψ_c with ψ_m.

It remains to find the residual strains in the region $a\le\rho\le\rho_{rc}$. In this region, the increment of strains consists of two portions, elastic and plastic. The elastic portion is found by substituting the stresses from Eqs. (19) and (52) into Hooke's law. As a result

$$\frac{\Delta\varepsilon_r^e}{k}=\frac{2}{\sqrt{3}}\left(\sin\varphi+\sin\psi\right)-\nu\left(\frac{\sin\varphi+\sin\psi}{\sqrt{3}}+\cos\varphi+\cos\psi\right)+$$
$$\sqrt{3}c\left[\tan\psi-\nu\frac{\sin\left(\psi+\pi/3\right)}{\cos\psi}\right]\left\{\cos\psi-\cos\psi_m\exp\left[\sqrt{3}\left(\psi_m-\psi\right)\right]\right\},$$
$$\frac{\Delta\varepsilon_\theta^e}{k}=\frac{\sin\varphi+\sin\psi}{\sqrt{3}}+\cos\varphi+\cos\psi-\frac{2\nu}{\sqrt{3}}\left(\sin\varphi+\sin\psi\right)+$$
$$\sqrt{3}c\left[\frac{\sin\left(\psi+\pi/3\right)}{\cos\psi}-\nu\tan\psi\right]\left\{\cos\psi-\cos\psi_m\exp\left[\sqrt{3}\left(\psi_m-\psi\right)\right]\right\}, \tag{61}$$
$$\frac{\Delta\varepsilon_z^e}{k}=-2\nu\left[\sin\left(\varphi+\frac{\pi}{6}\right)+\sin\left(\psi+\frac{\pi}{6}\right)\right]-$$
$$\frac{3}{2}c\nu\left(1+\sqrt{3}\tan\psi\right)\left\{\cos\psi-\cos\psi_m\exp\left[\sqrt{3}\left(\psi_m-\psi\right)\right]\right\}$$

Using Eqs. (10) and (51) it is possible to transform Eq. (17) to

$$\xi_r^p=\frac{\lambda_3\sigma_0}{3}\left(\sqrt{3}\sin\varphi-\cos\varphi\right),\quad \xi_\theta^p=\frac{2\lambda_3\sigma_0}{3}\cos\varphi,\quad \xi_z^p=-\frac{\lambda_3\sigma_0}{3}\left(\sqrt{3}\sin\varphi+\cos\varphi\right).$$

Eliminating λ_3 between these equations leads to

$$\xi_r^p = \frac{\left(\sqrt{3}\sin\varphi - \cos\varphi\right)}{2\cos\varphi}\xi_\theta^p, \quad \xi_z^p = -\frac{\left(\sqrt{3}\sin\varphi + \cos\varphi\right)}{2\cos\varphi}\xi_\theta^p. \tag{62}$$

It has been shown in Section 4.2 that φ is independent of q. Therefore, Eq. (62) can be immediately integrated with respect to q to give

$$\Delta\varepsilon_r^p = \frac{\left(\sqrt{3}\sin\varphi - \cos\varphi\right)}{2\cos\varphi}\Delta\varepsilon_\theta^p, \quad \Delta\varepsilon_z^p = -\frac{\left(\sqrt{3}\sin\varphi + \cos\varphi\right)}{2\cos\varphi}\Delta\varepsilon_\theta^p. \tag{63}$$

The equation of strain compatibility is

$$\rho\frac{\partial\left(\Delta\varepsilon_\theta\right)}{\partial\rho} = \Delta\varepsilon_r - \Delta\varepsilon_\theta\,. \tag{64}$$

Using the equations $\Delta\varepsilon_r = \Delta\varepsilon_r^e + \Delta\varepsilon_r^p$ and $\Delta\varepsilon_\theta = \Delta\varepsilon_\theta^e + \Delta\varepsilon_\theta^p$ and eliminating $\Delta\varepsilon_r^p$ by means of Eq. (63) it is possible to rewrite Eq. (64) as

$$\rho\frac{\partial\left(\Delta\varepsilon_\theta\right)}{\partial\rho} = \frac{\sqrt{3}\left(\sin\varphi - \sqrt{3}\cos\varphi\right)}{2\cos\varphi}\Delta\varepsilon_\theta + \Delta\varepsilon_r^e - \frac{\left(\sqrt{3}\sin\varphi - \cos\varphi\right)}{2\cos\varphi}\Delta\varepsilon_\theta^e.$$

Replacing differentiation with respect to ρ with differentiation with respect to ψ by means of Eq. (20) gives

$$\begin{gathered}\frac{\partial\left(\Delta\varepsilon_\theta\right)}{\partial\psi} = \frac{\sqrt{3}\left(\sin\varphi - \sqrt{3}\cos\varphi\right)\cos\psi}{\left(\sqrt{3}\cos\psi - \sin\psi\right)\cos\varphi}\Delta\varepsilon_\theta + \\ \frac{2\cos\psi}{\left(\sqrt{3}\cos\psi - \sin\psi\right)}\left[\Delta\varepsilon_r^e - \frac{\left(\sqrt{3}\sin\varphi - \cos\varphi\right)}{2\cos\varphi}\Delta\varepsilon_\theta^e\right].\end{gathered} \tag{65}$$

Using Eq. (61) the last term of Eq. (65) can be represented as a function of ψ and φ. Moreover, the dependence of φ on ψ is known from the solution to Eq. (54). Therefore, Eq. (65) is a linear ordinary differential equation with respect to $\Delta\varepsilon_\theta$. Its solution can be written in terms of ordinary integrals. The boundary condition to Eq. (65) follows from the condition $\Delta\varepsilon_\theta = \Delta\varepsilon_\theta^e$ at $\psi = \psi_{rc}$. Then, using Eq. (61)

$$\begin{gathered}\frac{\Delta\varepsilon_\theta}{k} = \frac{\sin\varphi_c + \sin\psi_{rc}}{\sqrt{3}} + \cos\varphi_c + \cos\psi_{rc} - \frac{2\nu}{\sqrt{3}}\left(\sin\varphi_c + \sin\psi_{rc}\right) + \\ \sqrt{3}c\left[\frac{\sin\left(\psi_{rc} + \pi/3\right)}{\cos\psi_{rc}} - \nu\tan\psi_{rc}\right]\left\{\cos\psi_{rc} - \cos\psi_m\exp\left[\sqrt{3}\left(\psi_m - \psi_{rc}\right)\right]\right\}\end{gathered} \tag{66}$$

for $\psi=\psi_{rc}$. Once Eq. (65) satisfying the boundary condition (66) has been solved, $\Delta\varepsilon_r$ and $\Delta\varepsilon_z$ are found by means of the equations $\Delta\varepsilon_r = \Delta\varepsilon_r^e + \Delta\varepsilon_r^p$, $\Delta\varepsilon_\theta = \Delta\varepsilon_\theta^e + \Delta\varepsilon_\theta^p$, $\Delta\varepsilon_z = \Delta\varepsilon_z^e + \Delta\varepsilon_z^p$ and Eqs.

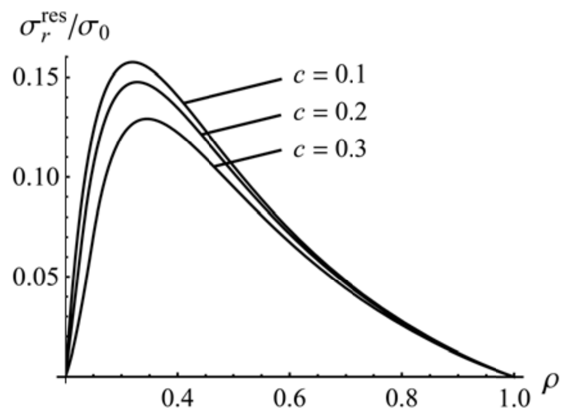

Fig. 3 Radial distribution of the residual radial stress at several values of c

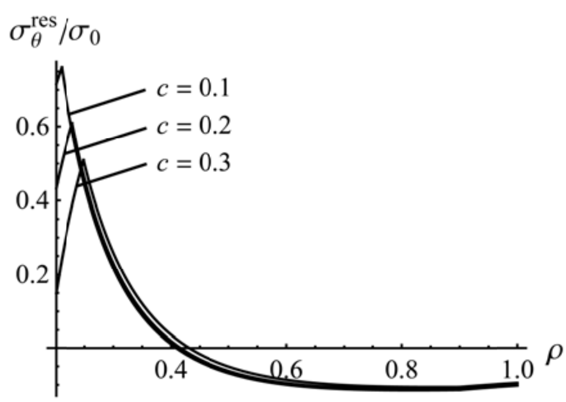

Fig. 4 Radial distribution of the residual circumferential stress at several values of c

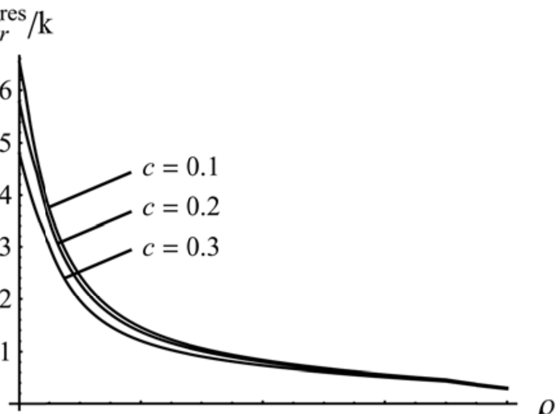

Fig. 5 Radial distribution of the residual radial strain at several values of c

(61) and (63). Substituting this solution and the solution for strains at the end of loading (Section 3.3) into Eq. (43) supplies the distribution of the residual strains in the region $a \le \rho \le \rho_{rc}$.

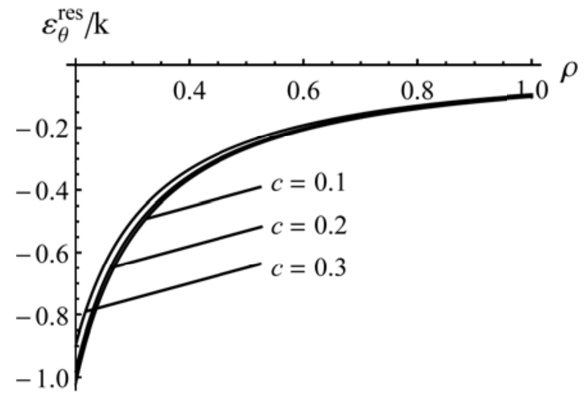

Fig. 6 Radial distribution of the residual circumferential strain at several values of c

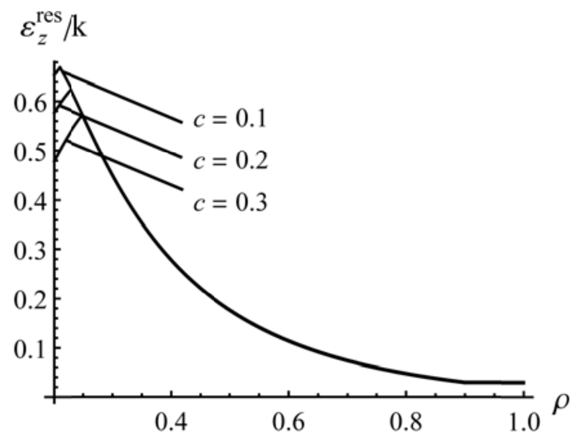

Fig. 7 Radial distribution of the residual axial strain at several values of c

5. Illustrative example

This section illustrates the effect of c on the distribution of residual stresses and strains. In all calculations, ν=0.3, a=0.2 and ρ_m=0.8. Eq. (38) has been solved numerically for ψ_m. Then, p_m has been determined from Eq. (46). The variation of the residual radial and circumferential stresses with ρ at several values of c is depicted in Figs. 3 and 4, respectively. The associated residual strain distributions are shown in Fig. 5 (radial strain), Fig. 6 (circumferential strain), and Fig. 7 (axial strain).

6. Conclusions

A new model that takes into account the Bauschinger effect, but neglects strain hardening during loading has been proposed. These mathematical properties of the model represent reasonably well a class of materials such as high strength steels (Franklin and Morrison 1960, Milligan *et al.* 1966, Findley and Reed 1983, Rees 2006). It is assumed that the material obeys the von Mises yield criterion during loading. Prager's law (Prager 1956) is adopted to describe the

Bauschinger effect. The model is restricted to plane stress problems. However, its generalization on three-dimensional problems is straightforward.

The model has been adopted to calculate the distribution of residual stresses and strains in a thin hollow disc subject to external pressure. In particular, the effect of the parameter C involved in Eq. (7) on these distributions is illustrated in Figs. 3 to 7. Note that c shown in the figures is proportional to C, as follows from Eq. (14).

The solution found is semi-analytic. In particular, a numerical technique is only necessary to solve transcendental equations, such as, for example, Eq. (38) for ψ_m if ρ_m is given. Therefore, the solution may serve as a benchmark problem to verify the accuracy of numerical codes.

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