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Plane strain bending of a bimetallic sheet at large strains

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Abstract. This paper deals with the pure bending of incompressible elastic perfectly plastic two-layer sheets under plane strain conditions at large strains. Each layer is classified by its yield stress, shear modulus of elasticity and its initial percentage thickness in relation to the whole sheet. The solution found is semi-analytic. In particular, a numerical technique is only necessary to solve transcendental equations. The general solution is cumbersome because different analytic expressions for the radial and circumferential stresses should be adopted in different regions of the whole sheet. In particular, there are several alternative ways a plastic region (or plastic regions) can propagate. However, for any given set of material and process parameters the solution to the problem consists of a sequence of rather simple analytic expressions connected by transcendental equations. The general solution is illustrated by a simple example.

Keywords: plane strain bending; bimetallic sheet; elastic/perfectly plastic material; large strains; analytic solution

1. Introduction

The pure plane strain bending of a sheet at large strains is one of the classical problems of plasticity. In particular, its solution for rigid perfectly plastic materials is available in monographs on plasticity (Hill 1950, Chakrabarty 1987). A general approach to analysis of this process has been proposed in Alexandrov *et al*. (2006). This approach is applicable to any isotropic incompressible material. However, previous solutions based on it are restricted to homogeneous sheets (Alexandrov *et al*. 2006, Alexandrov and Hwang 2010, 2011). A review of theoretical solutions for the pure plane strain bending of homogeneous sheets found by other methods has been provided in Zhu (2007).

A number of solutions are also available for bending of multi-layer sheets. An elastic solution for curved bars has been proposed in Lo and Conway (1975). This solution has been used by Arslan and Sulu (2014) to determine the onset of plastic yielding assuming Tresca's yield criterion. Several plastic solutions are based on the assumption that the through thickness stress

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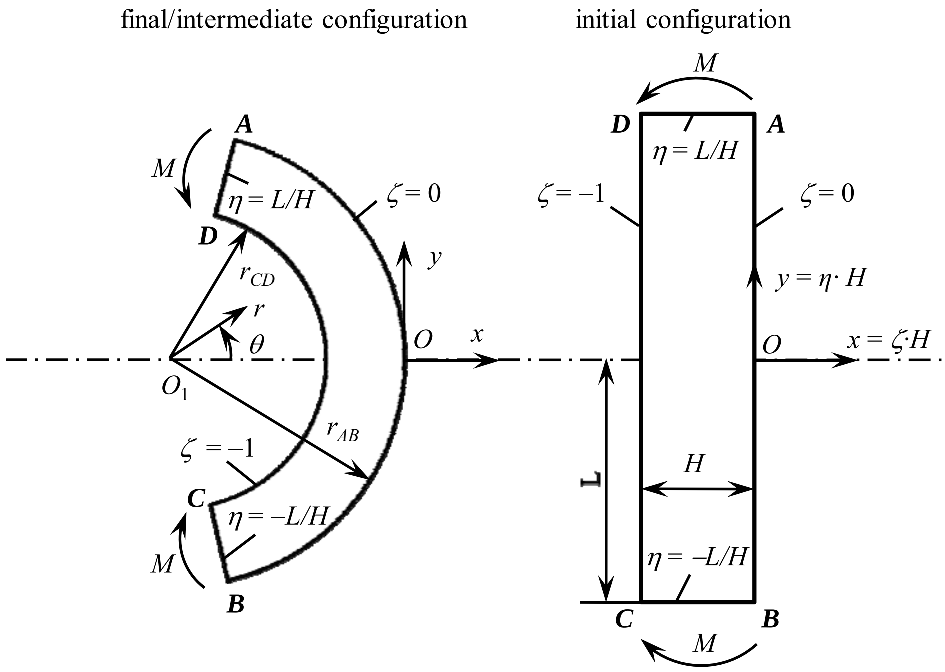

Fig. 1 Schematic illustration of pure bending

vanishes (Yuen 1996, Kagzi *et al.* 2015). The range of validity of such solutions is restricted to nearly flat sheets. The pure plane strain bending of bonded laminated metals has been studied in Verguts and Sowerby (1975). This solution is numerical. In particular, distinct computer programs have been written to solve each of various cases considered in the paper.

This present work extends the general method proposed in Alexandrov *et al.* (2006) to a bimetallic sheet assuming that each layer is made of an elastic perfectly plastic material. The solution is semi-analytical. In particular, a numerical technique is only necessary to solve transcendental equations. The general solution is cumbersome because different analytic expressions for the radial and circumferential stresses should be adopted in different regions of the sheet. However, for any given set of material and process parameters the solution is rather simple. The general solution is illustrated by a simple example.

2. Preliminaries

The plane strain bending process transforms an initial rectangle into a circular sector in the plane of flow by a bending moment, M (Fig. 1). The thickness and width of the initial rectangle are denoted by H and $2L$, respectively. A general approach to analyzing the plane strain pure bending process of incompressible materials at large strains has been proposed in Alexandrov *et al.* (2006). An advantage of this approach is that the general solution describing the kinematics of the process is independent of the specific material model chosen. In particular, bending of multi-layer sheets can be treated. For completeness, the main results presented in Alexandrov *et al.* (2006) are briefly discussed in this section. It is convenient to introduce two coordinate systems, (x,y) and (ζ,η). The former is an Eulerian Cartesian coordinate system. This coordinate system can be chosen such that the initial shape is defined by the equations $x=-H$, $x=0$, and $y=\pm L$. It is evident that the process is symmetric relative to the x-axis. (ζ,η) is a Lagrangian coordinate system such that $x=\zeta H$ and $y=\eta H$ at the initial instant. The normal stresses in this coordinate system are denoted by σ_ζ and σ_η. It has been shown in Alexandrov *et al*. (2006) that the mapping between the (x,y) and (ζ,η) coordinates is given by

$$\frac{x}{H}=\sqrt{\frac{\zeta}{a}+\frac{s}{a^2}}\cos(2a\eta)-\frac{\sqrt{s}}{a},\quad \frac{y}{H}=\sqrt{\frac{\zeta}{a}+\frac{s}{a^2}}\sin(2a\eta)\,. \tag{1}$$

Here a is a function of the time, t, and s is a function of a. The specific form of the function $a(t)$ is immaterial for rate-independent materials. The function $s\,(a)$ should be found from the solution. It is assumed that $a=0$ at the initial instant. Then, the function $s\,(a)$ must satisfy the condition

$$s=\frac{1}{4} \tag{2}$$

for $a=0$. It is possible to verify by inspection that the mapping (1) satisfies the equation of incompressibility and that principal strain rate trajectories coincide with coordinate curves of the (ζ,η) coordinate system. This implies that these coordinate curves are principal stress trajectories for coaxial models (models which postulate that the principal stress and principal strain rate directions coincide) and thus σ_ζ and σ_η are the principal stresses in the plane of flow. Therefore, the boundary of the deforming sheet is free of shear stresses, which is a boundary condition for the pure bending process, and the only stress boundary conditions to be satisfied are

$$\sigma_\zeta=0 \tag{3}$$

at $\zeta=-1$ and $\zeta=0$ (Fig. 1). Using Eq. (1) it is possible to find the total principal strain components as

$$\varepsilon_\zeta=-\varepsilon_\eta=-\frac{1}{2}\ln\left[4(\zeta a+s)\right]. \tag{4}$$

It is also convenient to introduce a moving cylindrical coordinate system (r,θ) by the following transformation equations (Fig. 1)

$$\frac{r}{H}=\sqrt{\frac{\zeta}{a}+\frac{s}{a^2}}\qquad\text{and}\qquad \theta=2a\eta\,. \tag{5}$$

In this coordinate system the boundaries of the deforming sheet are given by the following equations (Fig. 1)

$$\frac{r}{H}=\frac{r_{AB}}{H}=\frac{\sqrt{s}}{a},\quad \frac{r}{H}=\frac{r_{CD}}{H}=\sqrt{\frac{s}{a^2}-\frac{1}{a}},\quad \theta=\theta_0=\pm\frac{2aL}{H}. \tag{6}$$

It follows from these relations that the current thickness of the sheet is

$$h=r_{AB}-r_{CD}=H\left(\frac{\sqrt{s}}{a}-\sqrt{\frac{s}{a^2}-\frac{1}{a}}\right). \tag{7}$$

3. Material model

The classical Eulerian theory of finite elastoplasticity presented, for example, in Xiao *et al.* (2006) is used. Since Eq. (1) results in the equation of incompressibility, it is necessary to assume that Poisson's ratio is 1/2. The total principal strain rates are

$$\xi_\zeta = \xi_\zeta^e + \xi_\zeta^p, \quad \xi_\eta = \xi_\eta^e + \xi_\eta^p. \tag{8}$$

Here the superscript e denotes the elastic portion of the total strain rates and the superscript p denotes the plastic portion of the total strain rates. Let τ_ζ and τ_η be the principal deviatoric stresses in the plane of flow. The elastic portion of the principal strain rates is characterized by a rate constitutive equation in the form

$$\dot{\tau}_\zeta = 2G\xi_\zeta^e, \quad \dot{\tau}_\eta = 2G\xi_\eta^e \tag{9}$$

where the superimposed dot denotes the material derivative and G is the shear modulus of elasticity. Note that in general the left hand side of Eq. (9) should involve an objective stress rate. However, in the case of deformation described by Eq. (1), any objective corotational rate reduces to the material derivative. Any plane strain yield criterion for isotropic incompressible material can be written as

$$\left|\sigma_\zeta - \sigma_\eta\right| = \frac{2\sigma_Y}{\sqrt{3}} \tag{10}$$

where σ_Y is the yield stress in tension. In the case of perfectly plastic materials σ_Y is constant. Eq. (10) is supplemented with the associated flow rule for the plastic portion of the principal strain rates, ξ_ζ^p and ξ_η^p. Since the material is incompressible and the state of strain is plane

$$\tau_\zeta + \tau_\eta = 0. \tag{11}$$

Substituting this equation into Eq. (10) gives the yield criterion in the form

$$\left|\tau_\zeta\right| = \left|\tau_\eta\right| = \frac{\sigma_Y}{\sqrt{3}}. \tag{12}$$

Since the equation of incompressibility has been already satisfied by the mapping (1), the associated flow rule combined with the yield criterion (10) imposes no additional restrictions on the values of ξ_ζ^p and ξ_η^p.

It is evident from Eq. (5) that

$$\begin{gathered}\sigma_\zeta \equiv \sigma_r, \quad \sigma_\eta \equiv \sigma_\theta, \quad \tau_\zeta \equiv \tau_r, \quad \tau_\eta \equiv \tau_\theta, \quad \varepsilon_\zeta \equiv \varepsilon_r, \quad \varepsilon_\eta \equiv \varepsilon_\theta, \\ \xi_\zeta \equiv \xi_r, \quad \xi_\eta \equiv \xi_\theta, \quad \xi_\zeta^e \equiv \xi_r^e, \quad \xi_\eta^e \equiv \xi_\theta^e, \quad \xi_\zeta^p \equiv \xi_r^p, \quad \xi_\eta^p \equiv \xi_\theta^p.\end{gathered} \tag{13}$$

Here the subscripts r and θ denote the quantities in the cylindrical coordinate system. Then, the boundary conditions (3) become

$$\sigma_r = 0 \tag{14}$$

for $\zeta = -1$ and $\zeta = 0$.

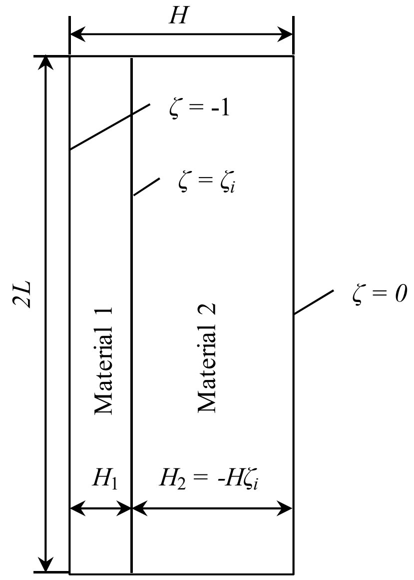

Fig. 2 Initial configuration of a bimetallic sheet

Both the radial and circumferential stresses are continuous across elastic/plastic boundaries (Hill 1950). These conditions can be written as $[\sigma_r]=0$ and $[\sigma_\theta]=0$ across elastic/plastic boundaries. Here [...] denotes the amount of jump in the quantity enclosed in the brackets. In the case under consideration the continuity in σ_r and σ_θ implies the continuity in τ_r. Therefore, using Eq. (13)

$$\left[\sigma_r\right]=0 \quad \text{and} \quad \left[\tau_r\right]=\left[\tau_\zeta\right]=0 \tag{15}$$

across elastic/plastic boundaries.

4. Statement of the problem

Consider the pure plane strain bending of a bimetallic sheet of initial thickness H. The initial thicknesses of the layers are denoted by H_1 and H_2 (Fig. 2). Then, the ζ-coordinate of the bimetallic interface is $\zeta_i=-H_2/H$. The radial stress must be continuous across this interface. Therefore

$$\left[\sigma_r\right]=0 \tag{16}$$

at $\zeta=\zeta_i$. The distributions of the shear modulus and yield stress are

$$G=\begin{cases} G_2 & \text{in the range} \quad \zeta_i \le \zeta \le 0 \\ G_1 & \text{in the range} \quad -1 \le \zeta \le \zeta_i \end{cases} \tag{17}$$

and

$$\sigma_Y=\begin{cases} \sigma_{Y2} & \text{in the range} \quad \zeta_i \le \zeta \le 0 \\ \sigma_{Y1} & \text{in the range} \quad -1 \le \zeta \le \zeta_i \end{cases} \tag{18}$$

Both σ_{Y1} and σ_{Y2} are constant. It is seen from Eq. (5) that the r-coordinate of the bimetallic interface is

$$\frac{r_i}{H} = \sqrt{\frac{\zeta_i}{a} + \frac{s}{a^2}} . \tag{19}$$

The bending moment M and its dimensionless representation m are given by

$$M = \int_{r_{CD}}^{r_{AB}} \sigma_\theta r dr, \quad m = \frac{2\sqrt{3}M}{\sigma_{Y2} H^2} . \tag{20}$$

Note that m=1 for a sheet wholly made of a rigid perfectly plastic material whose yield stress is σ_{Y2} (Hill 1950). Using Eq. (5) it is possible to transform Eq. (20) to

$$m = \frac{\sqrt{3}}{a} \int_{-1}^{0} \frac{\sigma_\theta}{\sigma_{Y2}} d\zeta . \tag{21}$$

It is assumed that the state of stress and strain is independent of θ. Then, the system of equations to solve consists of the constitutive equations and the only equilibrium equation of the form

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = 0 . \tag{22}$$

Using the identity $\sigma_r - \sigma_\theta \equiv \tau_r - \tau_\theta$ along with Eqs. (11) and (13) it is possible to transform Eq. (22) to

$$\frac{\partial \sigma_r}{\partial r} + \frac{2\tau_r}{r} = 0 .$$

Replacing in this equation differentiation with respect to r with differentiation with respect to ζ by means of Eq. (5) yields

$$\frac{\partial \sigma_r}{\partial \zeta} + \frac{a\tau_r}{(a\zeta + s)} = 0 . \tag{23}$$

5. General solution

In general, the process of bending consists of several stages. The entire sheet is elastic if the bending moment is small enough. The range of a for this stage is $0 \le a \le a_e$. A plastic region propagates from one of the surfaces, ζ=0 or ζ=−1, as the value of a further increases. Another plastic region starts to develop from the other surface at $a=a_p$. Therefore, the range of a for the second stage is $a_e \le a \le a_p$. The range of a for the third stage is $a > a_p$. In a special case, two plastic regions start to develop from the surfaces ζ=0 and ζ=−1 simultaneously. In this case $a_e = a_p$ and the second stage does not exist. The values of a_e and a_p should be found from the solution.

It follows from Hooke's law along with Eqs. (4), (13) and (17) that

$$\tau_r = \begin{cases} -G_2 \ln\left[4(\zeta a + s)\right] & \text{in the range} \quad \zeta_i \le \zeta \le 0 \\ -G_1 \ln\left[4(\zeta a + s)\right] & \text{in the range} \quad -1 \le \zeta \le \zeta_i \end{cases},$$
$$\tau_\theta = \begin{cases} G_2 \ln\left[4(\zeta a + s)\right] & \text{in the range} \quad \zeta_i \le \zeta \le 0 \\ G_1 \ln\left[4(\zeta a + s)\right] & \text{in the range} \quad -1 \le \zeta \le \zeta_i \end{cases}. \tag{24}$$

in elastic regions. Substituting Eq. (24) into Eq. (23) gives

$$\frac{\partial \sigma_r}{G_2 \partial \zeta} - \frac{a \ln\left[4(\zeta a + s)\right]}{(a\zeta + s)} = 0 \tag{25}$$

in the range $\zeta_i \le \zeta \le 0$ and

$$\frac{\partial \sigma_r}{G_2 \partial \zeta} - \frac{a g_1 \ln\left[4(\zeta a + s)\right]}{(a\zeta + s)} = 0 \tag{26}$$

in the range $-1 \le \zeta \le \zeta_i$. Here $g_1 = G_1/G_2$. The general solution of Eq. (25) is

$$\frac{\sigma_r}{G_2} = \frac{1}{2} \ln^2\left[4(a\zeta + s)\right] + C_2 \tag{27}$$

where C_2 is a constant of integration. Analogously, the general solution of Eq. (26) is

$$\frac{\sigma_r}{G_2} = \frac{g_1}{2} \ln^2\left[4(a\zeta + s)\right] + C_1 \tag{28}$$

where C_1 is another constant of integration. Equations (27) and (28) are valid in elastic regions in the ranges $\zeta_i \le \zeta \le 0$ and $-1 \le \zeta \le \zeta_i$, respectively.

It is seen from Eqs. (12), (13) and (18) that $\tau_r = \sigma_{Y1}/\sqrt{3}$ or $\tau_r = -\sigma_{Y1}/\sqrt{3}$ in plastic regions in the range $-1 \le \zeta \le \zeta_i$ and $\tau_r = \sigma_{Y2}/\sqrt{3}$ or $\tau_r = -\sigma_{Y2}/\sqrt{3}$ in plastic regions in the range $\zeta_i \le \zeta \le 0$. Substituting these values of τ_r into Eq. (23) and integrating give

$$\frac{\sigma_r}{G_2} = -\frac{k_1 g_1}{\sqrt{3}} \ln(a\zeta + s) + C_3 \tag{29}$$

if $\tau_r = \sigma_{Y1}/\sqrt{3}$ in the range $-1 \le \zeta \le \zeta_i$

$$\frac{\sigma_r}{G_2} = \frac{k_1 g_1}{\sqrt{3}} \ln(a\zeta + s) + C_4 \tag{30}$$

if $\tau_r = -\sigma_{Y1}/\sqrt{3}$ in the range $-1 \le \zeta \le \zeta_i$

$$\frac{\sigma_r}{G_2} = -\frac{k_2}{\sqrt{3}} \ln(a\zeta + s) + C_5 \tag{31}$$

if $\tau_r = \sigma_{Y2}/\sqrt{3}$ in the range $\zeta_i \le \zeta \le 0$, and

$$\frac{\sigma_r}{G_2} = \frac{k_2}{\sqrt{3}} \ln\left(a\zeta + s\right) + C_6 \tag{32}$$

if $\tau_r = -\sigma_{Y2}/\sqrt{3}$ in the range $\zeta_i \le \zeta \le 0$. Here $k_1 = \sigma_{Y1}/G_1$, $k_2 = \sigma_{Y2}/G_2$ and C_3, C_4, C_5, and C_6 are constants of integration. The circumferential stress is found from the equation

$$\sigma_\theta = \sigma_r - 2\tau_r. \tag{33}$$

Here Eqs. (11) and (13) have been used.

Let ζ_g be the ζ-coordinate of a generic elastic/plastic boundary. Substituting Eqs. (12) and (24) into Eq. (15) for $[\tau_r]$ leads to

$$\sqrt{3}\ln\left[4\left(a\zeta_g + s\right)\right] = -k_2 \tag{34}$$

if $\zeta_i \le \zeta_g \le 0$ and $\tau_r > 0$ at $\zeta = \zeta_g$

$$\sqrt{3}\ln\left[4\left(a\zeta_g + s\right)\right] = k_2 \tag{35}$$

if $\zeta_i \le \zeta_g \le 0$and $\tau_r < 0$ at $\zeta = \zeta_g$

$$\sqrt{3}\ln\left[4\left(a\zeta_g + s\right)\right] = -k_1 \tag{36}$$

if $-1 \le \zeta_g \le \zeta_i$ and $\tau_r > 0$ at $\zeta = \zeta_g$

$$\sqrt{3}\ln\left[4\left(a\zeta_g + s\right)\right] = k_1 \tag{37}$$

if $-1 \le \zeta_g \le \zeta_i$ and $\tau_r < 0$ at $\zeta = \zeta_g$.

5.1 Purely elastic solution

In this case Eq. (27) is valid in the range $0 \ge \zeta \ge \zeta_i$ and Eq. (28) in the range $-1 \le \zeta \le \zeta_i$. The solution (27) must satisfy the boundary condition (14) at $\zeta = 0$. Therefore

$$\frac{\sigma_r}{G_2} = \frac{1}{2}\ln\left[16s\left(\zeta a + s\right)\right]\ln\left(\frac{\zeta a + s}{s}\right) \tag{38}$$

in the range $0 \ge \zeta \ge \zeta_i$. The solution (28) must satisfy the boundary condition (14) at $\zeta = -1$. Therefore

$$\frac{\sigma_r}{G_2} = \frac{g_1}{2}\ln\left[16\left(s - a\right)\left(\zeta a + s\right)\right]\ln\left(\frac{\zeta a + s}{s - a}\right). \tag{39}$$

in the range $\zeta_i \ge \zeta \ge -1$. Then, it follows from Eqs. (16), (38) and (39) that

$$g_1 \ln\left[16\left(s - a\right)\left(\zeta_i a + s\right)\right]\ln\left(\frac{\zeta_i a + s}{s - a}\right) = \ln\left[16s\left(\zeta_i a + s\right)\right]\ln\left(\frac{\zeta_i a + s}{s}\right). \tag{40}$$

This equation determines s as a function of a in the purely elastic range and its solution should be found numerically. If plastic yielding is initiated at the surface $\zeta = 0$ then substitution of Eqs. (24) and (18) into Eq. (12) leads to

$$s_{e2} = \frac{1}{4}\exp\left(\frac{k_2}{\sqrt{3}}\right) \tag{41}$$

where s_{e2} is the value of s corresponding to the initiation of the plastic region at $\zeta=0$. The corresponding value of a is denoted by a_{e2}. The value of a_{e2} is determined from Eq. (40) in which s should be replaced with the right hand side of Eq. (41).

If plastic yielding is initiated at the surface $\zeta=-1$ then substitution of Eqs. (24) and (18) into Eq. (12) leads to

$$s_{e1} = a_{e1} + \frac{1}{4}\exp\left(-\frac{k_1}{\sqrt{3}}\right) \tag{42}$$

where a_{e1} and s_{e1} are the values of a and s, respectively, corresponding to the initiation of the plastic region at $\zeta=-1$. The value of a_{e1} is determined from Eq. (40) in which s should be replaced with the right hand side of Eq. (42). The value of a_e is determined from the equation $a_e = \min\{a_{e1}, a_{e2}\}$. The purely elastic solution is not valid for $a>a_e$.

5.2 Elastic/plastic solution with one plastic region

Let ζ_p be the ζ-coordinate of the elastic/plastic boundary. It is necessary to consider two cases separately; namely, plastic yielding occurs in the region $\zeta_p\le\zeta\le0$ and plastic yielding occurs in the region $-1\le\zeta\le\zeta_p$.

5.2.1 Plastic yielding occurs in the region $\zeta_p\le\zeta\le0$

In this case $a_{e1}>a_{e2}$ and $\tau_r<0$ in the plastic region. It is necessary to examine three cases; namely, (*i*) $\zeta_p>\zeta_i$, (*ii*) $\zeta_p=\zeta_i$, and (*iii*) $\zeta_p<\zeta_i$. In all these cases Eq. (28) is valid in the vicinity of the surface $\zeta=-1$ and Eq. (32) in the vicinity of the surface $\zeta=-1$. Therefore, these equations and the boundary conditions (14) combine to give

$$C_1 = -\frac{g_1}{2}\ln^2\left[4(s-a)\right] \quad \text{and} \quad C_6 = -\frac{k_2}{\sqrt{3}}\ln s\,. \tag{43}$$

Consider case (*i*). In this case Eq. (28) is valid in the range $-1\le\zeta\le\zeta_i$, Eq. (27) in the range $\zeta_i\le\zeta\le\zeta_p$, and Eq. (32) in the range $\zeta_p\le\zeta\le0$. Substituting Eqs. (27) and (28) into Eq. (16) and using Eq. (43) result in

$$C_2 = \frac{(g_1-1)}{2}\ln^2\left[4(a\zeta_i+s)\right] - \frac{g_1}{2}\ln^2\left[4(s-a)\right]. \tag{44}$$

Since $\tau_r<0$ at $\zeta=\zeta_p$ and $\zeta_i\le\zeta_p\le0$, Eq. (35) is valid and becomes

$$\sqrt{3}\ln\left[4(a\zeta_p+s)\right] = k_2\,. \tag{45}$$

Substituting Eqs. (28) and (32) into Eq. (15) for $[\sigma_r]$ supplies the equation for determining the function s (a). In particular, eliminating in this equation C_6 by means of Eq. (43), C_2 by means of Eq. (44), and ζ_p by means of Eq. (45) gives

$$\frac{k_2^2}{6} - \frac{k_2}{\sqrt{3}}\ln(4s) + \frac{g_1}{2}\ln^2\left[4(s-a)\right] - \frac{(g_1-1)}{2}\ln^2\left[4(a\zeta_i+s)\right] = 0\,. \tag{46}$$

This equation should be solved numerically to find the function s (a). This solution is valid if $\zeta_i\le\zeta_p$. Replacing ζ_p with ζ_i in Eq. (45) yields

$$\sqrt{3}\ln\left[4\left(a\zeta_i+s\right)\right]=k_2. \tag{47}$$

This equation can be used to eliminate s in Eq. (46). The resulting equation should be solved for a numerically. The solution is denoted by a_1. The range of a for case (*i*) is $a_{e2}\le a\le a_1$.

In case (*ii*) the elastic/plastic boundary coincides with the bi-material interface. Therefore, Eq. (28) is valid in the range $-1\le\zeta\le\zeta_i$ and Eq. (32) in the range $\zeta_p\le\zeta\le 0$. Substituting these equations into Eq. (16) and eliminating C_1 and C_6 by means of Eq. (43) result in the following equation for determining the function $s\,(a)$

$$\ln^2\left[4\left(a\zeta_i+s\right)\right]-\ln^2\left[4\left(s-a\right)\right]-\frac{2k_2}{g_1\sqrt{3}}\ln\left(\frac{a\zeta_i+s}{s}\right)=0. \tag{48}$$

This equation should be solved numerically. The range of validity of this solution is restricted by the condition that the yield criterion is not violated in material 1 at $\zeta=\zeta_i$ (Fig. 2). This condition may be represented as $\tau_r=-\sigma_{Y1}/\sqrt{3}$ at $\zeta=\zeta_i$. Then, it follows from Eq. (24) for τ_r in the range $-1\le\zeta\le\zeta_i$ that

$$\sqrt{3}\ln\left[4\left(a\zeta_i+s\right)\right]=k_1. \tag{49}$$

This equation can be used to eliminate s in Eq. (48). The resulting equation should be solved for a numerically. The solution is denoted by a_2. The range of a for case (*ii*) is $a_1\le a\le a_2$.

In case (*iii*) Eq. (28) is valid in the range $-1\le\zeta\le\zeta_p$, Eq. (30) in the range $\zeta_p\le\zeta\le\zeta_i$, and Eq. (32) in the range $\zeta_i\le\zeta\le 0$. Substituting Eqs. (28) and (30) into Eq. (16) and using Eq. (43) result in

$$C_4=\frac{\left(k_2-k_1g_1\right)}{\sqrt{3}}\ln\left(a\zeta_i+s\right)-\frac{k_2}{\sqrt{3}}\ln s. \tag{50}$$

Since $\tau_r<0$ at $\zeta=\zeta_p$ and $-1\le\zeta_p\le\zeta_i$, Eq. (37) is valid and becomes

$$\sqrt{3}\ln\left[4\left(a\zeta_p+s\right)\right]=k_1. \tag{51}$$

Substituting Eqs. (28) and (30) into Eq. (15) for $[\sigma_r]$ supplies the equation for determining the function $s\,(a)$. In particular, eliminating in this equation C_1 by means of Eq. (43), C_4 by means of Eq. (50), and ζ_p by means of Eq. (51) gives

$$\frac{k_1^2g_1}{6}+\frac{k_2}{\sqrt{3}}\ln\left(\frac{a\zeta_i+s}{s}\right)-\frac{k_1g_1}{\sqrt{3}}\ln\left[4\left(a\zeta_i+s\right)\right]+\frac{g_1}{2}\ln^2\left[4\left(s-a\right)\right]=0. \tag{52}$$

This equation should be solved numerically to find the function $s\,(a)$.

The solution of Eqs. (46), (48) and (52) determines s at any value of a in the range $a>a_{e2}$. This function is denoted by $s=s_2\,(a)$. However, this solution is valid if and only if no plastic yielding occurs at $\zeta=-1$. The corresponding condition follows from Eqs. (12), (13) and (18) in the form $\tau_r=\sigma_{Y1}/\sqrt{3}$ at $\zeta=-1$. Substituting this equation into Eq. (24) for τ_r in the range $-1\le\zeta\le\zeta_i$ leads to

$$\sqrt{3}\ln\left[4\left(s-a\right)\right]=-k_1. \tag{53}$$

Replacing s in this equation with $s_2\,(a)$ gives the equation for a. The solution of this equation is denoted by a_{p2}. It is evident that $a_p=a_{p2}$ in this case. The solution with one plastic region derived in

Section 5.2.1 is valid in the range $a_{e2} \le a \le a_{p2}$.

5.2.2 Plastic yielding occurs in the region $-1 \le \zeta \le \zeta_p$

In this case $a_{e1} \le a_{e2}$ and $\tau_r > 0$ in the plastic region. It is necessary to examine three cases; namely, (*i*) $\zeta_p < \zeta_i$, (*ii*) $\zeta_p = \zeta_i$, and (*iii*) $\zeta_p > \zeta_i$. In all these cases Eq. (27) is valid in the vicinity of the surface $\zeta = 0$ and Eq. (29) in the vicinity of the surface $\zeta = -1$. Therefore, these equations and the boundary conditions (14) combine to give

$$C_2 = -\frac{1}{2}\ln^2(4s) \quad \text{and} \quad C_3 = \frac{k_1 g_1}{\sqrt{3}}\ln(s-a). \tag{54}$$

Consider case (*i*). In this case Eq. (27) is valid in the range $\zeta_i < \zeta \le 0$, Eq. (28) in the range $\zeta_p \le \zeta \le \zeta_i$, and Eq. (29) in the range $-1 \le \zeta \le \zeta_p$. Substituting Eqs. (27) and (28) into Eq. (16) and using Eq. (54) result in

$$C_1 = \frac{(1-g_1)}{2}\ln^2\left[4(a\zeta_i + s)\right] - \frac{1}{2}\ln^2(4s). \tag{55}$$

Since $\tau_r > 0$ at $\zeta = \zeta_p$ and $-1 \le \zeta_p \le \zeta_i$, Eq. (36) is valid and becomes

$$\sqrt{3}\ln\left[4(a\zeta_p + s)\right] = -k_1. \tag{56}$$

Substituting Eqs. (28) and (29) into Eq. (15) for $[\sigma_r]$ supplies the equation for determining the function $s(a)$. In particular, eliminating in this equation C_3 by means of Eq. (54), C_1 by means of Eq. (55), and ζ_p by means of Eq. (56) gives

$$\frac{k_1^2 g_1}{6} + \frac{k_1 g_1}{\sqrt{3}}\ln\left[4(s-a)\right] + \frac{1}{2}\ln^2(4s) + \frac{(g_1-1)}{2}\ln^2\left[4(a\zeta_i + s)\right] = 0. \tag{57}$$

This equation should be solved numerically to find the function $s(a)$. This solution is valid if $\zeta_i \ge \zeta_p$. Replacing ζ_p with ζ_i in Eq. (56) yields

$$\sqrt{3}\ln\left[4(a\zeta_i + s)\right] = -k_1. \tag{58}$$

This equation can be used to eliminate s in Eq. (57). The resulting equation should be solved for a numerically. This solution is denoted by a_3. The range of a for case (*i*) is $a_{e1} \le a \le a_3$.

In case (*ii*) the elastic/plastic boundary coincides with the bi-material interface. Therefore, Eq. (27) is valid in the range $\zeta_i \le \zeta \le 0$ and Eq. (29) in the range $-1 \le \zeta \le \zeta_i$. Substituting these equations into Eq. (16) and eliminating C_2 and C_3 by means of Eq. (54) result in the following equation for determining the function $s(a)$

$$\ln^2\left[4(a\zeta_i + s)\right] - \ln^2(4s) + \frac{2k_1 g_1}{\sqrt{3}}\ln\left(\frac{a\zeta_i + s}{s-a}\right) = 0. \tag{59}$$

This equation should be solved numerically. The range of validity of this solution is restricted by the condition that the yield criterion is not violated in material 2 at $\zeta = \zeta_i$ (Fig. 2). This condition may be represented as $\tau_r = \sigma_{Y2}/\sqrt{3}$ at $\zeta = \zeta_i$. Then, it follows from Eq. (24) for τ_r in the range $\zeta_i \le \zeta \le 0$ that

$$\sqrt{3}\ln\left[4(a\zeta_i + s)\right] = -k_2. \tag{60}$$

This equation can be used to eliminate s in Eq. (59). The resulting equation should be solved for a numerically. This solution is denoted by a_4. The range of a for case (ii) is $a_3 \le a \le a_4$.

In case (iii) Eq. (27) is valid in the range $\zeta_p \le \zeta \le 0$, Eq. (31) in the range $\zeta_i \le \zeta \le \zeta_p$, and Eq. (29) in the range $-1 \le \zeta \le \zeta_i$. Substituting Eqs. (29) and (31) into Eq. (16) and using Eq. (54) result in

$$C_5 = \frac{(k_2 - k_1 g_1)}{\sqrt{3}} \ln(a\zeta_i + s) + \frac{k_1 g_1}{\sqrt{3}} \ln(s - a). \tag{61}$$

Since $\tau_r > 0$ at $\zeta = \zeta$ and $\zeta_i \le \zeta_p \le 0$, Eq. (34) is valid and becomes

$$\sqrt{3} \ln\left[4(a\zeta_p + s)\right] = -k_2. \tag{62}$$

Substituting Eqs. (27) and (31) into Eq. (15) for $[\sigma_r]$ supplies the equation for determining the function s (a). In particular, eliminating in this equation C_2 by means of Eq. (54), C_5 by means of Eq. (61), and ζ_p by means of Eq. (62) gives

$$\frac{k_2^2}{6} + \frac{k_2}{\sqrt{3}} \ln\left[4(a\zeta_i + s)\right] + \frac{k_1 g_1}{\sqrt{3}} \ln\left(\frac{s - a}{a\zeta_i + s}\right) + \frac{1}{2} \ln^2(4s) = 0. \tag{63}$$

This equation should be solved numerically to find the function s (a).

The solution of Eqs. (57), (59) and (63) determines s at any value of a in the range $a > a_{e1}$. This function is denoted by $s = s_1$ (a). However, this solution is valid if and only if no plastic yielding occurs at $\zeta = 0$. The corresponding condition follows from Eqs. (12), (13), and (18) in the form $\tau_r = -\sigma_{Y2} / \sqrt{3}$ at $\zeta = 0$. Substituting this equation into Eq. (24) for τ_r in the range $\zeta_i \le \zeta \le 0$ leads to

$$\sqrt{3} \ln(4s) = k_2. \tag{64}$$

Replacing s in this equation with s_1 (a) gives the equation for a. The solution of this equation is denoted by a_{p1}. It is evident that $a_p = a_{p1}$ in this case. The solution with one plastic region derived in Section 5.2.2 is valid in the range $a_{e1} \le a \le a_{p1}$.

5.3 Elastic/plastic solution with two plastic regions

Let ζ_{p1} and ζ_{p2} be the ζ-coordinates of the elastic/plastic boundaries. Plastic yielding occurs in the regions $-1 \le \zeta \le \zeta_{p1}$ and $\zeta_{p2} \le \zeta \le 0$. It is necessary to consider five cases separately; namely, $-1 \le \zeta_{p1} < \zeta_i$ and $\zeta_i < \zeta_{p2} \le 0$, $\zeta_{p1} > \zeta_i$ and $\zeta_{p1} < \zeta_{p2} \le 0$, $\zeta_{p2} < \zeta_i$ and $-1 \le \zeta_{p1} < \zeta_{p2}$, $\zeta_{p1} = \zeta_i$ and $\zeta_i < \zeta_{p2} \le 0$, and $\zeta_{p2} = \zeta_i$ and $-1 \le \zeta_{p1} < \zeta_i$. In all these cases Eq. (29) is valid in the vicinity of the surface $\zeta = -1$ and Eq. (32) in the vicinity of the surface $\zeta = 0$. These equations and the boundary conditions (14) combine to give

$$C_3 = \frac{k_1 g_1}{\sqrt{3}} \ln(s - a) \qquad \text{and} \qquad C_6 = -\frac{k_2}{\sqrt{3}} \ln s. \tag{65}$$

5.3.1 Case $-1 \le \zeta_{p1} < \zeta_i$ and $\zeta_i < \zeta_{p1} \le 0$

In this case Eq. (29) is valid in the region $-1 \le \zeta \le \zeta_{p1}$, Eq. (28) in the region $\zeta_{p1} \le \zeta \le \zeta_i$, Eq. (27) in the region $\zeta_i \le \zeta \le \zeta_{p2}$, and Eq. (32) in the region $\zeta_{p2} \le \zeta \le 0$. Substituting Eqs. (27) and (28) into Eq. (16) leads to

$$C_1 = C_2 + \frac{(1-g_1)}{2}\ln^2\left[4\left(a\zeta_i + s\right)\right]. \tag{66}$$

Since $\tau_r>0$ at $\zeta=\zeta_{p1}$ and $-1\le\zeta_{p1}\le\zeta_i$, Eq. (36) is valid and becomes

$$\sqrt{3}\ln\left[4\left(a\zeta_{p1}+s\right)\right] = -k_1\,. \tag{67}$$

Substituting Eqs. (28) and (29) into Eq. (15) for $[\sigma_r]$ leads to

$$\frac{g_1}{2}\ln^2\left[4\left(a\zeta_{p1}+s\right)\right] + \frac{k_1 g_1}{\sqrt{3}}\ln\left(a\zeta_{p1}+s\right) + C_1 - C_3 = 0\,. \tag{68}$$

Eliminating in this equation C_3 by means of Eq. (65) and ζ_{p1} by means of Eq. (67) gives

$$C_1 = \frac{g_1 k_1^2}{6} + \frac{k_1 g_1}{\sqrt{3}}\ln\left[4\left(s-a\right)\right]. \tag{69}$$

Since $\tau_r<0$ at $\zeta=\zeta_{p2}$ and $\zeta_i\le\zeta_{p2}\le 0$, Eq. (35) is valid and becomes

$$\sqrt{3}\ln\left[4\left(a\zeta_{p2}+s\right)\right] = k_2\,. \tag{70}$$

Substituting Eqs. (27) and (32) into Eq. (15) for $[\sigma_r]$ leads to

$$\frac{1}{2}\ln^2\left[4\left(a\zeta_{p2}+s\right)\right] - \frac{k_2}{\sqrt{3}}\ln\left(a\zeta_{p2}+s\right) + C_2 - C_6 = 0\,. \tag{71}$$

Eliminating in this equation C_6 by means of Eq. (65) and ζ_{p2} by means of Eq. (70) gives

$$C_2 = \frac{k_2^2}{6} - \frac{k_2}{\sqrt{3}}\ln\left(4s\right). \tag{72}$$

Using Eqs. (69) and (72) it is possible to transform Eq. (66) to

$$\frac{\left(g_1 k_1^2 - k_2^2\right)}{6} + \frac{k_1 g_1}{\sqrt{3}}\ln\left[4\left(s-a\right)\right] + \frac{k_2}{\sqrt{3}}\ln\left(4s\right) - \frac{\left(1-g_1\right)}{2}\ln^2\left[4\left(a\zeta_i+s\right)\right]. \tag{73}$$

The solution of this equation determines s as a function of a. This solution should be found numerically. Its range of validity is controlled by the conditions $\zeta_{p1}=\zeta_i$ or $\zeta_{p2}=\zeta_i$. Using Eqs. (67) and (70) these conditions are expressed as

$$\sqrt{3}\ln\left[4\left(a\zeta_i+s\right)\right] = -k_1 \qquad \text{or} \qquad \sqrt{3}\ln\left[4\left(a\zeta_i+s\right)\right] = k_2\,. \tag{74}$$

Since the function $s(a)$ has been determined, each of these conditions is an equation for a. The solution of Eq. $(74)^1$ is denoted by a_5 and the solution of Eq. $(74)^2$ by a_6. The solution derived in Section 5.3.1 is valid only if $a\le\min\{a_5,a_6\}$.

5.3.2 Case $\zeta_{p1}>\zeta_i$ and $\zeta_{p1}<\zeta_{p2}\le 0$

In this case Eq. (29) is valid in the region $-1\le\zeta\le\zeta_i$, Eq. (31) in the region $\zeta_i\le\zeta\le\zeta_{p1}$, Eq. (27) in the region $\zeta_{p1}\le\zeta\le\zeta_{p2}$, and Eq. (32) in the region $\zeta_{p2}\le\zeta\le 0$. Substituting Eqs. (29) and (31) into Eq. (16) and eliminating C_3 by means of Eq. (65) lead to

$$C_5 = \frac{k_1 g_1}{\sqrt{3}} \ln(s-a) + \frac{(k_2 - k_1 g_1)}{\sqrt{3}} \ln(a\zeta_i + s). \tag{75}$$

Since $\tau_r>0$ at $\zeta=\zeta_{p1}$ and $\zeta_i\leq\zeta_{p1}\leq 0$, Eq. (34) is valid and becomes

$$\sqrt{3}\ln\left[4\left(a\zeta_{p1} + s\right)\right] = -k_2. \tag{76}$$

Substituting Eqs. (27) and (31) into Eq. (15) for $[\sigma_r]$ leads to

$$C_2 = \frac{k_1 g_1}{\sqrt{3}} \ln\left(\frac{s-a}{a\zeta_i + s}\right) + \frac{k_2}{\sqrt{3}} \ln\left[4\left(a\zeta_i + s\right)\right] + \frac{k_2^2}{6}. \tag{77}$$

Since $\tau_r<0$ at $\zeta=\zeta_{p2}$ and $\zeta_i\leq\zeta_{p2}\leq 0$, Eq. (35) is valid and becomes

$$\sqrt{3}\ln\left[4\left(a\zeta_{p2} + s\right)\right] = k_2. \tag{78}$$

Substituting Eqs. (27) and (32) into Eq. (15) for $[\sigma_r]$ and eliminating C_6 and ζ_{p2} by means of Eqs. (65) and (78), respectively, lead to

$$C_2 = \frac{k_2^2}{6} - \frac{k_2}{\sqrt{3}} \ln(4s). \tag{79}$$

Eqs. (77) and (79) combine to give

$$k_1 g_1 \ln\left(\frac{s-a}{a\zeta_i + s}\right) + k_2 \ln\left[16s\left(a\zeta_i + s\right)\right] = 0. \tag{80}$$

This equation determines the function s (a). Its solution should be found numerically.

5.3.3 Case $\zeta_{p2}<\zeta_i$ and $-1\leq\zeta_{p1}<\zeta_{p2}$

In this case Eq. (29) is valid in the region $-1\leq\zeta<\zeta_{p1}$, Eq. (28) in the region $\zeta_{p1}\leq\zeta\leq\zeta_{p2}$, Eq. (30) in the region $\zeta_{p2}\leq\zeta\leq\zeta_i$, and Eq. (32) in the region $\zeta_i\leq\zeta\leq 0$. Substituting Eqs. (30) and (32) into Eq. (16) and eliminating C_6 by means of Eq. (65) lead to

$$C_4 = \frac{(k_2 - k_1 g_1)}{\sqrt{3}} \ln(a\zeta_i + s) - \frac{k_2}{\sqrt{3}} \ln s. \tag{81}$$

Since $\tau_r<0$ at $\zeta=\zeta_{p2}$ and $-1\leq \zeta_{p2}<\zeta_i$, Eq. (37) is valid and becomes

$$\sqrt{3}\ln\left[4\left(a\zeta_{p2} + s\right)\right] = k_1. \tag{82}$$

Substituting Eqs. (28) and (30) into Eq. (15) for $[\sigma_r]$ leads to

$$C_1 = \frac{k_2}{\sqrt{3}} \ln\left(\frac{a\zeta_i + s}{s}\right) - \frac{k_1 g_1}{\sqrt{3}} \ln\left[4\left(a\zeta_i + s\right)\right] + \frac{g_1 k_1^2}{6}. \tag{83}$$

Since $\tau_r>0$ at $\zeta=\zeta_{p1}$ and $-1\leq \zeta_{p1}<\zeta_i$, Eq. (36) is valid and becomes

$$\sqrt{3}\ln\left[4\left(a\zeta_{p1} + s\right)\right] = -k_1. \tag{84}$$

Substituting Eqs. (28) and (29) into Eq. (15) for $[\sigma_r]$ and eliminating C_3 and ζ_{p1} by means of Eqs. (65) and (84), respectively, lead to

$$C_1 = \frac{g_1 k_1^2}{6} + \frac{g_1 k_1}{\sqrt{3}} \ln\left[4(s-a)\right]. \tag{85}$$

Eqs. (83) and (85) combine to give

$$k_1 g_1 \ln\left[16(s-a)(a\zeta_i + s)\right] - k_2 \ln\left(\frac{a\zeta_i + s}{s}\right) = 0. \tag{86}$$

This equation determines the function s (a). Its solution should be found numerically.

5.3.4 Case $\zeta_{p1}=\zeta_i$ and $\zeta_i<\zeta_{p2}\le 0$

In this case Eq. (29) is valid in the region $-1\le\zeta\le\zeta_i$, Eq. (27) in the region $\zeta_i\le\zeta\le\zeta_{p2}$, and Eq. (32) in the region $\zeta_{p2}\le\zeta\le 0$. Substituting Eqs. (27) and (29) into Eq. (16) and eliminating C_3 by means of Eq. (65) lead to

$$C_2 = \frac{k_1 g_1}{\sqrt{3}} \ln\left(\frac{s-a}{a\zeta_i + s}\right) - \frac{1}{2}\ln^2\left[4(a\zeta_i + s)\right]. \tag{87}$$

Since $\tau_r<0$ at $\zeta=\zeta_{p2}$ and $\zeta_i\le\zeta_{p2}\le 0$, Eq. (35) is valid and becomes

$$\sqrt{3}\ln\left[4(a\zeta_{p2} + s)\right] = k_2. \tag{88}$$

Substituting Eqs. (27) and (32) into Eq. (15) for $[\sigma_r]$ leads to

$$C_2 = \frac{k_2^2}{6} - \frac{k_2}{\sqrt{3}}\ln(4s). \tag{89}$$

Eqs. (87) and (89) combine to give

$$\frac{k_2^2}{6} - \frac{k_2}{\sqrt{3}}\ln(4s) - \frac{k_1 g_1}{\sqrt{3}}\ln\left(\frac{s-a}{a\zeta_i + s}\right) + \frac{1}{2}\ln^2\left[4(a\zeta_i + s)\right] = 0. \tag{90}$$

This equation determines the function s (a). Its solution should be found numerically. The range of validity of this solution is restricted by the condition that the yield criterion is not violated in material 2 at $\zeta=\zeta_i$ (Fig. 2). This condition may be represented by Eq. (60). This equation can be used to eliminate s in Eq. (90). The resulting equation should be solved for a numerically.

5.3.5 Case $\zeta_{p2}=\zeta_i$ and $-1\le\zeta_{p1}<\zeta_i$

In this case Eq. (29) is valid in the region $-1\le\zeta\le\zeta_{p1}$, Eq. (28) in the region $\zeta_{p1}\le\zeta\le\zeta_i$, and Eq. (32) in the region $\zeta_i\le\zeta\le 0$. Substituting Eqs. (28) and (32) into Eq. (16) and eliminating C_6 by means of Eq. (65) lead to

$$C_1 = \frac{k_2}{\sqrt{3}}\ln\left(\frac{a\zeta_i + s}{s}\right) - \frac{g_1}{2}\ln^2\left[4(a\zeta_i + s)\right]. \tag{91}$$

Since $\tau_r>0$ at $\zeta=\zeta_{p1}$ and $-1\le\zeta_{p1}\le\zeta_i$, Eq. (36) is valid and becomes

$$\sqrt{3}\ln\left[4(a\zeta_{p1} + s)\right] = -k_1. \tag{92}$$

Substituting Eqs. (28) and (29) into Eq. (15) for $[\sigma_r]$ leads to

$$C_1 = \frac{k_1^2 g_1}{6} + \frac{k_1 g_1}{\sqrt{3}} \ln\left[4(s-a)\right]. \tag{93}$$

Eqs. (91) and (93) combine to give

$$\frac{k_1^2 g_1}{6} + \frac{k_1 g_1}{\sqrt{3}} \ln\left[4(s-a)\right] - \frac{k_2}{\sqrt{3}} \ln\left(\frac{a\zeta_i + s}{s}\right) + \frac{g_1}{2} \ln^2\left[4(a\zeta_i + s)\right] = 0 . \tag{94}$$

This equation determines the function $s(a)$. Its solution should be found numerically. The range of validity of this solution is restricted by the condition that the yield criterion is not violated in material 1 at $\zeta=\zeta_i$ (Fig. 2). This condition may be represented by Eq. (49). This equation can be used to eliminate s in Eq. (94). The resulting equation should be solved for a numerically.

6. Bending moment, stresses and geometric parameters

It is seen from Eqs. (26)-(32) that the through thickness distribution of the radial stress is known if the constants of integration involved in these equations are known. These constants have been found for any stage of the process. Therefore, the through thickness distribution of the radial stress can be calculated at any value of a with no difficulty. Using this distribution and Eq. (24) the through thickness distribution of the circumferential stress is found from Eq. (33). Then, the dimensionless bending moment is determined from Eq. (21) as a function of a. It is worthy of note that the integral involved in Eq. (21) can be evaluated in terms of elementary functions at any stage of the process. However, the final expression is cumbersome. In order to find the radial and circumferential stresses as functions of r at a given value of a, it is necessary to use Eq. (5). Geometric parameters of the sheet after any amount of deformation are given by Eqs. (6) and (7). In addition, the current thicknesses of the individual layers are determined from Eqs. (6) and (19) as

$$h_1 = r_i - r_{CD} = H\left(\sqrt{\frac{\zeta_i}{a} + \frac{s}{a^2}} - \sqrt{\frac{s}{a^2} - \frac{1}{a^2}}\right), \quad h_2 = r_{AB} - r_i = H\left(\frac{\sqrt{s}}{a} - \sqrt{\frac{\zeta_i}{a} + \frac{s}{a^2}}\right). \tag{95}$$

Here h_1 is the current thickness of the layer made of material 1 and h_2 is the current thickness of the layer made of material 2 (Fig. 2).

7. Illustrative example

Take, for example, $g_1=3$, $k_1=0.0012$ and $k_2=0.0008$. Solving Eqs. (40), (41) and (42) numerically gives $a_{e1}>a_{e2}$ for any value of ζ_i. Therefore, the elastic/plastic solution given in Section 5.2.1 should be used in the range $a>a_{e2}$. The subsequent analysis depends on the value of ζ_i. Numerical calculation was conducted for $\zeta_i=-0.2$, $\zeta_i=-0.4$ and $\zeta_i=-0.6$. If $\zeta_i=-0.2$ and $\zeta_i=-0.4$ then the solution given in Section 5.3.5 is valid in the range $a_p \le a \le a_f$. If $\zeta_i=-0.6$ then the solutions given in Sections 5.3.1 and 5.3.5 are valid, sequentially, in the range $a_p \le a \le a_f$. Here $a_f=\pi/40$. It is seen from Eq. (6) that this value of a corresponds to the final configuration in which $\theta_0=\pi/2$ if $L/H=10$ (Fig. 1). The numerical solution is illustrated in Figs. 3 to 5. In particular, the variation of the

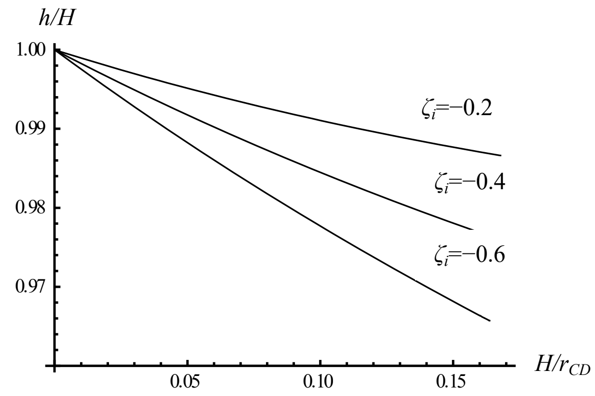

Fig. 3 Variation of the thickness of the sheet with H/r_{CD}

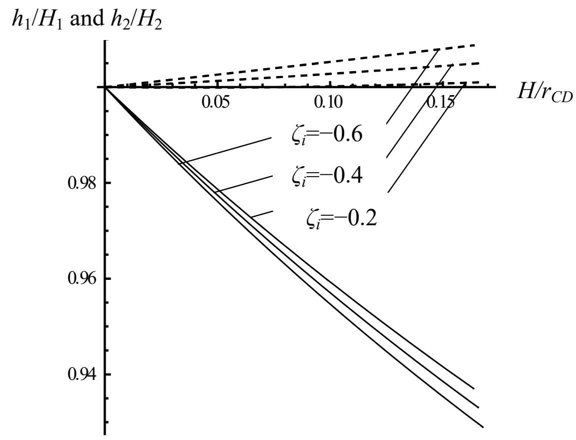

Fig. 4 Variation of the thickness of the sheet with H/r_{CD}

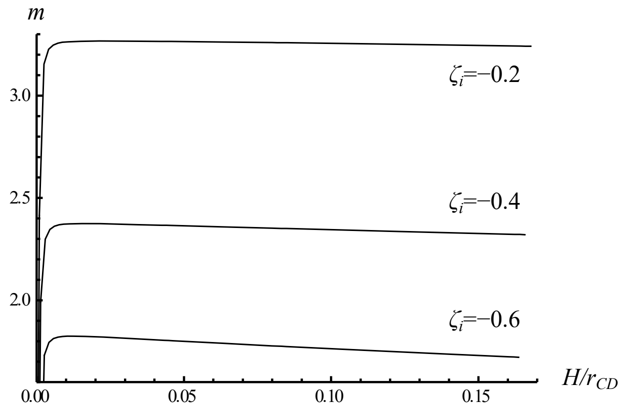

Fig. 5 Variation of the dimensionless bending moment with H/r_{CD}

thickness of the sheet with H/r_{CD} is depicted in Fig. 3 for $\zeta_i=-0.2$, $\zeta_i=-0.4$ and $\zeta_i=-0.6$. It is seen from this figure that the thickness changes as the deformation proceeds. It is worthy of note that the thickness of homogeneous sheets of rigid perfectly plastic material is constant in the plane strain pure bending (Hill, 1950). The variation of the thickness of each layer with H/r_{CD} is shown in Fig. 4. The solid lines correspond to the layer made of material 2 and the broken lines to the layer made of material 1 (Fig. 2). It is seen from this figure that the thickness of the layer made of material 2 changes more significantly than the thickness of the layer made of material 1. This is not surprisingly because material 2 is softer than material 1. Finally, the dependence of the dimensionless bending moment on H/r_{CD} is depicted in Fig. 5. It is seen from this figure that this moment rapidly increases at the very beginning of the process and then is almost constant.

8. Conclusions

The theoretical model of the plane strain bending process of bimetallic sheets presented in this paper deals only with elastic perfectly plastic materials. An advantage of this model is that the solution is semi-analytic. In particular, a numerical technique is only necessary to solve several transcendental equations. Therefore, a very high accuracy of numerical solutions can be easily achieved and such solutions can serve as a simple benchmark test for numerical packages. A necessity of such tests in metal forming applications has been pointed out in Roberts *et al*. (1992).

The model developed can be extended to strain hardening materials using the solution given in Alexandrov and Hwang (2010), to a class of anisotropic materials using the solution given in Alexandrov and Hwang (2009) and to bending under tension using the solution given in Alexandrov *et al*. (2011).

The illustrative example demonstrates that the current thickness of the sheet is sensitive to the initial thickness of each layer (Fig. 3). The thickness of the softer layer decreases and the thickness of the harder layer increases as the deformation proceeds (Fig. 4). The bending moment rapidly increases at the very beginning of the process and then becomes almost constant (Fig. 5). This qualitative behavior of the solution is similar to that found for homogeneous sheets in Alexandrov *et al*. (2006).

Acknowledgments

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