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Exact vibration of Timoshenko beam combined with multiple mass spring sub-systems

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Abstract. This paper deals with the analysis of the natural frequencies, mode shapes of an axially loaded beam system carrying ends consisting of non-concentrated tip masses and three spring-two mass sub-systems. The influence of system design and sub-system parameters on the combined system characteristics is the major part of this investigation. The effect of material properties, rotary inertia and shear deformation of the beam system is included. The end masses are elastically supported against rotation and translation at an offset point from the point of attachment. Sub-systems are attached to center of gravity eccentric points out of the beam span. The boundary conditions of the ordinary differential equation governing the lateral deflections and slope due to bending of the beam system including developed shear force frequency dependent terms, due to the sub–system suspension, have been formulated. Exact formulae for the modal frequencies and the modal shapes have been derived. Based on these formulae, detailed parametric studies are carried out. The geometrical and mechanical parameters of the system under study have been presented in non-dimensional analysis. The applied mathematical model is presented to cover wide range of mechanical, naval and structural engineering applications.

Keywords: vibration frequencies; exact solution; Timoshenko beam; eccentric mass; sub-system; combined system

1. Introduction

The study of vibration of beams with generalized end conditions and integrated with discrete sub-systems is crucial, because it has a wide range of applications in mechanical, aerospace and structural engineering. Timoshenko (1922) was the first how presented the vibration problem of beams with rotary inertia and shear deformation effect. Huang (1961) investigated the frequency equations and normal modes of free flexural vibrations of uniform beams including the effect of shear and rotary inertia for classical end conditions. Cowper (1966) derived a solution for the shear deformation coefficient for the different beam cross sections. The effects of axial load on the natural frequencies have been investigated by (Saito and Otomi 1979, Kounadis 1980, Sato 1991, Takahashi 1980, Grossi and Laura 1982, Bokaian 1990, Naguleswaran 2004, Ari-Gur and

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Elishakoff 1990). Kounadis (1980) studied the equation of motion of vibrating Timoshenko beam-column system. Sato (1991) studied also the governing equations of motion for vibration and stability of Timoshenko beam based on Hamilton's principle. Bokaian (1990) derived a frequency equation for Bernoulli-Euler beam with several classical boundary conditions. When a beam loaded axially in compression it may go through an instability by buckling. The beam instability and the critical buckling load have been investigated by (Naguleswaran 2004, Ari-Gur and Elishakoff 1990). Naguleswaran (2004) studied the transverse vibration of an uniform Bernoulli-Eulerbeam under linearly varying fully tensile, partly tensile or fully compressive axial force distribution. He also investigated the buckling of the beam when subjected to compressive axial load. Beams with generalized end conditions have been considered by (Chang 1993, To 1982, Farghaly 1992, Farghaly 1993, Farghaly and Shebl 1995). Farghaly derived an exact frequency equation for uniform cantilever Bernoulli-Euler-beam with an elastically mounted nonconcetrated tip mass. The beam system is subjected to a constant axial tensile or compression load acting at the center of the tip mass (Farghaly 1992). Farghaly and Shebl (1995) introduced an exact frequency equation for Timoshenko beam with a generalized end conditions. The combination between continious and discrete mass spring systems has many applications. The vibration of beams with an end or intermediate mass spring system has been studied by (Gürgöze 1996, Gürgöze and Batan 1996, Chen *et al.* 2015, Rossi *et al.* 1993). Snowdon (1966), Bergman and Nicholson (1985) have studied the free and forced vibration analyses of Bernoulli-Euler cantilever beam in which single mass spring damped sub-systems are attached. Rossi *et al.* (1993) presented an exact solution for classical thick beams carring spring mass system. They model studied the action of the spring on the beam, by equivalent transverse force. Gürgöze (1996) investigated the eigenfrequencies of a cantiliver uniform Bernoulli-Euler beam with attached tip mass carrying a single spring-mass sub-system. Gürgöze and Batan (1996) investigated a unifom beam with an intermediate spring-mass system. Very recently, Chen *et al.* (2015) presented a general exact solution for free vibration of a tensioned beam with any number of lateral and rotational two spring-mass damped sub-systems . His solution is also suitable for multi-span beam.

The survey of the above references indicates that the title problem with the applied mathematical model for exact vibration analysis of axially loaded Timoshenko beam system combined with three spring-two mass sub-systems beam has not been extensively investigated. In this paper, the exact natural frequency equation and the mode shapes of beam system have been investigated. This beam is carrying non-concentrated end masses of finite length, with attached three spring-two mass sub-systems acting at center of gravity of end masses out span. The problem statement is based on Timoshenko beam bending theory of elasticity. New exact formulae for the frequency equation and the modal shape have been derived, including the proposed sub-systems. Based on these formulae, detailed parametric study of the modal frequency parameters and the modal shapes have been introduced. This includes the effect of changing the non-dimensional values of sub-system masses, stiffness and location.

2. Applied mathematical model

Timoshenko beam system means here a uniform single span beam carrying two eccentric rigid masses elastically supported at an arbitrary offset point from the point of attachment at both ends. The combination between Timoshenko beam system and sub-system consisting of three springs-

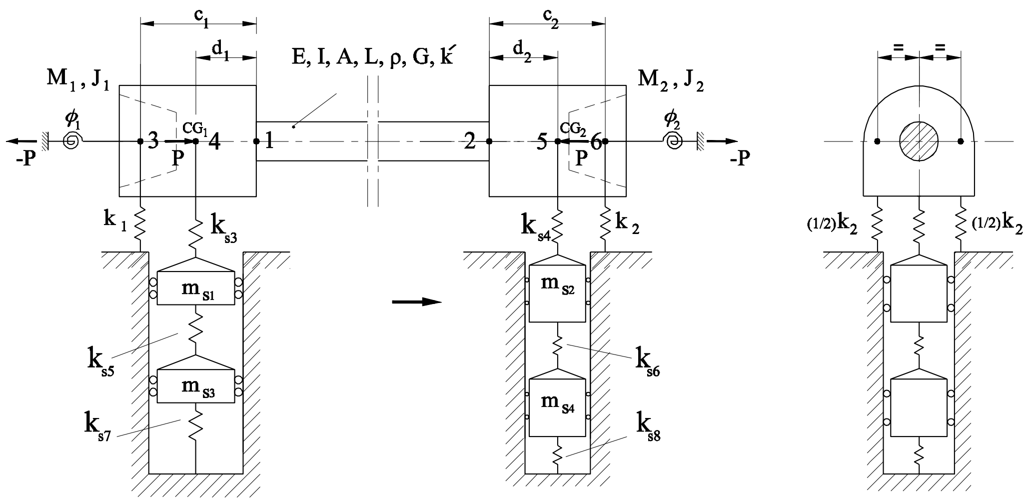
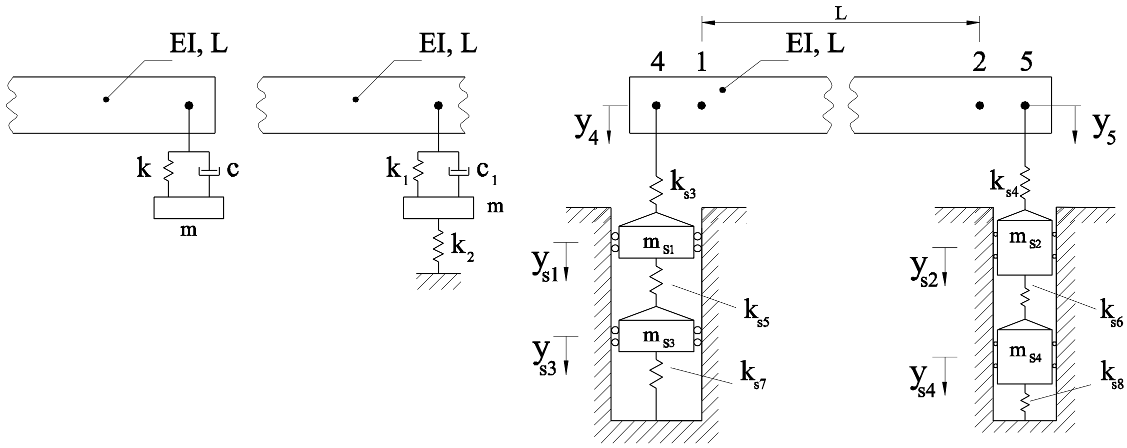


Fig. 1 Present combined system mathematical model



(a) (b) (c) Present undamped model

Fig. 2 Sub-system models

two mass proposed is satisfied through the attachment of the sub-system to center of gravity of end masses rigidly supported to the beam ends respectively. Fig. 1 shows the model to be investigated in this work.

Shown in Fig. 1, is an axially-loaded Timoshenko beam system with generalized end conditions including attached three spring-two mass sub-systems at center of gravity of end masses. A set of design, general, and specific variables have been considered, such as the elasticity modulus, moment of inertia, area, mass density, shear modulus of elasticity, shear shape factor, Poisson's ratio and length of beam span as, $E, I, A, \rho, G, k', v, L$ respectively. The translational, rotational spring stiffness and end masses are rigidly attached as shown to the beam ends, $k, \phi, M_1, M_2,$ respectively. The rotation and translational springs are acting at points at distance c_1 and c_2 from points of attachment 1, and 2 respectively. The system is subjected to constant axial tensile or compressive load acting at the end masses center of gravity at distance d_1 and d_2. A proposed sub–system consisting of three spring-two mass is attached to the end mass eccentric points as shown in Fig. 1. The sub-system masses m_{s1}, m_{s2}, m_{s3} and m_{s4} respectively, while k_{s3}, k_{s4}, k_{s5}, k_{s6}, k_{s7}, and k_{s8} represent the linear translational spring stiffness respectively. More details on the system non-dimensional parameters will be appeared in the next sections of this work.

3. End conditions with three spring-two mass sub-system

(Snowdon 1966, Bergman and Nicholson 1985) have studied the free and forced vibration analyses of Bernoulli-Euler cantilever beam in which single mass spring damped sub-systems are attached. Fig. 2(a) (Rossi *et al.* 1993) studied the exact analytical solution of Timoshenko beam carrying elastically mounted masses. Three beam configurations were presented, *P-P*; *C-P* and *C-C.* (Gürgöze 1996) studied the vibration problem of Bernoulli-Euler cantilever beam with tip mass and single spring mass (SS1) sub-system. Very recently, (Chen *et al.* 2015) in an interesting work studied the vibration problem of Bernoulli-Euler beam carrying two spring mass damped sub-system at any point along its span (Fig. 2(b)). The authors extended Chen's undamped (SS2) sub-system general model to another proposed one, consisting of three spring-two mass (SS3) attached to a vibrating Timoshenko beam system presented as shown in Fig. 1. The mathematical treatment of the present proposed model has not been appeared in literature and will be presented in details as follows:

Fig. 2(c) shows the four independent coordinates y_{s1}, y_{s2}, y_{s3} *and* y_{s4}. representing the four masses m_{s1}, m_{s2}, m_{s3} *and* m_{s4}. The shear forces acting at point 4 due to the sub-system 1 are

$$-k_{s3}(y_{s1} - y_4) - k_{s5}(y_{s1} - y_{s3}) = m_{s1}\,\ddot{y}_{s1}\ ; \tag{1}$$

$$-k_{s5}(y_{s3} - y_{s1}) - k_{s7}\,y_{s3} = m_{s3}\,\ddot{y}_{s3} \tag{2}$$

and the shear forces acting at point 5 due to the sub-system 2 are

$$-k_{s4}(y_{s2} - y_5) - k_{s6}(y_{s2} - y_{s4}) = m_{s2}\,\ddot{y}_{s2;} \tag{3}$$

$$-k_{s6}(y_{s4} - y_{s2}) - k_{s8}\,y_{s4} = m_{s4}\,\ddot{y}_{s4} \tag{4}$$

Using the following solution for Eqs. (1) to (4)

$$y_{s1} = Y_{s1}\,e^{j\omega t},\ y_{s2} = Y_{s2}\,e^{j\omega t},\ y_{s3} = Y_{s3}\,e^{j\omega t}, \text{ and }\ y_{s4} = Y_{s4}\,e^{j\omega t},$$

Therefore, the shear force Eqs. (1), (2), (3) and (4) can be written in the following non-dimensional forms

$$(Z_{s3} + Z_{s5} - \lambda^4\,\bar{m}_{s1})\,Y_{s1} - Z_{s3}\,Y_4 - Z_{s5}\,Y_{s3}\ = 0\ ; \tag{5}$$

$$(Z_{s5} + Z_{s7} - \lambda^4\,\bar{m}_{s3})\,Y_{s3} - Z_{s5}\,Y_{s1}\ = 0\ ; \tag{6}$$

$$(Z_{s4} + Z_{s6} - \lambda^4\bar{m}_{s2})\,Y_{s2} - Z_{s4}\,Y_5 - Z_{s6}\,Y_{s4}\ = 0 \quad \text{and} \tag{7}$$

$$(Z_{s6} + Z_{s8} - \lambda^4\bar{m}_{s4})\,Y_{s4} - Z_{s6}\,Y_{s2}\ = 0 \tag{8}$$

where

$$Z_{s3} = k_{s3}L^3/EI; \quad Z_{s5} = k_{s5}L^3/EI; \quad Z_{s7} = k_{s7}L^3/EI;$$
$$Z_{s4} = k_{s4}L^3/EI; \quad Z_{s6} = k_{s6}L^3/EI; \quad Z_{s8} = k_{s8}L^3/EI;$$

In fact, the connection of sub-system with the center of gravity of the eccentric end mass (point 4 and 5, (cf. Fig. 1)), will affect the shear force and bending moment balance of the system end conditions (at point 1 and 2). The additional developed shear force terms due to the sub-system connection are $k_{s3}\left[-\left(Y_{s1}-Y(0)\right)-d_1\,\Psi(0)\right]$ and $k_{s4}\left[\left(Y_{s2}-Y(1)\right)-d_2\,\Psi(1)\right]$ at point 1 and 2 respectively. On the other hand, the last two terms for the bending moment effect become, $k_{s3}\,d_1\left[\left(Y_{s1}-Y(0)\right)+d_1\,\Psi(0)\right]$ and $k_{s4}\,d_2\left[\left(Y_{s2}-Y(1)\right)-d_2\,\Psi(1)\right]$ respectively. Four governing equations for the present combined system shown in Fig. 1 can be written with special care as follows:
At point 1 ($\zeta=0$);

$$\left(k'GA\right)(Y'-\Psi)(0)=-[-\omega^2M_1d_1+k_1c_1]\,\Psi(0)+(-\omega^2M_1+k_1)\,Y(0)+P\,Y'(0)$$
$$+k_{s3}\left[-\left(Y_{s1}-Y(0)\right)-d_1\Psi(0)\right] \tag{9}$$

$$EI\,\Psi'(0)=[-\omega^2(J_1+M_1d_1^2)+(\phi_1+k_1c_1^2)]\,\Psi(0)+(-\omega^2M_1d_1-k_1c_1)\,Y(0)-P\,d_1Y'(0)$$
$$+k_{s3}\,d_1\left[\left(Y_{s1}-Y(0)\right)+d_1\,\Psi(0)\right] \tag{10}$$

at point *2* ($\zeta=1$);

$$(k'GA)(Y'-\Psi)(1)=-[-\omega^2M_2d_2+k_2c_2]\,\Psi(1)-(-\omega^2M_2+k_2)\,Y(1)+P\,Y'(1)$$
$$+k_{s4}\left[\left(Y_{s2}-Y(1)\right)-d_2\,\Psi(1)\right] \tag{11}$$

$$EI\,\Psi'(1)=-[-\omega^2(J_2+M_2d_2^2)+(\phi_2+k_2c_2^2)]\Psi(1)-\left(-\omega^2M_2d_2+k_2c_2\right)Y(1)$$
$$+P\,d_2\,Y'(1)$$
$$+k_{s4}\,d_2\left[\left(Y_{s2}-Y(1)\right)-d_2\,\Psi(1)\right] \tag{12}$$

Using Eqs. (5) and (6) one can get the relation

$$Y_{s1}=\left[\frac{Z_{s3}Z_{57}}{Z_{35}\,Z_{57}-Z_{S5}^2}\right]Y_4$$
$$=\left[\frac{Z_{s3}Z_{57}}{Z_{35}\,Z_{57}-Z_{S5}^2}\right](Y\,(0)-d_1\Psi(0)) \tag{13}$$

here $Z_{35}=-\lambda^4\bar{m}_{s1}+Z_{s3}+Z_{s5}$ and $Z_{57}=-\lambda^4\bar{m}_{s3}+Z_{s5}+Z_{s7}$
Similarly, using Eqs. (7) and (8) one can get the relation.

$$Y_{s2}=\left[\frac{Z_{s4}Z_{68}}{Z_{46}Z_{68}-Z_{S6}^2}\right]Y_5$$
$$=\left[\frac{Z_{s4}Z_{68}}{Z_{46}Z_{68}-Z_{S6}^2}\right]\left(Y\,(1)+d_2\Psi(1)\right) \tag{14}$$

here $Z_{46}=-\lambda^4\bar{m}_{s2}+Z_{s4}+Z_{s6}$ and $Z_{68}=-\lambda^4\bar{m}_{s4}+Z_{s6}+Z_{s8}$

The non-dimensional form of Eqs. (9) to (12) together with Eqs. (13) and (14) can be derived and rearranged as follows

$$\bar{\eta}_{1s}\,L\Psi(0)+\theta_{1s}\,Y(0)-s_p\,Y'(0)=0 \tag{15}$$

$$L\Psi'(0)-\gamma_{1s}\,L\Psi(0)+\epsilon_{1s}\,Y(0)+p_1^2\,Y'(0)=0 \tag{16}$$

$$\bar{\eta}_{2s}\, L\Psi(1) - \theta_{2s}\, Y(1) - s_p Y'(1) = 0 \tag{17}$$

$$L\Psi'^{(1)} + \gamma_{2s}\, L\Psi(1) - \epsilon_{2s}\, Y(1) - p_2^2\, Y'(1) = 0 \tag{18}$$

Introducing the shear force frequency dependent terms

$$\tau_{1s} = Z_{s3}\left[\frac{-Z_{s5}^2 - Z_{s3}\cdot Z_{57} + Z_{35}\cdot Z_{57}}{-Z_{s5}^2 + Z_{35}\cdot Z_{57}}\right] \tag{19}$$

$$\tau_{2s} = Z_{s4}\left[\frac{-Z_{s6}^2 - Z_{s4}\cdot Z_{68} + Z_{46}\cdot Z_{68}}{-Z_{s6}^2 + Z_{46}\cdot Z_{68}}\right] \tag{20}$$

The different parameters in Eqs. (15)-(18) can be written as follows

$$\theta_{1s} = s^2\left(-\lambda^4 \bar{m}_1 + Z_1 + \tau_{1s}\right)$$

$$\epsilon_{1s} = -\left(-\lambda^4 \bar{m}_1\, \bar{d}_1 + Z_1 \bar{c}_1 + \tau_{1s} \bar{d}_1\right)$$

$$\bar{\eta}_{1s} = 1 + s^2 \epsilon_{1s}$$

$$\gamma_{1s} = -\lambda^4\left(\bar{J}_1 + \bar{m}_1\, \bar{d}_1^2\right) + \Phi_1 + Z_1 \bar{c}_1^2 + \tau_{1s}\, \bar{d}_1^2 \tag{21 a to d}$$

and

$$\theta_{2s} = s^2\left(-\lambda^4\, \bar{m}_2 + Z_2 + \tau_{2s}\right)$$

$$\epsilon_{2s} = -\left(-\lambda^4 \bar{m}_2\, \bar{d}_2 + Z_2 \bar{c}_2 + \tau_{2s} \bar{d}_2\right)$$

$$\bar{\eta}_{2s} = 1 + s^2 \epsilon_{2s}$$

$$\gamma_{2s} = -\lambda^4\left(\bar{J}_2 + \bar{m}_2\, \bar{d}_2^2\right) + \Phi_2 + Z_2 \bar{c}_2^2 + \tau_{2s}\, \bar{d}_2^2$$

$$s_p = 1 - s^2 p^2$$

$$p_1^2 = p^2 \bar{d}_1$$

$$p_2^2 = p^2 \bar{d}_2 \tag{22 a to g}$$

Where.

$\bar{m}_1 = M_1/m$; $\bar{m}_2 = M_2/m$; $\bar{J}_1 = J_1/mL^2$; $\bar{J}_2 = J_2/mL^2$; $\bar{d}_1 = d_1/L$; $\bar{d}_2 = d_2/L$; $\bar{c}_1 = c_1/L$; $\bar{c}_2 = c_2/L$; $Z_1 = k_1 L^3/EI$; $Z_2 = k_2 L^3/EI$; $\Phi_1 = \phi_1 L/EI$ and $\Phi_2 = \phi_2 L/EI$

We can conclude that, the four Eqs. (15)-(18) represent the present system end conditions. The system equation of motion and the general solution to get the system frequency equation and mode shape are presented in the next sections.

4. Equation of motion and solution

The decoupled differential equations of motion based on Timoshenko beam theory of elasticity and for axial load consideration presented by (Sato 1991), will be presented for harmonic solution in the following non dimensional form

$$Y''''(\zeta) + \alpha\, Y''(\zeta) + \beta^2\, Y(\zeta) = 0 \tag{23}$$

$$L\Psi''''(\zeta) + \alpha\, L\Psi''(\zeta) + \beta^2\, L\Psi(\zeta) = 0 \tag{24}$$

where

$$\alpha = \frac{\lambda^4(r^2+s^2)-p^2(\lambda^4\, r^2 s^2-1)}{(1-s^2\, p^2)}, \qquad \beta^2 = \frac{\lambda^4(\lambda^4 r^2 s^2-1)}{(1-s^2\, p^2)}$$

and

$$p^2 = PL^2/EI, \lambda^4 = \rho A L^4 \omega^2/EI,\; s^2 = 2r^2(1+v)/k', r^2 = I/AL^2$$

The general solution for Eqs. (23) and (24) for small harmonic oscillations are

$$Y(\zeta) = A_1\, sin\, a\zeta + A_2\, cos\, a\zeta + A_3\, sinh\, b\zeta + A_4\, cosh\, b\zeta \tag{25}$$

and

$$L\Psi(\zeta) = -\left(\frac{\delta_1}{a}\right)A_1\, cos\, a\zeta + \left(\frac{\delta_1}{a}\right)A_2\, sin\, a\zeta + \left(\frac{\delta_2}{b}\right)A_3\, cosh\, b\zeta + \left(\frac{\delta_2}{b}\right)A_4\, sinh\, b\zeta \tag{26}$$

where

$$a^2 = \left(\frac{\alpha}{2}\right) + \left(\left(\frac{\alpha}{2}\right)^2 - \beta^2\right)^{1/2} \quad \text{and} \quad b^2 = -\left(\frac{\alpha}{2}\right) + \left(\left(\frac{\alpha}{2}\right)^2 - \beta^2\right)^{1/2}$$

One can obtain the expression of δ_1 and δ_2 in Eq. (26) in the form

$$\delta_1 = s^2\lambda^4 - a^2(1-s^2p^2) \tag{27}$$

$$\delta_2 = s^2\lambda^4 + b^2(1-s^2p^2) \tag{28}$$

5. Frequency and modal shape equations

The system frequency equation can be derived using Eqs. (25), (26) and the two end conditions at ζ = 0, and ζ =1, Eqs. (15)-(18). This yields to four equations in unknowns A_1, A_2, A_3, and A_4. These equations form four homogeneous equations in four unknowns. For the problem to have a non-trivial solution, the four unknowns cannot all be zero. Hence the determinant of the coefficients matrix of the system of equations must be vanished and may be written as follows

$$\text{det.}\begin{vmatrix} \epsilon_{11} & (\delta_1-\epsilon_{1s}) & e_{11} & (\delta_2-\epsilon_{1s}) \\ \epsilon_{21} & \theta_{1s} & e_{21} & \theta_{1s} \\ (\delta_1-\epsilon_{2s})\sin a - \epsilon_{12}\cos a & \epsilon_{12}\sin a + (\delta_1-\epsilon_{2s})\cos a & (\delta_2-\epsilon_{2s})sh\, b + e_{12}\text{ch}\, b & e_{12}\text{sh}\, b + (\delta_2-\epsilon_{2s})ch\, b \\ -\theta_{2s}\sin a - \theta_{2s}\sin a & \epsilon_{22}\sin a - \theta_{2s}\cos a & -\theta_{2s}\text{ sh}\, b + e_{22}\text{ch}\, b & e_{22}\text{sh}\, b - \theta_{2s}\text{ ch}\, b \end{vmatrix} = 0 \tag{29}$$

Therefore, the expansion of this determinant gives the new exact frequency equation including the proposed sub-systems which can be written in the form

(SST). sin *a* sinh *b*+(SCT). sin *a* cosh *b*+(CST). cos *a* sinh *b*+(CCT) . cos *a* cosh *b*+(ABT)= 0 (30)

Here,

$$(\mathrm{SST}) = \theta_{1s}\theta_{2s}(\delta_1-\delta_2)^2 + \big(-e_{11}\theta_{1s} + e_{21}(\delta_1-\epsilon_{1s})\big)\big(e_{12}\theta_{2s} + e_{22}(\delta_1-\epsilon_{2s})\big) - \big(\epsilon_{11}\,\theta_{1s} - \epsilon_{21}(\delta_2-\epsilon_{1s})\big)\big(\epsilon_{12}\,\theta_{2s} + \epsilon_{22}(\delta_2-\epsilon_{2s})\big) + (e_{21}\epsilon_{11} - e_{11}\epsilon_{21})(e_{12}\epsilon_{22} - e_{22}\epsilon_{12})$$

$$(\mathrm{SCT}) = \theta_{1s}(\delta_2-\delta_1)\ \big(e_{12}\theta_{2s} + e_{22}(\delta_{1-}\epsilon_{2s})\big) + \theta_{2s}(\delta_2-\delta_1)\ \big(-e_{11}\theta_{1s} + e_{21}(\delta_1-\epsilon_{1s})\big) + \big(\epsilon_{11}\,\theta_{1s} - \epsilon_{21}(\delta_2-\epsilon_{1s})\big)(\epsilon_{12}\,e_{22} - \epsilon_{22}e_{12}) + \big(\epsilon_{12}\,\theta_{2s} + \epsilon_{22}(\delta_2-\epsilon_{2s})\big)(\epsilon_{11}\,e_{21} - \epsilon_{21}e_{11})$$

$$(\mathrm{CST}) = \theta_{1s}(\delta_2-\delta_1)\big(\epsilon_{12}\,\theta_{2s} + \epsilon_{22}(\delta_2-\epsilon_{2s})\big) + \theta_{2s}(\delta_2-\delta_1)\big(+\epsilon_{11}\,\theta_{1s} - \epsilon_{21}(\delta_2-\epsilon_{1s})\big) + \big(-e_{11}\,\theta_{1s} + e_{21}(\delta_1-\epsilon_{1s})\big)(\epsilon_{22}e_{12} - \epsilon_{12}e_{22}) - \big(e_{12}\theta_{2s} + e_{22}(\delta_1-\epsilon_{2s})\big)(\epsilon_{11}e_{21} - \epsilon_{21}e_{11})$$

$$(\mathrm{CCT}) = -\theta_{1s}(\delta_2-\delta_1)(\epsilon_{12}\,e_{22} - \epsilon_{22}e_{12}) - \theta_{2s}(\delta_2-\delta_1)(\epsilon_{11}\,e_{21} - \epsilon_{21}\,e_{11}) + \big(-e_{11}\,\theta_{1s} + e_{21}(\delta_1-\epsilon_{1s})\big)\big(\epsilon_{12}\theta_{2s} + \epsilon_{22}(\delta_2-\epsilon_{2s})\big) + \big(\epsilon_{11}\,\theta_{1s} - \epsilon_{21}(\delta_2-\epsilon_{1s})\big)\big(e_{12}\theta_{2s} + e_{22}(\delta_2-\epsilon_{2s})\big)$$

$$(\mathrm{ABT}) = \big(e_{11}\,\theta_{1s} - e_{21}(\delta_2-\epsilon_{1s})\big)\big(\epsilon_{12}\,\theta_{2s} + \epsilon_{22}(\delta_1-\epsilon_{2s})\big) - \big(\epsilon_{11}\,\theta_{1s} - \epsilon_{21}(\delta_1-\epsilon_{1s})\big)\big(e_{12}\,\theta_{2s} + e_{22}(\delta_2-\epsilon_{2s})\big)$$

The constants of the mode shape Eqs. (25), (26) letting $A_4 = 1$, can be written in the form

$$A_i = [ST_i \sin a + CT_i \cos a + SHT_i \sinh b + CHT_i \cosh b]/\Delta\,,\ \mathrm{i} = 1, 2 \text{ and } 3 \tag{31}$$

here

$$\begin{aligned}
\Delta &= ST_\Delta \sin a + CT_\Delta \cos a + SHT_\Delta\,\mathrm{sh}\, b + CHT_\Delta\,\mathrm{ch}\, b\,;\\
ST_\Delta &= -(\delta_1-\epsilon_{2s})\big(\theta_{1s}e_{11} - e_{21}(\delta_1-\epsilon_{1s})\big) + \epsilon_{12}(e_{11}\epsilon_{21} - e_{21}\epsilon_{11})\\
CT_\Delta &= \epsilon_{12}\big(\theta_{1s}e_{11} - e_{21}(\delta_1-\epsilon_{1s})\big) + (\delta_1-\epsilon_{2s})(e_{11}\epsilon_{21} - e_{21}\epsilon_{11})\\
SHT_\Delta &= (\delta_2-\epsilon_{2s})\big(\theta_{1s}\epsilon_{11} - \epsilon_{21}(\delta_1-\epsilon_{1s})\big)\\
CHT_\Delta &= e_{12}\big(\theta_{1s}\epsilon_{11} - \epsilon_{21}(\delta_1-\epsilon_{1s})\big);
\end{aligned}$$

$$\begin{aligned}
A_1 &= -(ST_1 \sin a + CT_1 \cos a + SHT_1\,\mathrm{sh}\, b + CHT_1\,\mathrm{ch}\, b)/\Delta;\\
ST_1 &= \epsilon_{12}\big(\theta_{1s}e_{11} - e_{21}(\delta_2-\epsilon_{1s})\big)\\
CT_1 &= (\delta_1-\epsilon_{2s})\big(\theta_{1s}e_{11} - e_{21}(\delta_2-\epsilon_{1s})\big)\\
SHT_1 &= -e_{12}\big(\theta_{1s}e_{11} - e_{21}(\delta_1-\epsilon_{1s})\big) + \theta_{1s}(\delta_2-\delta_1)(\delta_2-\epsilon_{2s})\\
CHT_1 &= -(\delta_2-\epsilon_{2s})\big(\theta_{1s}e_{11} - e_{21}(\delta_1-\epsilon_{1s})\big) + \theta_{1s}e_{12}(\delta_2-\delta_1)
\end{aligned}$$

$$\begin{aligned}
A_2 &= -(ST_2 \sin a + CT_2 \cos a + SHT_2\,\mathrm{sh}\, b + CHT_2\,\mathrm{ch}\, b)/\Delta;\\
ST_2 &= -(\delta_1-\epsilon_{2s})\big(\theta_{1s}e_{11} - e_{21}(\delta_2-\epsilon_{1s})\big)\\
CT_2 &= \epsilon_{12}\big(\theta_{1s}e_{11} - e_{21}(\delta_2-\epsilon_{1s})\big)\\
SHT_2 &= (\delta_2-\epsilon_{2s})\big(\theta_{1s}\epsilon_{11} - \epsilon_{21}(\delta_2-\epsilon_{1s})\big) + e_{12}(e_{11}\epsilon_{21} - e_{21}\epsilon_{11})\\
CHT_2 &= e_{12}\big(\theta_{1s}\epsilon_{11} - \epsilon_{21}(\delta_2-\epsilon_{1s})\big) + (\delta_2-\epsilon_{2s})(e_{11}\epsilon_{21} - e_{21}\epsilon_{11});
\end{aligned}$$

$$\begin{aligned}
A_3 &= -(ST_3\ \sin a + CT_3 \cos a + SHT_3\,\mathrm{sh}\, b + CHT_3\,\mathrm{ch}\, b)/\Delta;\\
ST_3 &= -\epsilon_{12}\big(\theta_{1s}\epsilon_{11} - \epsilon_{21}(\delta_2-\epsilon_{1s})\big) - \theta_{1s}(\delta_2-\delta_1)(\delta_1-\epsilon_{2s})\\
CT_3 &= -(\delta_1-\epsilon_{2s})\big(\theta_{1s}\epsilon_{11} - \epsilon_{21}(\delta_2-\epsilon_{1s})\big) + \theta_{1s}\epsilon_{12}(\delta_2-\delta_1)\\
SHT_3 &= e_{12}\big(\theta_{1s}\epsilon_{11} - \epsilon_{21}(\delta_1-\epsilon_{1s})\big)\\
CHT_3 &= (\delta_2-\epsilon_{2s})\big(\theta_{1s}\epsilon_{11} - \epsilon_{21}(\delta_1-\epsilon_{1s})\big)
\end{aligned}$$

where,

$$\epsilon_{11} = \gamma_{1s}(\delta_1/a) + p_1^2 a \qquad \epsilon_{12} = \gamma_{2s}(\delta_1/a) + p_2^2 a$$
$$e_{11} = -\gamma_{1s}(\delta_2/b) + p_1^2 b \qquad e_{12} = \gamma_{2s}(\delta_2/b) - p_2^2 b$$
$$\epsilon_{21} = -\bar{\eta}_{1s}(\delta_1/a) - s_p a \qquad \epsilon_{22} = \bar{\eta}_{2s}(\delta_1/a) + s_p a$$
$$e_{21} = \bar{\eta}_{1s}(\delta_2/b) - s_p b \qquad e_{22} = \bar{\eta}_{2s}(\delta_2/b) - s_p b \tag{32 a to h}$$

Eqs. (30) and (31) represent the combined system closed form exact frequency and mode shape equations, respectively. It is interesting to note that the size of the coefficients matrix of the present beam system models shown in Fig. 1 is only (4×4) without need to extra rows and columns due to the presence of sub-systems. This modification represents one of the positive outputs of this mathematical treatment.

6. Reliability of results and discussion

The function obtained by using a highly transcendental Eq. (30) shows rapid oscillations attaining very large values between successive roots. The slope of the roots, therefore very closed to vertical. PC-Matlab version (2012b), software has been used for all the computational processes in this work. The results and a brief discussion are presented. All the system design parameters $\Phi_1, \Phi_2,$ $Z_1,$ $Z_2,$ $Z_{s3},$ $Z_{s4},$ $Z_{s5},$ Z_{s6} , Z_{s7} , Z_{s8} , $p^2,$ $r^2,$ $s^2,$ $\bar{c}_1,$ $\bar{c}_2$, $\bar{d}_1,$ $\bar{d}_2,$ $\bar{J}_1,$ $\bar{J}_2,$ $\bar{m}_1,$ $\bar{m}_2,$ $\bar{m}_{s1},$ $\bar{m}_{s2}$, $\bar{m}_{s3}$ *and* $\bar{m}_{s4}$ are considered in all of the chosen comparative and presented application examples. These 25 non-dimensional system parameters otherwise specified values, have one of the two values *vs*=10 *E*–12 and *vl*=10 *E*+12 equivalent to zero and infinity respectively. It is interesting to note that, Eq. (19) and Eq. (20) can be simplified to those presented by (Rossi *et al.* 1993, Gürgöze 1996, Chen *et al.* 2015) as follows:
when $Z_{s7} = vl,$ $\bar{m}_{s3} = 0, Z_{s8} = vl,$ $\bar{m}_{s4} = 0$, Eqs. (19) and (20) are reduced to those derived by (Chen *et al.* 2015), for undamped model (see Fig. 2(b)) in the forms

$$\tau_{1s} = Z_{s3} \left(\frac{-\lambda^4 \bar{m}_{s1} + Z_{s5}}{-\lambda^4 \bar{m}_{s1} + Z_{s3} + Z_{s5}}\right) \tag{33}$$

$$\text{and} \quad \tau_{2s} = Z_{s6} \left(\frac{-\lambda^4 \bar{m}_{s2} + Z_{s4}}{-\lambda^4 \bar{m}_{s2} + Z_{s4} + Z_{s6}}\right) \tag{34}$$

when $Z_{s5} = Z_{s7} = \bar{m}_{s3} = 0, Z_{s6} = Z_{s8} = \bar{m}_{s4} = 0$, Eqs. (19), (20) are reduced to those derived by (Rossi *et al*. 1993, Gürgöze 1996) (see Fig. 2(a)) in the forms

$$\tau_{1s} = \left(\frac{-\lambda^4 \bar{m}_{s1} Z_{s3}}{-\lambda^4 \bar{m}_{s1} + Z_{s3}}\right); \tag{35}$$

$$\text{and} \quad \tau_{2s} = \left(\frac{-\lambda^4 \bar{m}_{s2} Z_{s4}}{-\lambda^4 \bar{m}_{s2} + Z_{s4}}\right); \tag{36}$$

6.1 Verification of previous results

As can be seen from Table 1, that for $\bar{m}_{s1}$and $Z_{s3} \neq 0$. Good agreement between the results using the frequency Eq. (30), with those obtained by (Gürgöze 1996), while interesting results are observed when values of $\bar{m}_{s1} = \bar{Z}_{s3} = 0$, in which no variation are observed in the natural frequency parameters and the results are equivalent to the well known classical *C*-*F* beam as shown in this table. This means that the system model is valid even the sub-system is included in

Table 1 Comparison of results using Eq. (30) with those obtained by (Gürgöze 1996), for C-F beam carrying an end spring mass sub-system

		Z_{s3}					
		0		1		10	
$\bar{m}_{s1}$	λ_i^2	(Gürgöze)	Present	(Gürgöze)	Present	(Gürgöze)	Present
0	1	-	3.5160	-	3.5160	-	3.5160
	2	-	22.0347	-	22.0347	-	22.0347
1	1	-	3.5160	0.8594	0.8594	1.4194	1.4193
	2	-	22.0347	4.0711	4.0711	7.4474	7.4469
5	1	-	3.5160	0.3867	0.3867	0.6700	0.6700
	2	-	22.0347	4.0466	4.0461	7.0604	7.0601
10	1	-	3.5160	0.2736	0.2737	0.4771	0.4771
	2	-	22.0347	4.0431	4.0431	7.0123	7.0120

- not included in (Gürgöze 1996)

the combined system.

6.2 Results for the present combined system

To deeply understand the effect of the sub-system on the natural frequencies of the beam, the sub-system models shown in Fig. 2(c) are considered. The frequency equations of the discrete system are investigated separately considering point 4 and 5 are fixed, as follows:

The governing equations for the RH sub-system are

$$(k_{s4} + k_{s6} - \omega^2 m_{s2})\, Y_{s2} - k_{s6}\, Y_{s4} = 0$$
$$-k_{s6}\, Y_{s2} + (k_{s6} + k_{s8} - \omega^2 m_{s4})\, Y_{s4} = 0 \tag{37}$$

Multiplying these equations by the beam stiffness $\frac{L^3}{EI}$ in order to simply compare the discrete system results with those of the combined system, these equations may be written in the form

$$(-\lambda^4 \bar{m}_{s2} + Z_{s4} + Z_{s6})\, Y_{s2} - Z_{s6}\, Y_{s4} = 0$$
$$-Z_{s6}\, Y_{s2} + (-\lambda^4 \bar{m}_{s4} + Z_{s6} + Z_{s8}) \tag{38}$$

For nontrivial solution, one can obtain the frequency equation of RH sub-system in the from

$$(-\lambda^4 \bar{m}_{s2} + Z_{s4} + Z_{s6})(-\lambda^4 \bar{m}_{s4} + Z_{s6} + Z_{s8}) - Z^2{}_{s6} = 0 \tag{39}$$

and similarly for LH sub-system

$$(-\lambda^4 \bar{m}_{s1} + Z_{s3} + Z_{s5})(-\lambda^4 \bar{m}_{s2} + Z_{s5} + Z_{s7}) - Z_{s5}^2 = 0 \tag{40}$$

The roots of Eqs. (39), (40) represent the natural frequency parameters for the individual sub-systems (indicated in Tables from 2 to 9 by *D*1, *D*2, *D*3 and *D*4). The roots of Eq. (30) represent the combined natural frequency parameters for system model shown in Fig. 1, λ_i values in all of the following examples are included when the sub-system model considering points 4 and 5 are transversally vibrated. Three undamped models SS1, SS2, SS3 shown in Fig. 2(a), (b) and (c) are

Table 2 First six combined natural frequencies for (C-F) beam with end sub-system SS1 and their discrete modes

		Eq. (39)		λ_i , Eq. (30)					
$\bar{m}_{s2}$	Z_{s4}	D 1	D 2	1	2	3	4	5	6
0	All			1.8751	4.6941	7.8548	10.9955	14.1372	17.2788
0.5	0			1.8751	4.6941	7.8548	10.9955	14.1372	17.2788
	1	1.1892		1.0979	2.0261	4.7038	7.8568	10.9963	14.1375
	10	2.1147		1.3725	2.8158	4.7978	7.8758	11.0031	14.1407
	100	3.7606		1.4151	3.8901	5.7607	8.0962	11.0760	14.1737
	1000	6.6874		1.4195	4.0907	7.0355	9.7081	12.0410	14.6052
	10000	11.8921		1.4199	4.1091	7.1762	10.2520	13.3072	16.3110
	100000	21.1474		1.4200	4.1109	7.1889	10.2940	13.4105	16.5299
	1E+12	1189.2071		1.4200	4.1111	7.1903	10.2984	13.4210	16.5503
1	0			1.8751	4.6941	7.8548	10.9955	14.1372	17.2788
	1	1.0000		0.9271	2.0177	4.7038	7.8568	10.9963	14.1375
	10	1.7783		1.1914	2.7289	4.7957	7.8757	11.0031	14.1407
	100	3.1623		1.2419	3.7785	5.6876	8.0900	11.0754	14.1736
	1000	5.6234		1.2473	4.0069	6.9615	9.6321	11.9952	14.5931
	10000	10.0000		1.2479	4.0287	7.1183	10.2060	13.2661	16.2707
	100000	17.7828		1.2479	4.0309	7.1326	10.2517	13.3765	16.5011
	1E+12	1000.0000		1.2479	4.0311	7.1341	10.2566	13.3878	16.5227
5	0			1.8751	4.6941	7.8548	10.9955	14.1372	17.2788
	1	0.6687		0.5626	2.0115	4.7038	7.8568	10.9963	14.1375
	10	1.1892		0.8185	2.6571	4.7942	7.8757	11.0031	14.1407
	100	2.1147		0.8642	3.6706	5.6302	8.0853	11.0750	14.1736
	1000	3.7606		0.8694	3.9222	6.8943	9.5687	11.9597	14.5838
	10000	6.6874		0.8716	3.9472	7.0651	10.1656	13.2310	16.2367
	100000	11.8921		0.8717	3.9497	7.0808	10.2146	13.3474	16.4768
	1E+12	668.7403		0.8700	3.9500	7.0825	10.2199	13.3592	16.4994

considered. SS1 and SS2 is a single-degree of freedom systems having D1 or D3 only, while SS3 is a two–degree of freedom system having resonance frequencies (D1 and D2) or (D3 and D4) as shown in the next examples of this section. Tables 2, 3 and 4 present the results of the first part of this section.

Table 2, presents the first six natural frequencies for a cantilever beam with end sub-system consists of spring-mass (SS1). The values of the stiffness's discussed in this table are $(Z_{s4}) = 0, 1,\ 10,$ 100, 1000, 100000 and 10E+12 respectively. The values of the non-dimensional masses are $(\bar{m}_{s2}) =$ 0, 0.5, 1 and 5. Table 3 introduces similar analysis but the attached sub-system model is SS2 and finally Table 4 shows the analysis of a clamped free beam with sub-system model of three spring-two mass SS3 attached to point 5 as shown in Fig. 2(c).

As can be seen from Table 2, the well-known values of the classical C-F beam natural

Table 3 First six combined natural frequencies for (C-F) beam with end sub-system SS2 and their discrete modes.

		Eq. (39)		λ_i , Eq. (30)					
$\bar{m}_{s2}$	$Z_{s4} = Z_{s6}$	D 1	D 2	1	2	3	4	5	6
0	0			1.8751	4.6941	7.8548	10.9955	14.1372	17.2788
	1			1.9464	4.6989	7.8558	10.9959	14.1373	17.2789
	10			2.3668	4.7433	7.8651	10.9993	14.1389	17.2797
	100			3.4009	5.2012	7.9642	11.0342	14.1551	17.2885
	1000			3.8687	6.6683	8.9969	11.4576	14.3386	17.3836
	10000			3.9209	7.0325	10.0960	13.0717	15.9014	18.5749
	100000			3.9260	7.0650	10.1995	13.3275	16.4468	19.5543
	1E+12			3.9266	7.0686	10.2102	13.3518	16.4934	19.6350
0.5	0			1.8751	4.6941	7.8548	10.9955	14.1372	17.2788
	1	1.4142		1.3554	2.0284	4.7038	7.8568	10.9963	14.1375
	10	2.5149		2.0367	2.8835	4.7979	7.8758	11.0031	14.1407
	100	4.4721		3.2016	4.1686	5.7925	8.0968	11.0760	14.1737
	1000	7.9527		3.8651	6.1770	7.2846	9.7496	12.0508	14.6063
	10000	14.1421		3.9208	7.0299	10.0318	11.6622	13.4201	11.6623
	100000	25.1487		3.9234	7.0463	10.0855	11.6736	13.4989	16.5497
	1E+12	1414.2136		3.9266	7.0686	10.2102	13.3518	16.4934	19.6350
1	0			1.8751	4.6941	7.8548	10.9955	14.1372	17.2788
	1	1.1892		1.1455	2.0182	4.7038	7.8568	10.9963	14.1375
	10	2.1147		1.7981	2.7477	4.7958	7.8757	11.0031	14.1407
	100	3.7606		2.9521	3.8700	5.6952	8.0902	11.0754	14.1736
	1000	6.6874		3.8604	5.4912	7.0146	9.6422	11.9975	14.5934
	10000	11.8921		3.9208	7.0265	9.6515	10.4155	13.2859	16.2775
	100000	21.1474		3.9260	7.0650	10.1988	13.3218	16.3820	17.7040
	1E+12	1189.2071		3.9266	7.0686	10.2102	13.3518	16.4934	19.6350
5	0			1.8751	4.6941	7.8548	10.9955	14.1372	17.2788
	1	0.7953		0.7692	2.0115	4.7038	7.8568	10.9963	14.1375
	10	1.4142		1.2435	2.6578	4.7942	7.8757	11.0031	14.1407
	100	2.5149		2.1047	3.6742	5.6304	8.0853	11.0750	14.1736
	1000	4.4721		3.6479	3.9990	6.8960	9.5691	11.9598	14.5838
	10000	7.9527		3.9205	6.6095	7.1129	10.1680	13.2316	16.2370
	100000	14.1421		3.9260	7.0648	10.1931	11.8648	13.3599	16.4791
	1E+12	795.2707		3.9266	7.0686	10.2102	13.3518	16.4934	19.6350

frequencies are observed for all values of $Z_{s4}\ and\ \bar{m}_{s2} = 0$. This means that the sub-system SS1 has no effect on the system. Also for all cases in which $Z_{s4} = 0$ and with variable $\bar{m}_{s2}$ same observations are recorded. Increasing the value of the sub-system mass $\bar{m}_{s2}$ results in decrease the natural frequency parameter λ_i and of the separate sub-system model *D1*. When the value of Z_{s4} tends to ∞ a clamped-free beam carrying an end mass $\bar{m}_{s2} = 0.5$, 1 and 5, λ_i are satisfied. An increase in Z_{s4} results in an increase in the combined natural frequency parameters are observed.

Table 3, shows the variations in λ_i vs the variation in both $\bar{m}_{s2}$ and $(Z_{s4} = Z_{s6})$. As can be seen, that when $\bar{m}_{s2} = Z_{s4} = Z_{s6} = 0$, classical values of λ_i for *C-F* beam are observed. In this case, the sub-system SS2 has no effect on the beam system. For the case in which the value of

Table 4 First six combined natural frequencies for (C-F) beam with end sub-system SS3 and their discrete modes.

		Eq. (39)		λ_i, Eq. (30)					
$\bar{m}_{s2} = \bar{m}_{s4}$	$Z_{s4} = Z_{s6} = Z_{s8}$	D 1	D 2	1	2	3	4	5	6
0	0			1.8751	4.6941	7.8548	10.9955	14.1372	17.2788
	1			1.9236	4.6973	7.8554	10.9958	14.1373	17.2788
	10			2.2418	4.7267	7.8617	10.9981	14.1383	17.2794
	100			3.2184	5.0354	7.9264	11.0211	14.1491	17.2853
	1000			3.8395	6.4671	8.6647	11.2885	14.2661	17.3469
	10000			3.9180	7.0139	10.0335	12.9074	15.5752	18.1743
	100000			3.9257	7.0633	10.1940	13.3151	16.4223	19.5102
	1E+12			3.9266	7.0686	10.2102	13.3518	16.4934	19.6350
0.5	0			1.8751	4.6941	7.8548	10.9955	14.1372	17.2788
	1	1.1892	1.5651	1.1423	1.5423	2.0288	4.7038	7.8568	10.9963
	10	2.1147	2.7832	1.7224	2.5631	2.9367	4.7979	7.8758	11.0031
	100	3.7606	4.9492	2.7326	3.8679	4.7729	5.8033	8.0969	11.0760
	1000	6.6874	8.8011	3.8181	5.1392	7.0103	8.4416	9.7717	12.0521
	10000	11.8921	15.6508	3.9178	6.9975	9.0782	10.3387	13.2711	15.0393
	100000	21.1474	27.8316	3.9257	7.0631	10.1923	13.2983	16.0506	16.8820
	1E+12	1189.20	1565.08	3.9266	7.0686	10.2102	13.3518	16.4934	19.6350
1	0			1.8751	4.6941	7.8548	10.9955	14.1372	17.2788
	1	1.0000	1.3161	1.0466	1.3050	2.0183	4.7038	7.8568	10.9963
	10	1.7783	2.3403	1.4890	2.2367	2.7538	4.7958	7.8757	11.0031
	100	3.1623	4.1618	2.4113	3.6803	4.0988	5.6962	8.0902	11.0754
	1000	5.6234	7.4008	3.7764	4.4730	6.8304	7.2513	9.6440	11.9976
	10000	10.0000	13.1607	3.9176	6.9562	7.8605	10.2022	12.6344	13.3249
	100000	17.7828	23.4035	3.9257	7.0630	10.1899	13.2146	14.0071	16.5122
	1E+12	1000	1316.07	3.9266	7.0686	10.2102	13.3518	16.4934	19.6350
5	0			1.8751	4.6941	7.8548	10.9955	14.1372	17.2788
	1	0.6687	0.8801	0.7401	0.9147	2.0115	4.7038	7.8568	10.9963
	10	1.1892	1.5651	1.0164	1.5188	2.6579	4.7942	7.8757	11.0031
	100	2.1147	2.7832	1.6712	2.6815	3.6748	5.6305	8.0853	11.0750
	1000	3.7606	4.9492	2.9296	3.9185	4.7824	6.8962	9.5691	11.9598
	10000	6.6874	8.8011	3.9156	5.2436	7.0622	8.5005	10.1690	13.2317
	100000	11.8921	15.6508	3.9257	7.0616	9.3163	10.2273	13.3438	15.1186
	1E+12	668.740	880.111	3.9266	7.0686	10.2102	13.3518	16.4934	19.6350

$\bar{m}_{s2} = 0$ and $(Z_{s4} = Z_{s6}) \Rightarrow \infty$, λ_i for clamed-pinned beam are recorded. An increase in Z_s values results in an increase in both *D1* and λ_i. Also an increase in $\bar{m}_{s2}$ decreases the values of λ_i and *D1*. In Table 4, the sub-system SS3 is considered and located at the free end of the *C-F* beam. The table shows the first six combined natural frequencies for $(\bar{m}_{s2} = \bar{m}_{s4})$ and $(Z_{s4} = Z_{s6} = Z_{s8})$. One can see that for all values of the sub-system masses and

$(Z_{s4} = Z_{s6} = Z_{s8})$ are zeros, λ_i for classical clamped-free beam are observed, which indicates that the sub-system has no effect on the beam system. On the other hand, when Z_s values approach infinity a clamped-pinned case is satisfied. An increase in Z_s increases the λ_i, *D*1 and *D*2. As expected the increase in the masses decrease the combined system natural frequencies and those for the discrete masses. The first section of Table 4, shows the results when $\bar{m}_{s2} = \bar{m}_{s4} = 0$. This means that the sub-system model is equivalent to three spring in series and the value of the equivalent spring is $Z_{eq} = 1/(\frac{1}{Z_{s4}} + \frac{1}{Z_{s6}} + \frac{1}{Z_{s8}})$. Selecting the case where $Z_{s4} = Z_{s6} = Z_{s8} = 10$ and $m_{s2} = m_{s4} = 0$, we can find that the equivalent model is a clamped free beam with end spring of Z_{eq} = 10/3. The first five natural frequencies for this case are [2.2418, 4.7267, 7.8617, 10.9981 and 14.1383], respectively, which equals to the third row of the first group in Table 4. In column *D*1 and *D*2 the values of $(Z_{s4} = Z_{s6} = Z_{s8})$ equal zero for all values of $\bar{m}_{s2} = \bar{m}_{s4}$.

For more understanding of the vibrational behavior of SS1, SS2 and SS3. Fig. 3(a), (b) and (c) present the first, third and fifth natural frequency parameters for the models considered in Tables 2, 3 and 4. The figures plot the variation of natural frequency parameter versus the change in the sub-system spring stiffness parameter Z_s. It is interesting to note from Fig. 3 that the combined natural frequencies of the second and third sub-systems model (SS2 and SS3) approach the same end as Z_s tends to infinity which is the clamped pinned results. As shows in Fig. 3(c) the individual variation in λ_i is diminished in the fifth mode from Z_{s4} equal 1 to Z_{s4}=100. This behavior may be appeared for Z_{s4}>100 especially for higher modes.

Table 5 presents the first five combined natural frequency parameter for clamped free beam with end mass having $\bar{m}_2 = 1$ and $Z_2 = vs$ and sub-system (SS3) having $Z_{s4} = Z_{s6} = 100$. The table introduces the effect of changing the location of the end mass center of gravity (sub-system

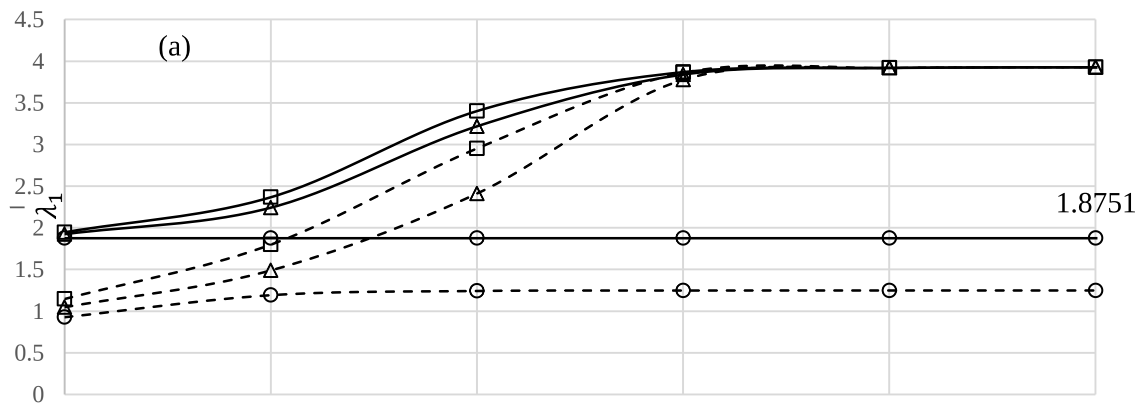


Fig. 3 (a) first mode, (b) third mode and (c) fifth mode variation vs the spring stiffness parameters: ○ for SS1, for SS2 and Δ for SS3 when located at the free end of *C-F* beam. Input parameters are $(z_{s4} = z_{s6})$ for SS2, $(z_{s4} = z_{s6} = z_{s8})$ for SS3. Two values of $\bar{m}_{s4}$ are distinguished by for $\bar{m}_{s4} = 0$; and $---$ for $\bar{m}_{s4} = 1$

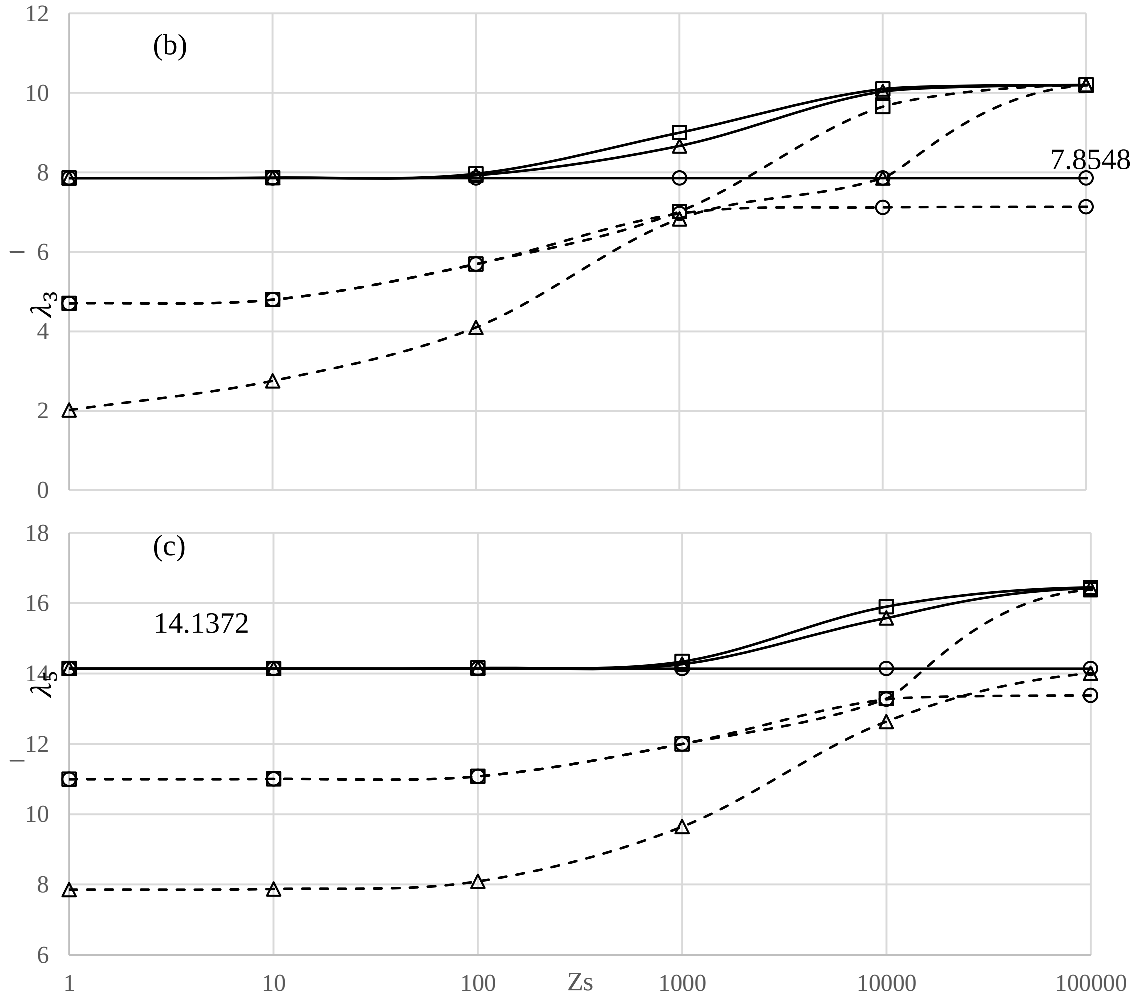


Fig. 3 Continued

point of attachment) from the beam end point by changing the value of $\bar{d}_2$. The effect of changing the Z_{s8} parameter is also investigated. Connecting the sub-system SS3 with pre-mentioned parameters to the point 5, see Fig. 1, results in adding two natural frequencies to the combined system. The value of the separate natural frequency determines the location of the additional mode through the beam modes. Comparing the natural frequency of $Z_{s8} = 1000$ & $\bar{d}_2 = 0$ and $Z_{s8} = 100000$ & $\bar{d}_2 = 0$ we can find that the fifth natural frequency in the first case equals to the fourth natural frequency in the second case. This may be explained by the fact that the value of the second discrete natural frequency, *D*2 is higher than the natural frequency investigated range. Therefore, the sub-system adds to the beam an extra one natural frequency.

Table 6 shows the first five combined natural frequencies parameter for (*C*-*F*) beam carrying an end mass $\bar{m}_2 = 1,$ and sub-system SS3 having $(Z_{s6} = Z_{s8}) = 100000$ for different values of $(Z_2 = Z_{s4})$ = 0, 10, 100, 1000, 100000 and *vl*, fixed value of $\bar{c}_2 = 0.5$ and five values of $\bar{d}_2 =$

Table 5 First five combined natural frequencies parameter and the discrete values for clamped- free beam with end mass having $\bar{m}_2 = 1$, $Z_2 = vs$ and $\bar{c}_2 = 0.5$ for five values of $\bar{d}_2 = 0$, 0.25, 0.375 and 0.5. carrying sub-system (SS3) , having $\bar{m}_{s2} = \bar{m}_{s4} = 1$ and $Z_{s4} = Z_{s6} = 100$

		Eq. (39)		λ_i , Eq. (30)				
Z_{s8}	$\bar{d}_2$	$D1$	$D2$	1	2	3	4	5
0	0	2.4860	4.0225	0.9777	3.0679	3.9691	4.2237	7.1367
	0.125			0.9015	3.0623	3.6308	4.1753	6.4408
	0.25			0.8381	3.0349	3.3957	4.1676	6.1048
	0.375			0.7849	2.9803	3.2732	4.1651	5.9357
	0.5			0.7400	2.9106	3.2199	4.1639	5.8379
10	0	2.5949	4.0334	1.4049	3.1092	3.9739	4.2268	7.1367
	0.125			1.3864	3.1004	3.6357	4.1806	6.4408
	0.25			1.3737	3.0655	3.4070	4.1734	6.1048
	0.375			1.3648	2.9992	3.2957	4.1709	5.9357
	0.5			1.3584	2.9211	3.2505	4.1698	5.8379
100	0	3.1623	4.1618	2.0850	3.4383	4.0290	4.2743	7.1367
	0.125			2.0908	3.3770	3.7144	4.2517	6.4408
	0.25			2.0947	3.2298	3.5909	4.2479	6.1048
	0.375			2.0975	3.0808	3.5584	4.2466	5.9357
	0.5			2.0995	2.9665	3.5468	4.2460	5.8379
1000	0	3.7079	5.7733	2.3761	3.8242	4.1586	5.7735	7.1367
	0.125			2.3891	3.5528	4.0283	5.7735	6.4408
	0.25			2.3975	3.2991	4.0053	5.7735	6.1048
	0.375			2.4033	3.1220	3.9982	5.7735	5.9357
	0.5			2.4076	2.9975	3.9951	5.7735	5.8379
100 000	0	3.7601	17.7872	2.4146	3.8506	4.1687	7.1367	10.2570
	0.125			2.4283	3.5622	4.0552	6.4408	9.3690
	0.25			2.4370	3.3051	4.0357	6.1048	9.0641
	0.375			2.4430	3.1270	4.0296	5.9357	8.9341
	0.5			2.4473	3.0022	4.0269	5.8379	8.8641

$0, 0.125,\ 0.25, 0.375$ and 0.5 are considered. As can be seen from this table that increasing the value of *(Z_2 = Z_{s4})* = from 0 to 1000, results in slight variations in $D1$ and $D2$ this is because the value is recognized of Z_{s4} is relatively small compared with the values of $Z_{s6} = Z_{s8} = 100000$. A significant increase in their value when Z_{s4}=10000 or *vl*.

Table 7 shows the results of the first natural frequency parameters. Tensile and compressive axial loads are acted individually at point 5 of Fig. 1 in which $\bar{d}_2 = 0.25$. Four values for $p^2 = vs,\ 0.25\pi^2,\ 0.50\pi^2$ and π^2 are considered. The input data for cases are given on the legend of Table 7. As expected, an increase in the tensile axial load increases λ_i , while an increase in the compressive axial load decreases λ_i .

Table 6 First five combined natural frequencies parameter for (*C-F*) beam carrying an end mass $\bar{m}_2 = 1$, $\bar{c}_2 = 0.5$ and sub-system SS3 having $Z_{s6} = Z_{s8} = 100000$ for different values of $(Z_2 = Z_{s4}) =$ 0, 10, 100, 1000, 100000 and *vl*, and five values of $\bar{d}_2 = 0,\ 0.125,\ 0.25,\ 0.375$ and 0.5

		Eq. (39)		λ_i, Eq. (30)				
$Z_2 = Z_{s4}$	$\bar{d}_2$	$D1$	$D2$	1	2	3	4	5
	0	13.9800	22.6201	1.2479	4.0311	7.1341	10.2566	13.3878
	0.125			1.1609	3.6068	6.4380	9.3687	12.3671
0	0.25	Fig.4 (a)		1.0856	3.3166	6.1028	9.0640	12.1042
	0.375			1.0209	3.1247	5.9344	8.9340	12.0016
	0.5			0.9650	2.9927	5.8370	8.8640	11.9480
	0	13.9806	22.6201	2.1941	4.3606	7.3085	10.3774	13.4802
	0.125			2.1889	3.8524	6.5169	9.4037	12.3850
10	0.25			2.1690	3.4418	6.1229	9.0704	12.1070
	0.375			2.1366	3.1725	5.9380	8.9349	12.0019
	0.5			2.0965	3.0056	5.8371	8.8640	11.9480
	0	13.9866	22.6207	3.0670	4.9181	7.7840	10.8051	13.8677
	0.125			3.0379	4.7636	7.0888	9.7124	12.5499
100	0.25	Fig.4 (b)		2.9997	4.4347	6.3284	9.1308	12.1322
	0.375			2.9579	4.0561	5.9758	8.9432	12.0051
	0.5			2.9279	3.7865	5.8388	8.8641	11.9480
	0	14.0456	22.6261	4.4017	5.6826	8.0373	11.0744	14.0461
	0.125			4.2133	5.7106	8.0260	10.8358	13.6280
1000	0.25			3.8655	5.7469	7.8415	9.8943	12.4360
	0.375			3.3217	5.7876	7.2542	9.0632	12.0400
	0.5			2.9505	5.8165	6.6840	8.8653	11.9481
	0	17.7828	23.4035	4.7234	7.8249	10.8144	11.8823	14.2152
	0.125			4.7178	7.8011	10.6910	11.8919	14.2611
100 000	0.25			4.7010	7.7293	10.3990	11.9080	14.3539
	0.375			4.608	7.347	9.6172	11.9314	14.5781
	0.5			2.9521	5.8283	8.8612	11.9466	15.0535
	0	21.1474	1000.000	4.7258	7.8347	10.8887	12.1313	14.2160
	0.125			4.7224	7.8203	10.8106	12.1098	14.2680
vl	0.25	Fig.4 (c)		4.7130	7.7788	10.6106	12.0683	14.3854
	0.375			4.6623	7.5569	9.9399	11.9971	14.6824
	0.5			2.9522	5.8284	8.8614	11.9468	15.0546

Selected cases are chosen from results shown in Table 6. These cases are underlined inside the table. Interesting modal shapes are presented in Fig. 4(a) shows the first five modal shapes between the point of attachment 1 and 2 for a clamped free beam. Fig. 4(b) shows that the elastic support affect slightly the modal shapes. This is because the $Z_2$and $Z_{s4} = 100$ are considered small elastic spring. An interesting modal shapes are observed in Fig. 4(c) in which the extension of the tangents at point 2 are intersected at the pin location at point 6 in which $Z_2 \rightarrow \infty$.

Table 7 First five combined natural frequencies parameter for (C-F) beam carrying an end mass $\bar{m}_2 = 1$, $\bar{c}_2 = 0.5, \bar{d}_2 = 0.25$ and sub-system SS3 having $Z_{s6} = Z_{s8} = 100000$ for different values of $(Z_2 = Z_{s4}) =$ vs, 10, 1000 and vl, and four values of $p^2 =$ vs, $0.25\pi^2, 0.50\pi^2$ and π^2.(tensile and compressive)

		Eq. (39)		λ_i, Eq. (30)				
$Z_2 = Z_{s4}$	p^2	$D1$	$D2$	1	2	3	4	5
vs	$-\pi^2$	13.9800	22.6201	1.7427	3.8030	6.4231	9.2987	12.2876
	$-0.5\pi^2$			1.5255	3.5851	6.2691	9.1836	12.1969
	$-0.25\pi^2$			1.3619	3.4589	6.1877	9.1244	12.1508
	vs			1.0856	3.3166	6.1028	9.0640	12.1042
	$0.25\pi^2$			-	3.1532	6.0143	9.0024	12.0570
	$0.5\pi^2$			-	2.9599	5.9216	8.9395	12.0093
	π^2			-	2.4101	5.7222	8.8095	11.9121
10	$-\pi^2$	13.9806	22.6201	2.3273	3.8882	6.4406	9.3046	12.2902
	$-0.5\pi^2$			2.2529	3.6862	6.2879	9.1898	12.1996
	$-0.25\pi^2$			2.2123	3.5705	6.2070	9.1307	12.1535
	vs			2.1690	3.4418	6.1229	9.0704	12.1070
	$0.25\pi^2$			2.1224	3.2961	6.0351	9.0089	12.0598
	$0.5\pi^2$			2.0717	3.1271	5.9433	8.9461	12.0121
	π^2			1.9502	2.6661	5.7458	8.8164	11.9150
1000	$-\pi^2$	14.0456	22.6261	4.0656	5.9569	7.9975	10.0581	12.6018
	$-0.5\pi^2$			3.9709	5.8559	7.9210	9.9769	12.5197
	$-0.25\pi^2$			3.9197	5.8025	7.8817	9.9358	12.4780
	vs			3.8655	5.7469	7.8415	9.8943	12.4360
	$0.25\pi^2$			3.8079	5.6889	7.8005	9.8526	12.3935
	$0.5\pi^2$			3.7464	5.6285	7.7585	9.8105	12.3507
	π^2			3.6092	5.4991	7.6716	9.7252	12.2637
vl	$-\pi^2$	21.1474	1000.00	4.9720	7.9952	10.7429	12.1499	14.5236
	$-0.5\pi^2$			4.8482	7.8896	10.6784	12.1087	14.4549
	$-0.25\pi^2$			4.7821	7.8349	10.6449	12.0884	14.4203
	vs			4.7130	7.7788	10.6106	12.0683	14.3854
	$0.25\pi^2$			4.6403	7.7214	10.5755	12.0484	14.3503
	$0.5\pi^2$			4.5637	7.6625	10.5395	12.0287	14.3150
	π^2			4.3967	7.5398	10.4649	11.9898	14.2437

Tables 8, 9 and 10 show the first five combined natural frequency parameter for symmetric models having $\bar{c}_1 = \bar{c}_2 = 0.5$ and $\bar{m}_1 = \bar{m}_2 = 1$. Two sub-systems are connected to points 4 ad 5 with $\bar{m}_{s1} = (\bar{m}_{s2} = \bar{m}_{s3} = \bar{m}_{s4}) = 1$, see Figs. 1 and 2. The values of $(Z_{s4} = Z_{s6} = Z_{s8} = Z_{s3} = Z_{s5} = Z_{s7} = Z_s)$, where Z_s equals to 0, 10, 100, 1000, 100000, vl respectively. The results of Table 10 are calculated at $\bar{d}_1 = \bar{d}_2 = 0$, the results of Table 9 are calculated at $\bar{d}_1 = \bar{d}_2 = 0.25$ while Table 8 for $\bar{d}_1 = \bar{d}_2 = 0.5$.

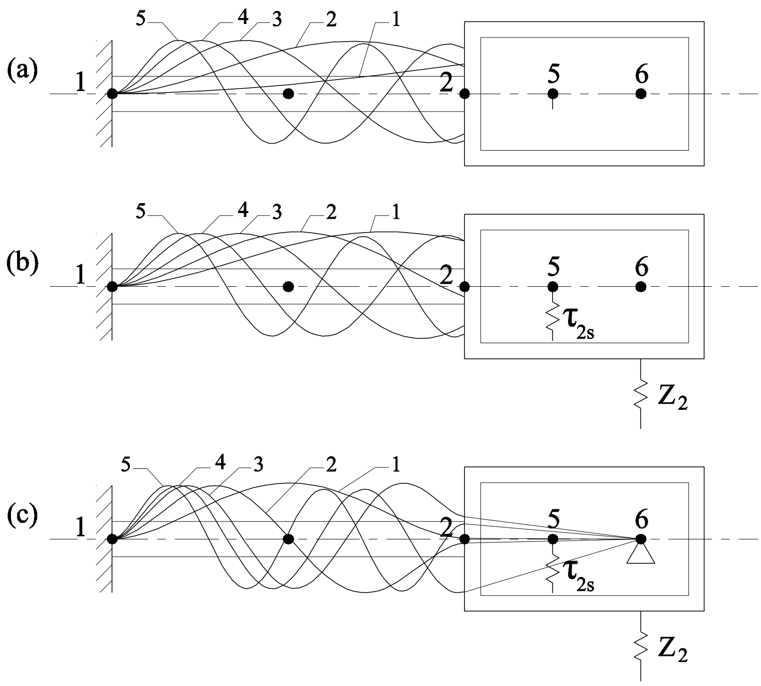


Fig. 4 first five modal shapes for *C-F* beam chosen form Table 6 for case of $\bar{d}_2 = 0.25$ and (a) $Z_2 = Z_{s4} = 0, (b)\ Z_{s4} = 100$ and (c) $Z_2 = Z_{s4} = vl$

The attached sub-systems to points 4 and 5 add four natural frequencies to the combined system. In the first group in Table 10, $Z_1 = Z_2 = 0$ and $Z_s = 0$, this represent the case of free-free beam with two end masses of ratios 1. In this case the sub-system has no effect on the combined system. Increasing the value of Z_s, the two attached sub-systems add four natural frequencies to the combined system as shown in the second group of Table 10. Increasing the value of Z_s results in an increase in the natural frequency. When the value of Z_s approaches infinity a pinned support is created at points 4 and 5. This explains why the sixth row in Table 10 represent the classical *P-P* beam configuration with classical natural frequency parameters $\pi, 2\pi, \dots, n\,\pi,$ respectively The same results are recognized in Tables 8 and 9, but the natural frequency deviates from the classical *P-P* because the sub-system point of attachment is not located at the beam end points.

When $Z_1 = Z_2$ approaches infinity a pinned support is created at points 3 and 6, and when Z_s approaches infinity a pinned support is created at points 4 and 5, in which the end mass is considered as a rigid wall. If the two hinges coincide to each other the beam behaves as pinned support at the location of attachment as shown in the last group of Table 8.

Selected cases are chosen from results shown in Tables 8, 9 and 10. These cases are underlined inside the tables. Using Eq. (31), interesting modal shapes are presented in Fig. 5 and Fig. 6. Fig. 5(a) shows the first five mode shapes between the beam ends 1 and 2. If we linearly extend the tangent to the mode shapes at point 1 and 2, this will intersect the pin location as shown in Fig. 5(a). On the other hand, when two pinned support are adjacent each other, the end mass becomes as a rigid wall in which the beam becomes *C-C* configurations. Fig. 5 (b) and (c) show the modal shapes for this case and the extension of tangents passes through the pinned support locations. It is

Table 8 First five combined natural frequency parameters λ_i and for the discrete masses of the sub-system for symmetric model having $\bar{c}_1 = \bar{c}_2 = 0.5$ and $\bar{d}_1 = \bar{d}_2 = 0.5$.$\bar{m}_1 = \bar{m}_2 = 1$. No axial load, and $\bar{m}_{s1} = \bar{m}_{s3} = \bar{m}_{s4} = \bar{m}_{s2} = 1$

		Eqs. (39), (40)		λ_i , Eq. (30)				
$Z_1 = Z_2$	$Z_{s3} = Z_{s5}$, $Z_{s7} = Z_{s4}$, $Z_{s6} = Z_{s8}$	$D1$, $D3$	$D2$, $D4$	1	2	3	4	5
0	0			1.9009	4.0971	6.8619	9.8303	12.8764
	10	1.7783	2.3403	1.1037	1.1796	1.7467	1.9784	2.0867
	100	3.1622	4.1617	1.6440	2.0969	2.1993	3.5048	3.5381
	1000	5.6234	7.4008	1.7149	3.6888	3.7592	4.1219	6.2694
	100000	17.7827	23.4034	1.7206	4.0572	6.8505	9.8244	11.8612
	vl	1000	1316.074	1.7207	4.0575	6.8512	9.8264	12.8746
10	0			1.4787	1.7596	2.0657	4.0986	6.8619
	10			1.4541	1.5516	1.7487	2.1031	2.1949
	100			1.6654	2.2249	2.3029	3.5225	3.5587
	1000			1.7151	3.7109	3.7842	4.1251	6.2728
	100000			1.7206	4.0572	6.8505	9.8244	11.8620
	vl			1.7207	4.0575	6.8512	9.8264	12.8746
100	0			1.7009	3.1145	3.1932	4.1178	6.8624
	10			1.6953	1.7551	1.7627	2.3293	3.1955
	100			1.7055	2.7574	2.7798	3.7017	3.7676
	1000			1.7163	3.8685	3.9845	4.1692	6.3041
	100000			1.7206	4.0572	6.8505	9.8244	11.8693
	vl			1.7207	4.0575	6.8512	9.8264	12.8746
1000	0			1.7188	4.0427	5.6127	5.6410	6.8705
	10			1.7187	4.0429	5.6266	5.6549	6.8706
	100			1.7188	3.1214	4.0423	4.1408	4.1437
	1000	Fig. 6(a)		1.7193	4.0450	4.9191	4.9294	6.6444
	100000			1.7206	4.0572	6.8505	9.8245	11.9412
	vl			1.7207	4.0575	6.8512	9.8264	12.8746
100000	0			1.7206	4.0574	6.8510	9.8260	12.8739
	10			1.7206	4.0574	6.8510	9.8260	12.8739
	100			1.7206	4.0574	6.8510	9.8260	12.8739
	1000			1.7206	4.0574	6.8510	9.8260	12.8739
	100000			1.7206	4.0574	6.8511	9.8261	12.8740
	vl			1.7207	4.0575	6.8512	9.8264	12.8746
vl	0			1.7207	4.0575	6.8512	9.8264	12.8746
	10			1.7207	4.0575	6.8512	9.8264	12.8746
	100			1.7207	4.0575	6.8512	9.8264	12.8746
	1000			1.7207	4.0575	6.8512	9.8264	12.8746
	100000			1.7207	4.0575	6.8512	9.8264	12.8746
	vl	Fig. 5(a)		1.7207	4.0575	6.8512	9.8264	12.8746

Table 9 First five combined natural frequency parameters λ_i and for the discrete masses of the sub-system for symmetric model having $\bar{c}_1 = \bar{c}_2 = 0.5$ and $\bar{d}_1 = \bar{d}_2 = 0.25$. $\bar{m}_1 = \bar{m}_2 = 1$. No axial load, and $\bar{m}_{s1} = \bar{m}_{s3} = \bar{m}_{s2} = \bar{m}_{s4} = 1$

		Eqs. (39), (40)		λ_i , Eq. (30)				
$Z_1 = Z_2$	$Z_{s3} = Z_{s5}$, $Z_{s7} = Z_{s4}$, $Z_{s6} = Z_{s8}$	$D1, D3$	$D2, D4$	1	2	3	4	5
0	0			2.3727	4.6532	7.3155	10.1863	13.1628
	10	1.7783	2.3403	1.1113	1.1746	1.8804	1.9732	2.3044
	100	3.1622	4.1617	1.8661	2.0880	2.4102	3.5001	3.5494
	1000	5.6234	7.4008	2.1356	3.6905	3.7710	4.6374	6.2633
	100000	17.7827	23.4034	2.1536	4.5768	7.2844	10.1658	11.8580
	vl	1000	1316.074	2.1537	4.5779	7.2872	10.1740	13.1567
10	0			1.3769	1.9826	2.7593	4.7290	7.3378
	10			1.3849	1.6305	1.8813	2.1731	2.3384
	100			1.8697	2.2837	2.6508	3.5217	3.6083
	1000			2.2502	3.7156	3.8341	4.7017	6.2690
	100000			2.2983	4.6240	7.3017	10.1727	11.8602
	vl			2.2987	4.6255	7.3049	10.1817	13.1605
100	0			1.5886	3.0776	4.1200	5.3645	7.5448
	10			1.5349	1.7589	1.8822	2.3300	2.3484
	100			1.8730	2.7715	3.1285	3.6364	4.0478
	1000			2.6093	3.8133	4.2742	5.1506	6.3117
	100000			2.9818	4.9799	7.4509	10.2340	11.8790
	vl			2.9863	4.9868	7.4596	10.2508	13.1953
1000	0			1.6173	3.5034	5.6440	7.5714	9.2233
	10			1.5590	1.7738	1.8824	2.3384	2.3498
	100			1.8737	2.9763	3.2298	3.7461	4.1623
	1000	Fig. 6(b)		2.8094	3.8831	5.1253	5.8223	6.4352
	100000			4.1594	6.3696	8.4427	10.6816	12.0192
	vl			4.1896	6.4498	8.5711	10.8968	13.5464
100000	0			1.6205	3.5599	5.9908	8.7389	11.6436
	10			1.5618	1.7738	1.8825	2.3384	2.3499
	100			1.8738	3.0031	3.2400	3.7661	4.1686
	1000			2.8398	3.8950	5.3605	5.9115	6.4878
	100000			4.6728	7.6177	10.0786	11.2628	12.3000
	vl			4.7230	7.8312	10.9462	14.0434	17.1188
vl	0			1.6206	3.5605	5.9943	8.7511	11.6739
	10			1.5618	1.7738	1.8825	2.3384	2.3499
	100			1.8738	3.0033	3.2401	3.7663	4.1686
	1000			2.8401	3.8952	5.3630	5.9123	6.4884
	100000			4.6796	7.6381	10.1091	11.2697	12.3058
	vl	Fig. 5(b)		4.7300	7.8532	10.9956	14.1372	17.2788

Table 10 First five combined natural frequency parameters λ_i and for the discrete masses of the sub-system for symmetric model having $\bar{c}_1 = \bar{c}_2 = 0.5$ and $\bar{d}_1 = \bar{d}_2 = 0$. $\bar{m}_1 = \bar{m}_2 = 1$. No axial load, and $\bar{m}_{s1} = \bar{m}_{s2} = \bar{m}_{s3} = \bar{m}_{s4} = 1$

		Eqs. (39), (40)		λ_i, Eq. (30)				
$Z_1 = Z_2$	$Z_{s3} = Z_{s5}$, $Z_{s7} = Z_{s4}$, $Z_{s6} = Z_{s8}$	$D1, D3$	$D2, D4$	1	2	3	4	5
0	0			3.3988	6.4273	9.5245	12.6424	15.7694
	10	1.7783	2.3403	1.1150	1.1607	1.9176	1.9600	2.3641
	100	3.1622	4.1617	1.9683	2.0637	3.1506	3.4830	3.6806
	1000	5.6234	7.4008	3.0006	3.6642	3.8684	6.0257	6.2391
	100000	17.782	23.403	3.1407	6.2752	9.3845	11.6598	11.8219
	vl	1000	1316.074	3.1416	6.2832	9.4248	12.5664	15.7080
10	0			1.2689	2.1975	4.0656	6.8006	9.7799
	10			1.3276	1.6865	1.9292	2.2441	2.3641
	100			1.9888	2.3566	3.3000	3.5200	4.0417
	1000			3.2050	3.7135	4.1907	6.0721	6.2571
	100000			3.6448	6.5849	9.5934	11.6621	11.8409
	vl			3.6477	6.5989	9.6531	12.7445	15.8539
100	0			1.4102	2.7764	4.9916	7.7137	10.6244
	10			1.4050	1.7470	1.9362	2.3215	2.3641
	100			1.9992	2.6275	3.3457	3.5861	4.1757
	1000			3.3165	3.7846	4.8308	6.1077	6.3118
	100000			4.4212	7.3666	10.2551	11.6672	11.9333
	vl			4.4304	7.4040	10.4233	13.4637	16.5214
1000	0			1.4291	2.8936	5.2758	8.1271	11.1370
	10			1.4161	1.7527	1.9374	2.3257	2.3641
	100			2.0009	2.6851	3.3512	3.6076	4.1828
	1000	Fig. 6(c)		3.3326	3.8047	5.0398	6.1141	6.3425
	100000			4.6809	7.7400	10.6249	11.6693	12.0299
	vl			4.6932	7.7928	10.9120	14.0310	17.1505
100000	0			1.4313	2.9080	5.3141	8.1888	11.2215
	10			1.4173	1.7533	1.9376	2.3262	2.3641
	100			2.0011	2.6922	3.3518	3.6105	4.1835
	1000			3.3345	3.8073	5.0677	6.1148	6.3475
	100000			4.7169	7.7972	10.6821	11.6696	12.0507
	vl			4.7297	7.8526	10.9947	14.1360	17.2774
vl	0			1.4313	2.9082	5.3145	8.1895	11.2224
	10			1.4174	1.7533	1.9376	2.3262	2.3641
	100			2.0011	2.6922	3.3518	3.6105	4.1835
	1000			3.3345	3.8074	5.0680	6.1148	6.3475
	100000			4.7173	7.7978	10.6827	11.6696	12.0509
	vl	Fig. 5(c)		4.7300	7.8532	10.9956	14.1372	17.2788

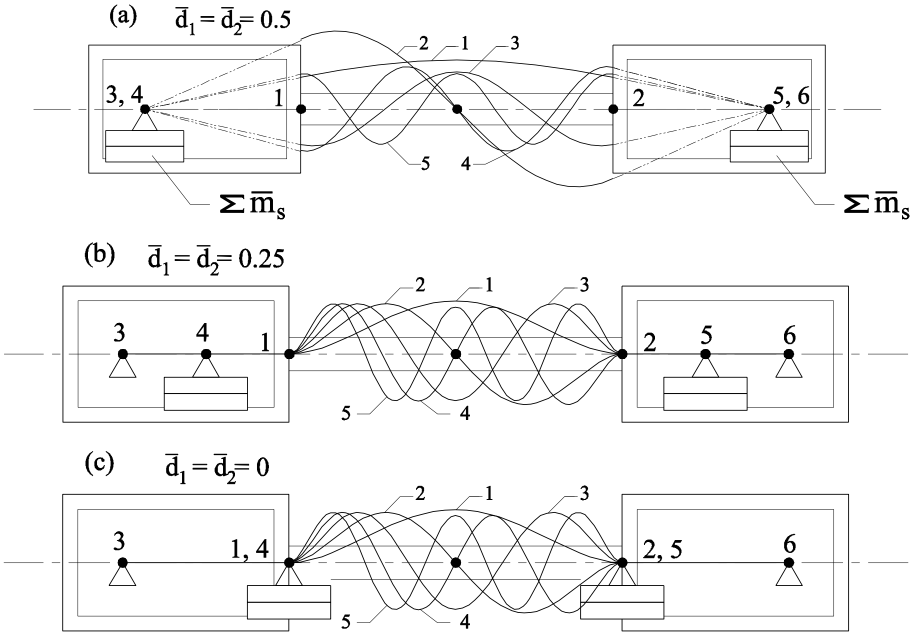


Fig. 5 first five modal shapes for symmetric model, $\bar{c}_1 = \bar{c}_2 = 0.5$ (a) $\bar{d}_1 = \bar{d}_2 = 0.5$, (b) $\bar{d}_1 = \bar{d}_2 = 0.25$ and (c) $\bar{d}_1 = \bar{d}_2 = 0$

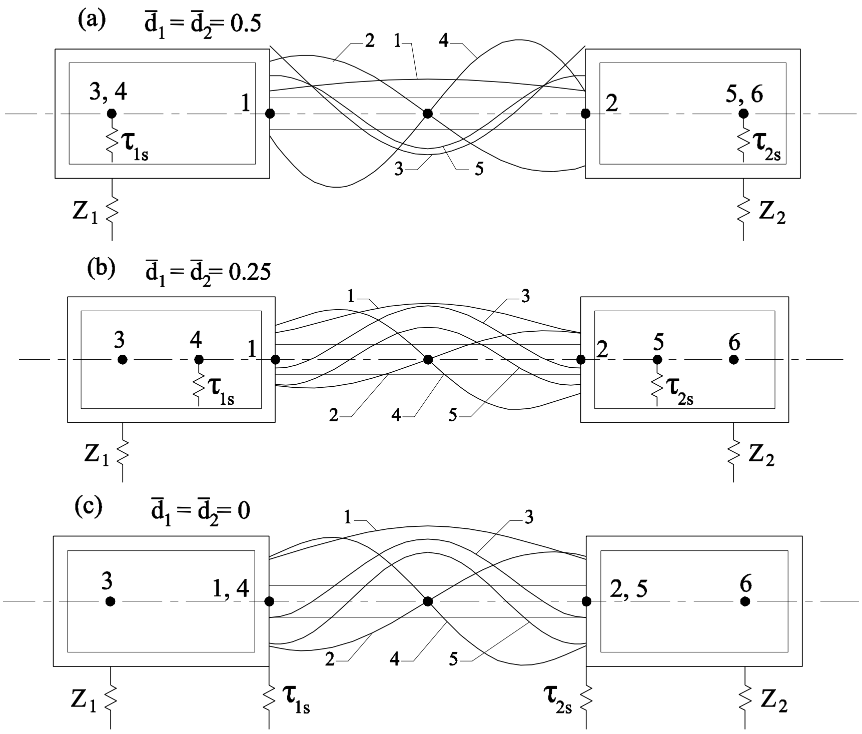


Fig. 6 first five modal shapes for elastically supported symmetric model, $\bar{c}_1 = \bar{c}_2 = 0.5$ (a) $\bar{d}_1 = \bar{d}_2 = 0.5$, (b) $\bar{d}_1 = \bar{d}_2 = 0.25$ and (c) $\bar{d}_1 = \bar{d}_2 = 0$

clear that the first five modals shapes are identical to those for *C-C* beam configurations. Fig. 6(a) shows the first five modal shapes for elastically supported end masses and carrying SS3 at points 4 and 5. As can be seen from Fig. 6(b) and Fig. 6(c), the modal shapes are influenced by the change in the spring stiffness and locations.

7. Conclusions

Exact vibration analysis of an axially loaded Timoshenko beam combined system with generalised end conditions including three spring-two mass sub-system are presented. The significance of this work may be drawn in the following contributions:

1- Mathematical model valid for, mechanical, naval and structural applications.
2- Three spring-two mass sub-system located at the center of both of the eccentric masses.
3- New frequency dependent shear force and bending moment terms are derived.
4- No need to extra columns and rows due to the sub-system presence.
5- Exact closed form frequency and mode shape equations for the combined system are derived.
6- Resonance frequencies are appeared equal to the degree of freedom of the located sub-system.
7- Previous results are verified.
8- Interesting results for combined system natural frequencies and mode shapes are presented.

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PL

Nomenclature

A	cross-section area of the beam.	Zs_3	stiffness parameter of k_{s3}.
a, b	polynomial roots	Zs_4	stiffness parameter of k_{s4}.
c_1, c_2	distance from k_1 to 1 and k_2 to 2.	Zs_5	stiffness parameter of k_{s5}
$\bar{c}_1, \bar{c}_2$	c_1 / L, c_2/L respectively	Zs_6	stiffness parameter of k_{s6}
d_2, d_1	distance between the mass center of gravity and the point of attachment	Zs_7	stiffness parameter of k_{s7}
$\bar{d}_2, \bar{d}_1$	ratio defined as d_2/L, d_1/L.	Zs_8	stiffness parameter of k_{s8}
E	Young's modulus of elasticity	γ_1, γ_2	set of non-dimensional terms defined as in Eqs. (21d), (22d).
G	shear modulus of rigidity	δ_1, δ_2	parameter defined as in Eqs. (31), (32)
I	moment of inertia of the beam cross section about the neutral axis.	$\bar{\eta}_{1s}$	set of non-dimensional terms defined as in Eq. (21c).
J_1, J_2	rotational moment of inertia of the end mass.	$\bar{\eta}_{2s}$	set of non-dimensional terms defined as in Eq. (22c).
$\bar{J}_1, \bar{J}_2$	ratio $(J_1/\rho A L^3), (J_2/\rho A L^3)$	$\epsilon_{1s}, \epsilon_{2s}$	set of non-dimensional terms defined as in Eqs. (21 b), (22 b).
$\acute{k}$	shear deformation shape coefficient	ϕ_1	LH end rotational spring stiffness.
k	elastic stiffness.	ϕ_2	RH end rotational spring stiffness
L	length of the beam (between points 1 & 2).	Φ_1	LH rotational rigidity parameter $\phi_1 L/EI$

m	mass of the beam.
$\bar{m}_1, \bar{m}_2$	$= M_1/m\,, M_2/m$ respectively.
M_1, M_2	end masses.
m_{s1}, m_{s2}	mass of sub-system 1 and 2.
m_{s3}, m_{s4}	mass of sub-system 1 and 2.
$\bar{m}_{s1}, \bar{m}_{s2}$	non-dimensional sub-system mass.
$\bar{m}_{s3}, \bar{m}_{s4}$	non-dimensional sub-system mass.
P	axial load.
p^2	axial load parameter (PL^2/EI).
r^2	rotary inertia parameter (I/AL^2).
s^2	shear deformation parameter ($Er^2/G\acute{k}$).
Y	non-dimensional lateral vibration.
x, y, ψ	system co-ordinate of the beam.
$y_{s1}, y_{s2},$	system co-ordinate of the masses
Z_1	LH lateral rigidity parameter (k_1L^3/EI).
Z_2	RH lateral rigidity parameter (k_2L^3/EI).
Φ_2	RH rotational rigidity parameter ($\phi_2 L/EI$).
λ^4	frequency parameter ($\rho AL^4\omega^2/EI$).
ω	circular frequency.
ψ	slope due to bending.
$L\Psi$	non-dimensional slope due to bending.
ν	Poisson's ratio.
ρ	mass density of the beam material.
θ_{1s}, θ_{2s}	set of non- dimensional terms defined as in Eqs. (21 a) and (22 a).
τ_{is}	nondimensional shear force parameter defined as in Eqs. (19) and (20).
ζ	non-dimensional beam length x/L.
C	Clamped (fixed) support
F	Free support.
P	Pinned (hinged) support.
vs	10 E-12.
vl	10 E+12.