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The finite element model research of the pre-twisted thin-walled beam

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Abstract. Based on the traditional mechanical model of thin-walled straight beam, the paper makes analysis and research on the pre-twisted thin-walled beam finite element numerical model. Firstly, based on the geometric deformation differential relationship, the Saint-Venant warping strain of pre-twisted thin-walled beam is deduced. According to the traditional thin-walled straight beam finite element mechanical model, the finite element stiffness matrix considering the Saint-Venant warping deformations is established. At the same time, the paper establishes the element stiffness matrix of the pre-twisted thin-walled beam based on the classic Vlasov Theory. Finally, by calculating the pre-twisted beam with elliptical section and I cross section and contrasting three-dimensional solid finite element using ANSYS, the comparison analysis results show that pre-twisted thin-walled beam element stiffness matrix has good accuracy.

Keywords: pre-twisted; thin-walled; coupling; warping; finite element model

1. Introduction

The pre-twisted beam, also known as naturally twisted beam, presents initially twisted shape in the natural state. Some researchers called it Naturally Twisted Beam (Zelenina and Zubov 2006). Static and dynamic analysis of naturally twisted beams has many important applications in mechanical and civil engineering, such as turbineblades, helicopter rotor blades, aircraft propeller blades, wind turbine blades, and so on. Dawson *et al*. (1971) found the natural frequencies of pre-twisted cantilever beams of uniform rectangular cross-section allowing for shear deformation and rotary inertia by the numerical integration. They also made some experiments in order to obtain the natural frequencies for beams of various breadth to depth ratios and lengths ranging from 3 to 20 in and pre-twist angle in the range 0-90°. Carnegie and Thomas investigated the effects of shear deformation and rotary inertia on the frequencies of flexural vibration of

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pre-twisted uniform and tapered cantilever beams by using the finite-difference method. Gupta and Rao used the finite element method to determine the natural frequencies of uniformly pre-twisted tapered cantilever beams. Subrahmanyam *et al.* applied the Reissner method and the total potential energy approach to calculate the natural frequencies and mode shapes of pre-twisted cantilever blading including shear deformation and rotary inertia. Rosen presented a survey paper as an extensive bibliography on the structural and dynamic aspects of pre-twisted beams. Chen and Keer studied the transverse vibration problems of a rotating twisted Timoshenko beam under axial loading, and investigated the effects of the twist angle, rotational speed, and axial force on natural frequencies by the finite element method. Chen and Ho introduced the differential transform to solve the free vibration problems of a rotating twisted Timoshenko beam under axial loading. Lin *et al*. derived the coupled governing differential equations and the general elastic boundary conditions for the coupled bending-bending forced vibration of a non-uniform pre-twisted Timoshenko beam by Hamilton's principle. They used a modified transfer matrix method to study the dynamic behaviour of a Timoshenko beam with arbitrary pre-twist angle. Banerjee developed a dynamic stiffness matrix and used it for free vibration analysis of a twisted beam. Raoand Gupta derived the stiffness and mass matrices of a rotating twisted and tapered Timoshenko beam element, and calculated the first four natural frequencies and mode shapes in bending–bending mode for cantilever beams. Dawe presented a Timoshenko beam finite element that has three nodes and two-degrees-of-freedom per node, which are the lateral deflection and the cross-sectional rotation.

As the prevailing research needs, the studies focused on material strength and vibration performance. However, as far as the authors are aware, no work has been reported in the existing literature on finite element formulation of pre-twisted thin-walled beam. On the traditional mechanical model of thin-walled straight beams, the classical theories are suitable for open and close thin-walled beam, the basic differential equation is given, but it is rather difficult to directly solve the basic differential equation of analytical solutions. Based on the energy principle, the finite element method is an effective method and the application is very widely. In the past decade, research scholars put forward using the naturally twisted beam in frames and support members of wall structures (Shadnam and Abbasnia 2002, Leung 2010). However, because of the existence of the naturally twisted angle ω, bending displacements coupled with each other, the finite element model will be different from traditional mechanical model of straight beam. As a result, research on the pre-twisted thin-walled beam performance has important theoretical and practical meaning for the engineers.

2. The finite element model based on the Saint-Venant warping deformation

2.1 The coupled elastic bending displacement

In the global coordinate system O_XYZ, the coupled elastic bending displacement behavior of the pre-twisted thin-walled beam is studied. At any position $Z=z$, we introduce local coordinate system $G\_\xi\eta z$, Gz axis and OZ axis are coincidence, $G\xi$ and $G\eta$ are the main bending axis of the section, the twisted angle of $G\_\xi\eta z$ relative to the O_XYZ is $\omega=kz$. The linear displacements are u and v along the axis $G\xi$ and $G\eta$, the rotations displacements are φ_ξ and φ_η. So at position $Z=z+dz$, the linear displacements are $u+u'dz$ and $v+v'dz$ along the axis $\overline{G}\overline{\xi}$ and $\overline{G}\overline{\eta}$, as shown in Fig. 1.

The incremental displacements of adjacent section in the local coordinate system are given by

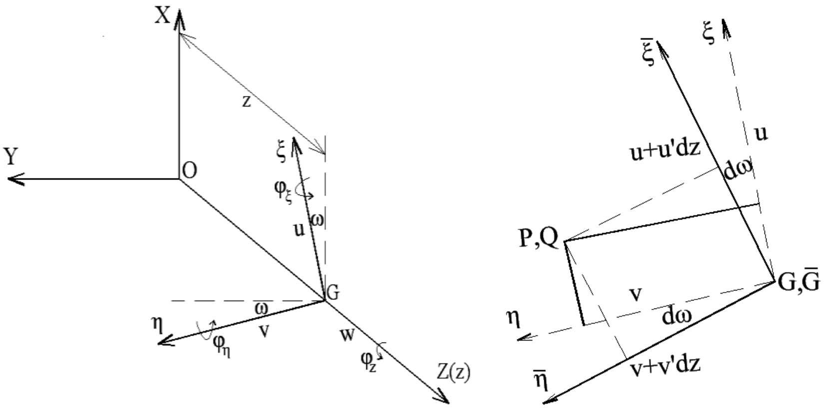


Fig. 1 Global and local coordinate of pre-twisted thin-walled beam

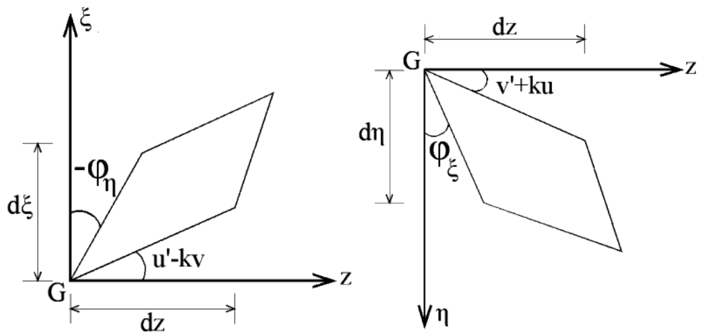


Fig. 2 Infinitesimal element shear strains of pre-twisted thin-walled beam

$$\Delta u = (u + u^{'}\mathrm{d}z)\cos \mathrm{d}\omega - (v + v^{'}\mathrm{d}z)\sin \mathrm{d}\omega - u \tag{1}$$

$$\Delta v = (v + v^{'}\mathrm{d}z)\cos \mathrm{d}\omega + (u + u^{'}\mathrm{d}z)\sin \mathrm{d}\omega - v \tag{2}$$

To the first order, cos$d\ \omega$=1, sin$d\ \omega$=$d\ \omega$ and $d\omega$=kdz, these are

$$\Delta u = (u' - kv)\mathrm{d}z \tag{3}$$

$$\Delta v = (v' + ku)\mathrm{d}z \tag{4}$$

From the definition of shear strain (Fig. 2), the following equations can be obtained

$$\gamma_{z\xi} = -\varphi_\eta + u' - kv,\ \gamma_{z\eta} = \varphi_\xi + v' + ku \tag{5}$$

Without considering shear deformation of the Euler beam

$$\varphi_\xi = -v' - ku\ ,\quad \varphi_\eta = u' - kv \tag{6}$$

Due to the bending effect, the axial displacement W of any point in $G\_\xi\eta$ plane along the GZ axis is given by

$$W = \eta\varphi_{\xi} - \xi\varphi_{\eta} \tag{7}$$

By introducing the global and local coordinate system conversion relation

$$\xi = X\cos\omega + Y\sin\omega,\ \ \eta = -X\sin\omega + Y\cos\omega \tag{8}$$

Substituting Eqs. (6) and (8) into (7), we can get

$$W = (-X\sin\omega + Y\cos\omega)(-v' - ku) - (X\cos\omega + Y\sin\omega)(u' - kv) \tag{9}$$

The normal strain is

$$\begin{aligned}\varepsilon_z = \frac{\partial W}{\partial z} &= (-X\sin\omega + Y\cos\omega)(-v'' - ku') - (X\cos\omega + Y\sin\omega)(u'' - kv') \\ &(-X\cos\omega - Y\sin\omega)k(-v' - ku) - (-X\sin\omega + Y\cos\omega)k(u' - kv)\end{aligned} \tag{10}$$

Substituting Eqs. (8) into (10), gives

$$\varepsilon_z = -\xi(u'' - 2kv' - k^2u) - \eta(v'' + 2ku' - k^2v) \tag{11}$$

2.2 The Saint-Venant warping deformations

The coordinate of any point P is (ξ, η) at Z=z position in the local coordinate system $G\_\xi\eta z$, and the projection point Q coordinates is $(\bar{\xi}, \bar{\eta})$. According to the Eqs. (3) and (4), we can get

$$\bar{\xi} = (\xi + k\eta\mathrm{dz}),\ \bar{\eta} = (-k\xi\mathrm{dz} + \eta) \tag{12}$$

Introducing the Saint-Venant warping displacement function $\varphi(\xi, \eta)$, and considering the effect on warping displacement to the axial displacement, we can get the axial displacement of the point P caused by warping deformation along the GZ axis in the plane $G\_\xi\eta$ is

$$W_{\omega} = -\alpha(z)\varphi(\xi,\eta) \tag{13}$$

The parameter $\alpha(z)$ represents the torsion angle of unit length. The axial displacement of point Q$(\bar{\xi}, \bar{\eta})$ caused by warping deformation along the GZ axis in the plane $\overline{G\_\xi\eta}$ is

$$\overline{W}_{\omega} = -\left(\alpha(z) + \alpha'(z)\mathrm{dz}\right)\varphi(\bar{\xi},\bar{\eta}) \tag{14}$$

Substituting the Eq. (12) into (13), we can get

$$\overline{W}_{\omega} = -\left(\alpha(z) + \alpha'(z)\mathrm{dz}\right)\times\varphi(\xi + k\eta\mathrm{dz}, -k\xi\mathrm{dz} + \eta) \tag{15}$$

Using the Taylor function expansion, and omitting the high-order item, we can get

$$\varphi(\xi + k\eta\mathrm{dz}, -k\xi\mathrm{dz} + \eta) = \varphi(\xi,\eta) + \left(k\eta\frac{\partial\varphi}{\partial\xi} - k\xi\frac{\partial\varphi}{\partial\eta}\right)\mathrm{dz} \tag{16}$$

So, we can get the normal strain caused by the section warping, the equation is

$$
\begin{aligned}
\varepsilon_{\omega} &= \frac{\overline{W_{\omega}} - W_{\omega}}{\mathrm{dz}} = \frac{\alpha(z)\varphi(\xi,\eta)}{\mathrm{dz}} \\
&- \frac{\left(\alpha(z) + \alpha^{'}(z)\mathrm{dz}\right)\left(\varphi(\xi,\eta) + \left(k\eta\dfrac{\partial\varphi}{\partial\xi} - k\xi\dfrac{\partial\varphi}{\partial\eta}\right)\mathrm{dz}\right)}{\mathrm{dz}}
\end{aligned}
\tag{17}
$$

Further expand the Eq. (17) and ignore the second order, we can get

$$
\varepsilon_{\omega} = -\alpha^{'}(z)\varphi(\xi,\eta) - k\alpha(z)\left(\eta\frac{\partial\varphi}{\partial\xi} - \xi\frac{\partial\varphi}{\partial\eta}\right) \tag{18}
$$

So, considering the effect of the biaxial bending and warping to axial displacement, we can get the normal strain

$$
\begin{aligned}
\varepsilon_{z} &= w' - \xi(\varphi_{\eta}^{'} + k\varphi_{\xi}) + \eta(\varphi_{\xi}^{'} - k\varphi_{\eta}) \\
&- \alpha^{'}(z)\varphi(\xi,\eta) - k\alpha(z)\left(\eta\frac{\partial\varphi}{\partial\xi} - \xi\frac{\partial\varphi}{\partial\eta}\right)
\end{aligned}
\tag{19}
$$

At the same time, we can get shear strain caused by the section warping, the equations are

$$
\gamma_{\omega\_z\xi} = \frac{\partial\left(-\alpha(z)\varphi(\xi,\eta)\right)}{\partial\xi} = -\alpha(z)\frac{\partial\varphi}{\partial\xi} \tag{20}
$$

$$
\gamma_{\omega\_z\eta} = \frac{\partial\left(-\alpha(z)\varphi(\xi,\eta)\right)}{\partial\eta} = -\alpha(z)\frac{\partial\varphi}{\partial\eta} \tag{21}
$$

Further, we can shear strain of the pre-twisted thin-walled beam considering the combined effect of torsion, biaxial bending and the warping, the equations are

$$
\gamma_{z\xi} = -\varphi_{\eta} + u' - kv - \eta\varphi_{z}^{'} - \alpha(z)\frac{\partial\varphi}{\partial\xi} \tag{23}
$$

$$
\gamma_{z\eta} = \varphi_{\xi} + v' + ku + \xi\varphi_{z}^{'} - \alpha(z)\frac{\partial\varphi}{\partial\eta} \tag{24}
$$

2.3 The displacement function

In this section, we use the following assumptions based on traditional thin-walled straight beam:

(1) The cross section shape remain the same;

(2) Only consider shear deformation by constrained torsion and ignore shear deformation by lateral load;

(3) The deformation is small and linear elastic;

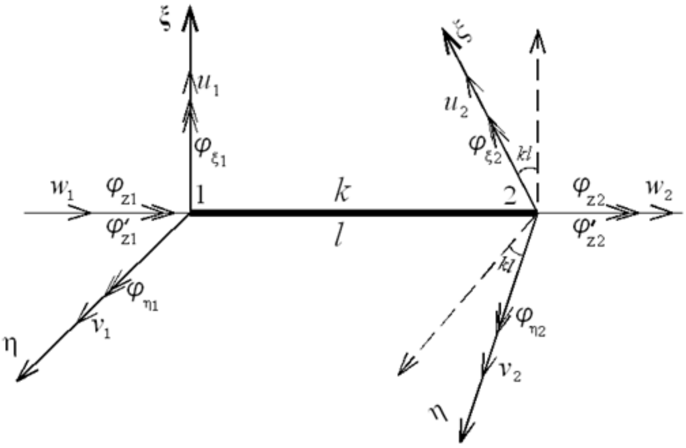


Fig. 3 The node displacement of pre-twisted thin-walled beam

(4) The warping deformation uses the classical Veslov thin-walled beam theory;

(5) Don't consider the coupling effect between the various deformation;

In the local coordinate system $G\_\xi\eta z$, the beam element uses two nodes with fourteen degrees of freedom, as shown in Fig. 3, each node has seven displacements vector, namely are u, v, w, φ_ξ, φ_η, φ_z and $\varphi_z^{'}$, and each node has also seven forces vector, namely are Q_ξ, Q_η, N, M_ξ, M_η, M_z and B. The axial displacement interpolation function uses Lagrange interpolation function of two nodes, regardless of the effect of biaxial bending and warping displacement. The bending displacement u and v use the cube polynomial interpolation function. The torsional angular displacement also uses the cube polynomial interpolation function. The specific expression is as following

$$\begin{cases} u = a_0 + a_1 z + a_2 z^2 + a_3 z^3 \\ v = b_0 + b_1 z + b_2 z^2 + b_3 z^3 \\ w = c_0 + c_1 z \\ \varphi_z = d_0 + d_1 z + d_2 z^2 + d_3 z^3 \end{cases} \tag{25}$$

According to the assumption (2), the relation equations between linear displacement and angle displacement will use the Eq. (6), namely, it can be expressed into the following

$$\frac{\mathrm{d}u}{\mathrm{d}z} = \varphi_\eta + kv\ ,\ \frac{\mathrm{d}v}{\mathrm{d}z} = -\varphi_\xi - ku \tag{26}$$

The matrix format of the Eqs. (25) and (26) are expressed into the following

$$\boldsymbol{u} = \overline{\boldsymbol{N}} \cdot \boldsymbol{C} \tag{27}$$

Among:

$$\boldsymbol{u} = \begin{bmatrix} u & v & w & \varphi_\xi & \varphi_\eta & \varphi_z & \varphi_z^{'} \end{bmatrix}^T$$

$$\boldsymbol{C} = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 & b_0 & b_1 & b_2 & b_3 & c_0 & c_1 & d_0 & d_1 & d_2 & d_3 \end{bmatrix}^T$$

Introduce the boundary conditions, the Eq. (27) can be expressed into the following

$$\boldsymbol{u} = \boldsymbol{N} \cdot \boldsymbol{u}_e \tag{28}$$

Among:
The joint displacement is:

$$\boldsymbol{u}_e = \begin{bmatrix} u_1 & v_1 & w_1 & \varphi_{\xi 1} & \varphi_{\eta 1} & \varphi_{z1} & \varphi_{z1}^{'} & u_2 & v_2 & w_2 & \varphi_{\xi 2} & \varphi_{\eta 2} & \varphi_{z2} & \varphi_{z2}^{'} \end{bmatrix}^T$$

The shape function matrix is:

$$\mathrm{N} = \begin{bmatrix} \mathbf{N}_1 \\ \mathbf{N}_2 \\ \mathbf{N}_3 \\ \mathbf{N}_4 \\ \mathbf{N}_5 \\ \mathbf{N}_6 \\ \mathbf{N}_7 \end{bmatrix} = \begin{bmatrix} N_1 & N_3 & 0 & 0 & N_5 & 0 & 0 & N_2 & N_4 & 0 & 0 & N_6 & 0 & 0 \\ -N_3 & N_1 & 0 & -N_5 & 0 & 0 & 0 & -N_4 & N_2 & 0 & -N_6 & 0 & 0 & 0 \\ 0 & 0 & N_7 & 0 & 0 & 0 & 0 & 0 & 0 & N_8 & 0 & 0 & 0 & 0 \\ A & -B & 0 & N_5^{'} & -kN_5 & 0 & 0 & C & -D & 0 & N_6^{'} & -kN_6 & 0 & 0 \\ B & A & 0 & kN_5 & N_5^{'} & 0 & 0 & D & C & 0 & kN_6 & N_6^{'} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & N_1 & N_5 & 0 & 0 & 0 & 0 & 0 & N_2 & N_6 \\ 0 & 0 & 0 & 0 & 0 & N_1^{'} & N_5^{'} & 0 & 0 & 0 & 0 & 0 & N_2^{'} & N_6^{'} \end{bmatrix}$$

$$N_1 = \frac{(2z+l)(l-z)^2}{l^3},\quad N_2 = \frac{z^2(3l-2z)}{l^3},\quad N_3 = \frac{kz(l-z)^2}{l^2},\quad N_4 = \frac{-kz^2(l-z)}{l^2},\quad N_5 = \frac{N_3}{k},\ N_6 = \frac{N_4}{k}$$

$$N_7 = \frac{l-z}{l},\quad N_8 = \frac{z}{l},\quad A = -kN_1 + \frac{\mathrm{d}N_3}{\mathrm{d}z},\quad B = kN_3 + \frac{\mathrm{d}N_1}{\mathrm{d}z},\quad C = -kN_2 + \frac{\mathrm{d}N_4}{\mathrm{d}z},\quad D = kN_4 + \frac{\mathrm{d}N_2}{\mathrm{d}z}$$

2.4 The element stiffness matrix

According to the Eq. (18), we can get the warping strain energy caused by warping deformation

$$U_\omega = \frac{1}{2}E\int_\Omega \left[\varphi_z^{''}(z)\varphi + k\varphi_z^{'}(z)\left(\eta\frac{\partial\varphi}{\partial\xi} - \xi\frac{\partial\varphi}{\partial\eta} \right) \right]^2 d\Omega \tag{29}$$

Further expand the Eq. (29), we can get

$$\begin{aligned} U_\varpi = {} & \frac{1}{2}\int_0^l \iint_A E(\varphi_z^{''})^2\varphi^2 \mathrm{d}z + \frac{1}{2}\int_0^l \iint_A Ek^2(\varphi_z^{'})^2 \left(\eta\frac{\partial\varphi}{\partial\xi} - \xi\frac{\partial\varphi}{\partial\eta} \right)^2 \mathrm{d}\xi\mathrm{d}\eta\mathrm{d}z \\ & + \frac{1}{2}\int_0^l \iint_A 2Ek\varphi_z^{''}(z)\varphi_z^{'}(z)\varphi\left(\eta\frac{\partial\varphi}{\partial\xi} - \xi\frac{\partial\varphi}{\partial\eta} \right)\mathrm{d}\xi\mathrm{d}\eta\mathrm{d}z \end{aligned} \tag{30}$$

Introduce the following parameters

$$I_\omega = \iint_A \varphi^2 \mathrm{d}\xi\mathrm{d}\eta,\ \bar{I}_\omega = \iint_A \left(\eta\frac{\partial\varphi}{\partial\xi} - \xi\frac{\partial\varphi}{\partial\eta} \right)^2 \mathrm{d}\xi\mathrm{d}\eta,\ \bar{I}_{\omega 1} = \iint_A 2\varphi\left(\eta\frac{\partial\varphi}{\partial\xi} - \xi\frac{\partial\varphi}{\partial\eta} \right)\mathrm{d}\xi\mathrm{d}\eta$$

According to the biaxial symmetry section features, we can get the following equation

$$\bar{I}_{\omega 1} = \iint_A 2\varphi\left(\eta\frac{\partial\varphi}{\partial\xi} - \xi\frac{\partial\varphi}{\partial\eta} \right)\mathrm{d}\xi\mathrm{d}\eta = 0 \tag{31}$$

Introduce the shape function, we can get the warping stiffness matrix

$$U_{\varpi} = \frac{1}{2}\boldsymbol{u}_e^T\left[\int_0^l EI_{\varpi}\left(\frac{\mathrm{d}^2\boldsymbol{N}_6}{\mathrm{dz}^2}\right)^T\left(\frac{\mathrm{d}^2\boldsymbol{N}_6}{\mathrm{dz}^2}\right)\mathrm{dz}\right]\boldsymbol{u}_e + \frac{1}{2}\boldsymbol{u}_e^T\left[\int_0^l Ek^2\bar{I}_{\varpi}\boldsymbol{N}_7^T\cdot\boldsymbol{N}_7\mathrm{dz}\right]\boldsymbol{u}_e \tag{32}$$

Also, we can get the axial strain energy using energy principle, as following

$$U_a = \frac{1}{2}\boldsymbol{u}_e^T\left[EA\int_0^l\left(\frac{\mathrm{d}\boldsymbol{N}_3}{\mathrm{dz}}\right)^T\left(\frac{\mathrm{d}\boldsymbol{N}_3}{\mathrm{dz}}\right)\mathrm{dz}\right]\boldsymbol{u}_e \tag{33}$$

And, we can get the shear strain energy as the warping displacement, as following

$$U_s = \frac{1}{2}G\int_0^l\iint_A(\varphi_z^{'})^2\left[\left(\frac{\partial\varphi}{\partial\xi}\right)^2 + \left(\frac{\partial\varphi}{\partial\eta}\right)^2\right]\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{dz} \tag{34}$$

Introduce the following parameters

$$\bar{J}_{\varpi} = \iint_A\left[\left(\frac{\partial\varphi}{\partial\xi}\right)^2 + \left(\frac{\partial\varphi}{\partial\eta}\right)^2\right]\mathrm{d}\xi\mathrm{d}\eta \tag{35}$$

Introduce the shape function, we can get the shear strain energy

$$U_s = \frac{1}{2}\boldsymbol{u}_e^T\left[G\bar{J}_{\varpi}\int_0^l\boldsymbol{N}_7^T\cdot\boldsymbol{N}_7\mathrm{dz}\right]\boldsymbol{u}_e \tag{36}$$

According to the Eq. (11), we can get the normal strain energy caused by bending deformation

$$U_b = \frac{1}{2}E\int_0^l\int_A[\xi(u''-2kv'-k^2u)+\eta(v''+2ku'-k^2v)]^2\mathrm{d}A\mathrm{dz} \tag{37}$$

Introduce the shape function, we can get the bending strain energy

$$\begin{aligned} U_b &= \frac{1}{2}\boldsymbol{u}_e^T\left[EI_{\xi}\int_0^l\left(\frac{\mathrm{d}^2\boldsymbol{N}_2}{\mathrm{dz}^2}+2k\frac{\mathrm{d}\boldsymbol{N}_1}{\mathrm{dz}}-k^2\boldsymbol{N}_2\right)^T\left(\frac{\mathrm{d}^2\boldsymbol{N}_2}{\mathrm{dz}^2}+2k\frac{\mathrm{d}\boldsymbol{N}_1}{\mathrm{dz}}-k^2\boldsymbol{N}_2\right)\mathrm{dz}\right]\boldsymbol{u}_e \\ &+\frac{1}{2}\boldsymbol{u}_e^T\left[EI_{\eta}\int_0^l\left(\frac{\mathrm{d}^2\boldsymbol{N}_1}{\mathrm{dz}^2}-2k\frac{\mathrm{d}\boldsymbol{N}_2}{\mathrm{dz}}-k^2\boldsymbol{N}_1\right)^T\left(\frac{\mathrm{d}^2\boldsymbol{N}_1}{\mathrm{dz}^2}-2k\frac{\mathrm{d}\boldsymbol{N}_2}{\mathrm{dz}}-k^2\boldsymbol{N}_1\right)\mathrm{dz}\right]\boldsymbol{u}_e \end{aligned} \tag{38}$$

So, we can get the element stiffness matrix of the pre-twisted beam, as following

$$\begin{aligned} \boldsymbol{K}^e &= EA\int_0^l\left(\frac{\mathrm{d}\boldsymbol{N}_3}{\mathrm{dz}}\right)^T\left(\frac{\mathrm{d}\boldsymbol{N}_3}{\mathrm{dz}}\right)\mathrm{dz} + EI_{\xi}\int_0^l\left(2k\frac{\mathrm{d}\boldsymbol{N}_1}{\mathrm{dz}}+\frac{\mathrm{d}^2\boldsymbol{N}_2}{\mathrm{dz}^2}-k^2\boldsymbol{N}_2\right)^T\cdot\left(2k\frac{\mathrm{d}\boldsymbol{N}_1}{\mathrm{dz}}+\frac{\mathrm{d}^2\boldsymbol{N}_2}{\mathrm{dz}^2}-k^2\boldsymbol{N}_2\right)\mathrm{dz} \\ &+EI_{\eta}\int_0^l\left(-2k\frac{\mathrm{d}\boldsymbol{N}_2}{\mathrm{dz}}+\frac{\mathrm{d}^2\boldsymbol{N}_1}{\mathrm{dz}^2}-k^2\boldsymbol{N}_1\right)^T\cdot\left(-2k\frac{\mathrm{d}\boldsymbol{N}_2}{\mathrm{dz}}+\frac{\mathrm{d}^2\boldsymbol{N}_1}{\mathrm{dz}^2}-k^2\boldsymbol{N}_1\right)\mathrm{dz} \\ &+EI_{\varpi}\int_0^l\left(\frac{\mathrm{d}^2\boldsymbol{N}_6}{\mathrm{dz}^2}\right)^T\left(\frac{\mathrm{d}^2\boldsymbol{N}_6}{\mathrm{dz}^2}\right)\mathrm{dz}+\left(k^2E\bar{I}_{\varpi}+G\bar{J}_{\varpi}\right)\int_0^l\boldsymbol{N}_7^T\cdot\boldsymbol{N}_7\mathrm{dz} \end{aligned} \tag{39}$$

3. The finite element model based on the classical Vlasov theory

However, it is very difficult to find the precise Saint-Venant warping function $\varphi(\xi, \eta)$ about the thin-walled beam, so, it is necessary to further study on the finite element mode of the pre-twisted thin-walled beam.

In this section, the pre-twisted beam still uses the above displacement function and the beam element still uses two nodes with fourteen degrees of freedom, as shown in Fig. 3. However, the warping deformation of the pre-twisted thin-walled beam will use the classical Vlasov Theory, the normal strain caused by the warping deformation is

$$\varepsilon_{\omega} = -\varphi_z^{''}(z)\varpi \tag{40}$$

The parameter ϖ (s) is the fan-shape area of the any point $M(z, s)$ in section, namely, it is the warping coordinate of the point M. So, we can get the warping strain energy, as following

$$U_{\omega} = \frac{1}{2}E\int_0^l \iint_A \left[-\varphi_z^{''}(z)\varpi\right]^2 \mathrm{d}\xi \mathrm{d}\eta \mathrm{dz} \tag{41}$$

Introduce the shape function, we can get the warping strain energy

$$U_{\omega} = \frac{1}{2}\boldsymbol{u}_e^T \left[\int_0^l EI_{\omega}\left(\frac{\mathrm{d}^2\boldsymbol{N}_6}{\mathrm{dz}^2}\right)^T \left(\frac{\mathrm{d}^2\boldsymbol{N}_6}{\mathrm{dz}^2}\right)\mathrm{dz}\right]\boldsymbol{u}_e \tag{42}$$

The parameter I_{ω} is the main fan-shape moment of inertia. Also, we can get the shear strain energy caused by the warping deformation, as following

$$U_s = \frac{1}{2}GI_p\int_0^l (\varphi_z^{'})^2 \mathrm{dz} \tag{43}$$

The parameter I_p is the torsional moment of inertia. Introduce the shape function, we can get the shear strain energy

$$U_s = \frac{1}{2}\boldsymbol{u}_e^T \left[GI_p\int_0^l \boldsymbol{N}_7^T \cdot \boldsymbol{N}_7 \mathrm{dz}\right]\boldsymbol{u}_e \tag{44}$$

So, we also can get the element stiffness matrix of the pre-twisted beam, as following

$$\begin{aligned}\boldsymbol{K}^e = {} & EA\int_0^l \left(\frac{\mathrm{d}\boldsymbol{N}_3}{\mathrm{dz}}\right)^T \left(\frac{\mathrm{d}\boldsymbol{N}_3}{\mathrm{dz}}\right)\mathrm{dz} + EI_{\xi}\int_0^l \left(2k\frac{\mathrm{d}\boldsymbol{N}_1}{\mathrm{dz}} + \frac{\mathrm{d}^2\boldsymbol{N}_2}{\mathrm{dz}^2} - k^2\boldsymbol{N}_2\right)^T \cdot \left(2k\frac{\mathrm{d}\boldsymbol{N}_1}{\mathrm{dz}} + \frac{\mathrm{d}^2\boldsymbol{N}_2}{\mathrm{dz}^2} - k^2\boldsymbol{N}_2\right)\mathrm{dz} \\ & + EI_{\eta}\int_0^l \left(-2k\frac{\mathrm{d}\boldsymbol{N}_2}{\mathrm{dz}} + \frac{\mathrm{d}^2\boldsymbol{N}_1}{\mathrm{dz}^2} - k^2\boldsymbol{N}_1\right)^T \cdot \left(-2k\frac{\mathrm{d}\boldsymbol{N}_2}{\mathrm{dz}} + \frac{\mathrm{d}^2\boldsymbol{N}_1}{\mathrm{dz}^2} - k^2\boldsymbol{N}_1\right)\mathrm{dz} \\ & + EI_{\omega}\int_0^l \left(\frac{\mathrm{d}^2\boldsymbol{N}_6}{\mathrm{dz}^2}\right)^T \left(\frac{\mathrm{d}^2\boldsymbol{N}_6}{\mathrm{dz}^2}\right)\mathrm{dz} + GI_p\int_0^l \boldsymbol{N}_7^T \cdot \boldsymbol{N}_7 \mathrm{dz}\end{aligned} \tag{45}$$

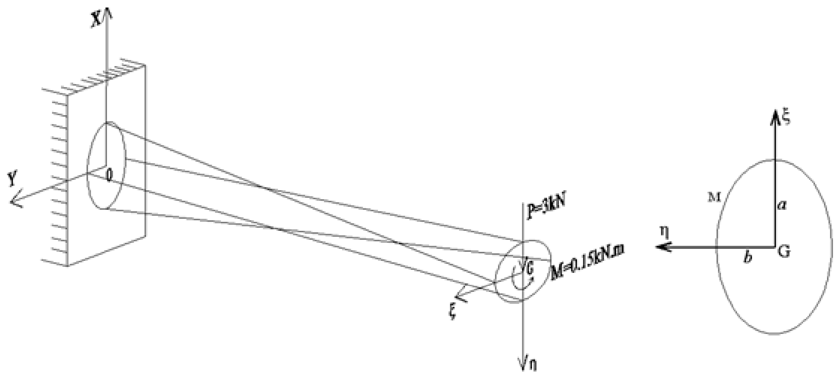


Fig. 4 the geometric parameter of pre-twisted ellipse beam (a=200 mm, b=100 mm)

4. Analysis example

4.1 The pre-twisted beam with ellipse cross section

4.1.1 Analysis Model

The analysis model in our example is a cantilever beam, the cantilever beam length l is 6000 mm, the cross section is ellipse, the radius are 100 mm and 200 mm, the pre-twisted angle ω is $\frac{\pi}{2}$, the steel material modulus E is 2.1×10^5 Mpa. The concentrated force P is 3 kN and the torque M is 0.15 kN.m.

The warping displacement function of beam with ellipse cross section is as following

$$\varphi(\xi,\eta) = -\frac{a^2-b^2}{a^2+b^2}\xi\eta \tag{46}$$

So, we can get section properties, as following

$$\begin{cases} A=\pi ab, I_\xi=\frac{1}{4}\pi ab^3, I_\eta=\frac{1}{4}\pi a^3 b, I_p=I_\xi+I_\eta=\frac{1}{4}\pi ab(a^2+b^2), J=\frac{\pi a^3 b^3}{a^2+b^2} \\ \bar{J}_\varpi=\frac{\pi ab(a^2-b^2)^2}{4(a^2+b^2)}, I_\varpi=\frac{\pi a^3 b^3(a^2-b^2)^2}{24(a^2+b^2)^2}, \bar{I}_\varpi=\frac{\pi ab(3a^4-2a^2b^2+3b^4)(a^2-b^2)^2}{24(a^2+b^2)^2} \end{cases} \tag{47}$$

4.1.2 Analysis result

The paper makes comparison analysis with the ANSYS software of using the solid element, the results are following Fig. 5. From the results of the comparison, we can get the following conclusion:

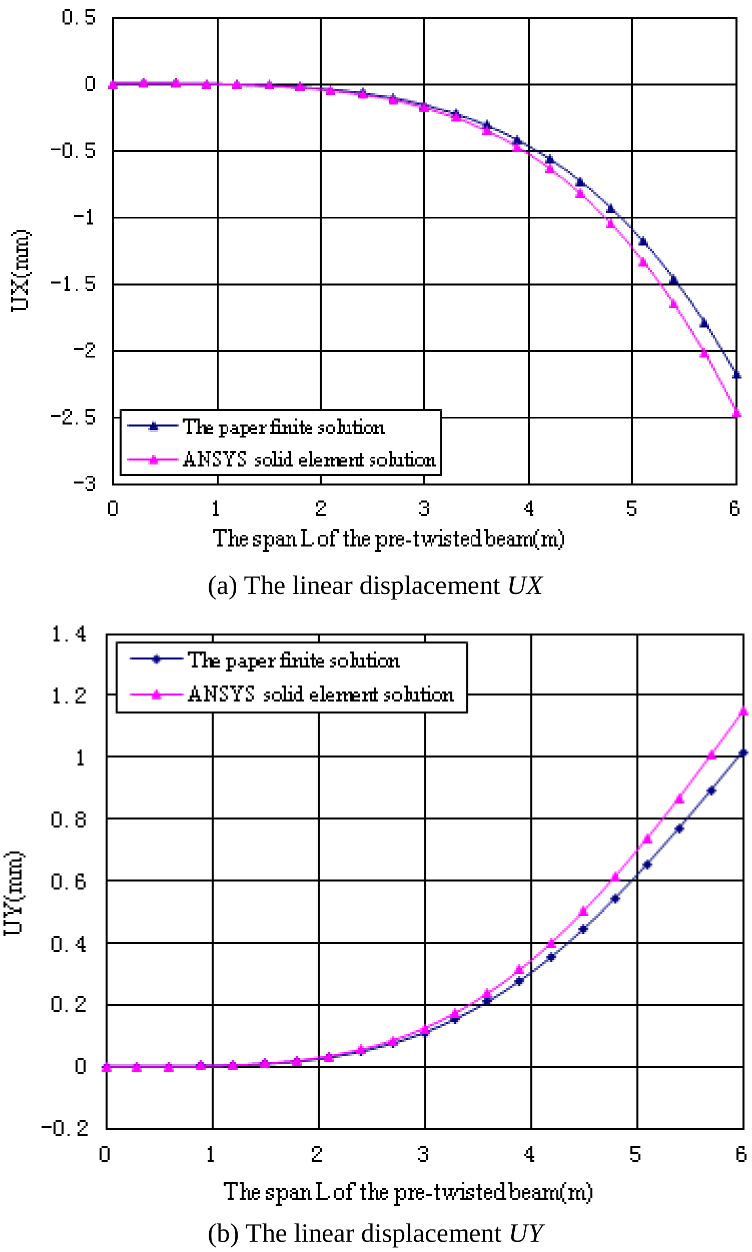
(1) Based on the pre-twisted beam element stiffness matrix derived by the classic Saint Venant warping displacement, the bending and warping displacements result is very close to result by ANSYS. The X, Y direction of bending displacement error is within 5%.

(2) The fixed end restrain torsion affection of cantilever beam is very strong and the warping displacement is very small. Free end restrain torsion effect is small, adjacent section warping

displacement is almost same.

4.2 The pre-twisted beam with I cross section

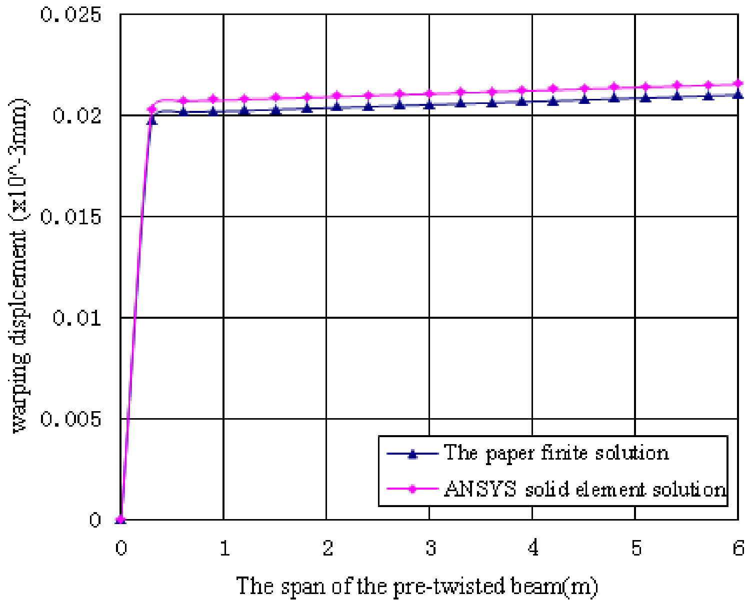
As previously mentioned, it is very difficult to find the precise Saint-Venant warping function $\varphi(\xi, \eta)$ about the thin-walled beam, so, the paper makes analysis using the classic Vlasov Theory by comparison with the ANSYS software of using the solid element.



(a) The linear displacement *UX*

(b) The linear displacement *UY*

Fig. 5 The elliptical section pre-twisted beam displacement comparison



(c) The point *C* warping displacement

Fig. 5 Continued

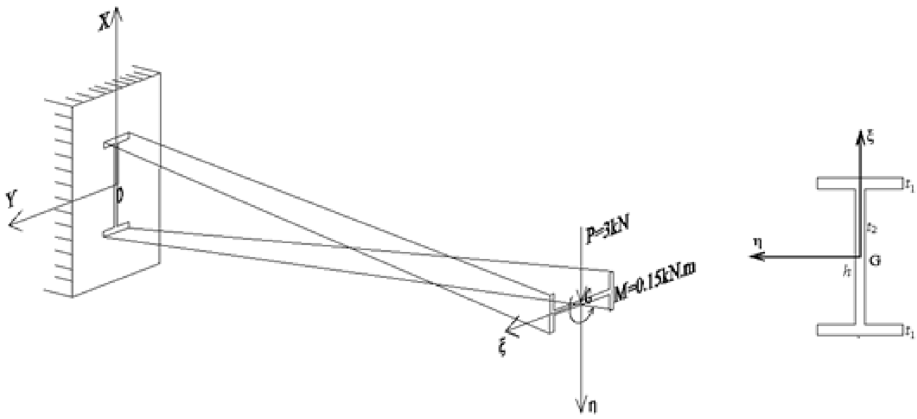


Fig. 6 the geometric parameter of pre-twisted beam with *I* cross section

The cantilever beam length *l* is 6000 mm, the cross section is *I* cross section (Fig. 6), the height h is 400 mm, web thickness is 8mm and the flange thickness is 10 mm. The pre-twisted angle ω is $\frac{\pi}{2}$. The steel material modulus *E* is 2.1×10^5 Mpa. The concentrated force *P* is 3 *k*N and the torque *M* is 0.15 *k*N.m.

From the results of the comparison (Fig. 7), we can get the following conclusion: Based on the pre-twisted beam element stiffness matrix derived by the classic Vlasov Theory, the warping displacements result is very close to result by ANSYS using the solid element, the displacement error is within 10.5%.

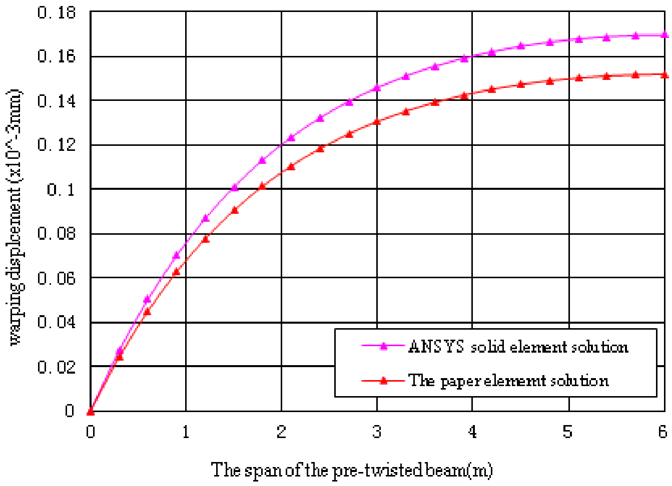


Fig. 7 The pre-twisted beam warping displacement comparison with *I* cross section

5. Conclusions

1. Based on the classical Saint-Venant warping displacement function, the paper strictly deduces the Saint-Venant warping strain of the pre-twisted thin-walled beam. Based on the three-dimensional deformation relationship and energy principle, the paper establishes the pre-twisted thin-walled beam element stiffness matrix.
2. Because it is very difficult to find the precise Saint-Venant warping function about the thin-walled beam, the paper establishes the pre-twisted thin-walled beam element stiffness matrix by considering the classic Vlasov Theory.
3. By comparison with ANSYS three-dimensional solid calculation analysis, the result shows that the pre-twisted beam element stiffness matrix has good precision. However, this article does not take into account the cross-section warping and shear deformation effects on the axial displacement, the pre-twisted beam precise displacement interpolation function needs further study.

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