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Natural vibration of the three-layered solid sphere with middle layer made of FGM: three-dimensional approach

Surkay D. Akbarov[1,2a], Hatam H. Guliyev[3b] and Nazmiye Yahnioglu[*4]

[1]*Department of Mechanical Engineering, Yildiz Technical University, Yildiz Campus, 34349, Besiktas, Istanbul, Turkey*
[2]*Institute of Mathematics and Mechanics of the National Academy of Sciences of Azerbaijan, AZ1141, Baku, Azerbaijan*
[3]*Institute of Geology and Geophysics of the National Academy of Sciences of Azerbaijan, AZ1073, Baku, Azerbaijan*
[4]*Department of Mathematical Engineering, Yildiz Technical University, Davutpasa Campus, 34220, Esenler Istanbul, Turkey*

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Abstract. The paper studies the natural oscillation of the three-layered solid sphere with a middle layer made of Functionally Graded Material (FGM). It is assumed that the materials of the core and outer layer of the sphere are homogeneous and isotropic elastic. The three-dimensional exact equations and relations of linear elastodynamics are employed for the investigations. The discrete-analytical method proposed by the first author in his earlier works is applied for solution of the corresponding eigenvalue problem. It is assumed that the modulus of elasticity, Poisson's ratio and density of the middle-layer material vary continuously through the inward radial direction according to power law distribution. Numerical results on the natural frequencies related to the torsional and spheroidal oscillation modes are presented and discussed. In particular, it is established that the increase of the modulus of elasticity (mass density) in the inward radial direction causes an increase (a decrease) in the values of the natural frequencies.

Keywords: functionally graded material; three-layered solid sphere; natural vibration; natural frequencies; torsional oscillation; spheroidal oscillation

1. Introduction

Study of the vibration of the hollow and especially solid spheres is required not only to answer the fundamental questions of elastodynamics and the dynamics of structural elements, but also for explanation and understanding of some natural phenomena, such as oscillations of the Earth caused by an earthquake. Furthermore, from the historical aspect these investigations were associated originally with interest in the oscillations of the earth (see, Love 1944). The first attempt in this field was made by Lamb (1882), in which the natural vibration of the solid sphere

∗Corresponding author, Professor, E-mail: nazmiye@yildiz.edu.tr
[a]Professor, E-mail: akbarov@yildiz.edu.tr
[b]Professor, E-mail: hatamguliyev@gmail.com

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made of homogeneous and isotropic elastic material was investigated by the use of Cartesian coordinates. Note that Lamb (1882) found the automodel-similarity solution of the governing field equations, so that all sought quantities are presented as a function of the distance of the point considered from the origin of the Cartesian coordinate system. Therefore the solution by Lamb gives the possibility of finding the values of the natural frequencies but this solution does not allow for finding of the modes of the natural vibrations from the standpoint of modern ideas. Chree (1889) further developed the mathematical treatment by Lamb (1882) with the use of spherical coordinates. Nevertheless, the results obtained in the paper by Lamb (1882) can be estimated as fundamental in the dynamics of the spherical elastic body and these results have a great significance not only in the theoretical, but also in the practical sense. Moreover, in the paper by Lamb (1882) it was established that the solid sphere has two types of uncoupled free vibrations, the first of which are torsional vibrations with rotatory motions of the sphere for which there is no radial displacement and no volumetric change. The second type of free vibration of the sphere is called spheroidal vibration which is characterized by the volumetric change of the sphere caused by the non-zero radial displacement. Lamb's results were used with subsequent investigations and attempts were made to apply them to describe the vibration of the Earth generated by earthquakes. Of note among these subsequent investigations are those carried out by Sato and Usami (1962a, 1962b), Sato *et al*. (1962) in which a detailed analysis of the natural frequencies and vibration modes of the homogeneous isotropic solid sphere was performed and tabulated. The results by Sato and Usami (1962a, 1962b), Sato *et al*. (1962) were also presented and discussed in the monograph by Eringen and Suhubi (1975).

Much later, Guz (1985a, 1985b) studied the natural vibration of the solid sphere with initially uniform volumetric loading by utilizing the three-dimensional linearized theory of elastic waves in initially stressed bodies for incompressible and compressible bodies. In these works, it was established that Lamb's result on the types of natural vibration of the sphere occurs also for the cases which have initial static stresses caused by uniformly-cubic loading.

Shah *et al*. (1969a, 1969b) investigated natural vibration of the hollow sphere (or spherical shell) made of homogeneous and isotropic elastic material by utilizing the three-dimensional exact equations of the linear theory of elastodynamics and numerical results for a wide range of thickness-to-radius ratios are given in graphical form. A detailed analysis of these and other related results which are associated with the free oscillations of the earth, are also given and discussed in the book by Lapwood and Usami (1981).

Note that recent investigations related to the free vibration of the solid and hollow spheres consider the more complicated problems connected with the geometries (see, for instance, the paper by Hasheminejad and Mirzaei (2011) and others listed therein) and properties of the material of the spheres (see, for instance, the paper by Sharma *et al*. (2012) and others listed therein). Moreover, note that in present time the study of dynamics of the structural elements made of non-homogeneous traditional and advanced materials such as FGM, piezoelectric materials and etc. are developed intensively. As an example for such investigations it can be taken papers by Asemi *et al*. (2014), Ipek (2015), Yun *et al*. (2010), Asgari and Akhlaghi (2011), Nihat and Koç (2015) and other ones listed therein.

By developing the theory on the bowels of the earth and by employing layered composite spherical constructions in various branches of modern industries, the necessity to investigate the corresponding dynamic problems related to oscillations of the layered hollow and solid spheres, appears. These investigations are made both within the scope of the various approximate shell theories and within the scope of the three-dimensional exact equations of elastodynamics. The accuracy of the approximate theories is examined in the paper by Grigorenko and Kilina (1989). The results are given in graphical form and it is established that the accuracy of the approximate shell theories decreases with increasing of the ratio *h/R*, where *h* is the thickness and *R* is the radius of the middle surface of the sphere.

Further, in the paper by Jiang *et al*. (1996), the natural vibration of layered hollow spheres is studied by using the three-dimensional exact equations of elastodynamics. Concrete numerical results are presented for the three-layered hollow sphere for various ratios of the densities and modulus of elasticities. These results are given in table form and, in particular cases, are compared with the known ones.

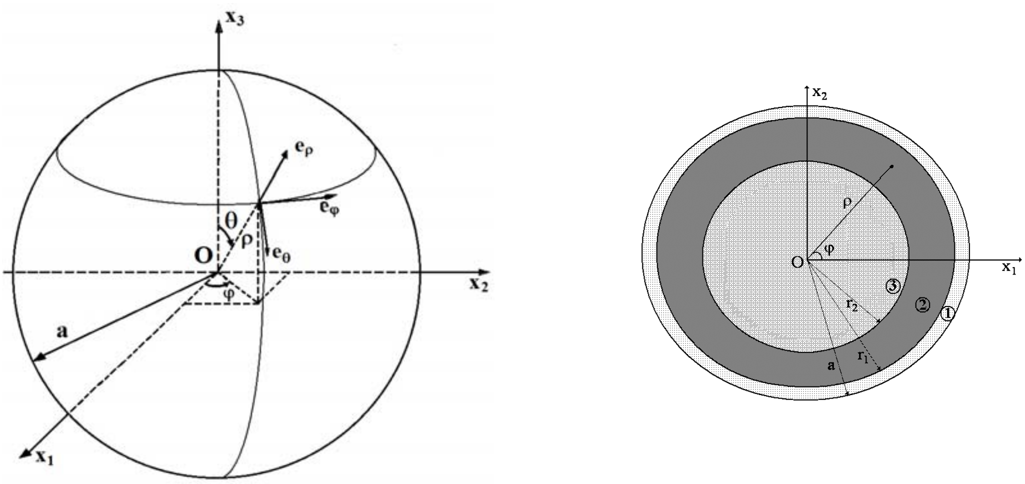
Note that in the foregoing investigations related to the vibration of the layered hollow sphere it was assumed that the layers' materials are homogeneous and isotropic. In the paper by Chen and Ding (2001) these investigations were developed for the cases where the materials of the layers of the layered hollow sphere are spherically isotropic (a special case of transversal isotropic materials) and homogeneous. Numerical results are presented and discussed for the three-layered case and the influence of the type of anisotropy of the layers' materials on the natural frequencies and vibration modes is established.

The modern level of studies on the bowels of the earth detailed for instance in the book by Anderson (2007), show that the mechanical properties, such as the modulus of elasticity and density of the mantle material increase continuously from the crust to the core. Moreover, in modern layered hollow spheres, the layers are made of Functionally Graded Materials (FGM) which give some advantages to these constructions in the application sense. These and many other reasons require study of the dynamics of the layered hollow spheres, the layers of which are made of FGM. Certain attempts in this field were recently made in the paper by Ye *et al*. (2014) in which the three-dimensional vibration analysis of a spherical shell which is obtained by cutting the complete hollow sphere by two parallel planes with arbitrary end conditions, was studied. It is assumed that the shell is a single-layered one with effective mechanical properties, the values of which change continuously in the thickness direction of the shell. These effective mechanical properties are determined through the mechanical properties of the ceramic and metal layers and their volumetric fraction in the shell, which also vary continuously through the thickness direction according to power law distribution. The exact three-dimensional relations between the strains and displacements, as well as between the stresses and strains are used in constructing the functional for employing the Rayleigh-Ritz method. The sought values are presented through the modified Fourier series for all coordinates. Numerical results on the natural frequencies and the influence of the FGM properties on these results are discussed.

This completes the review of the related investigations from which it follows that up to now there has not been any investigation related layered solid sphere. Accordingly, in the present paper an attempt is made for investigation of the natural vibration of the three-layered solid sphere, the middle layer of which is made of FGM by utilizing of the three-dimensional exact equations of elastodynamics. Also, it is assumed that the outer layer and core of the sphere are made of homogeneous isotropic material. The corresponding eigenvalue problem is solved by employing the discrete analytical method proposed by Akbarov (2006, 2015).

2. Formulation of the problem

Consider the three-layered solid sphere and with the center of the sphere we associate the



(a) Selected coordinate systems (b) The cross section of the sphere at x_3=0

Fig. 1 Selected coordinate systems and cross section of the sphere

Cartesian coordinate system $Ox_1x_2x_3$ and spherical coordinate system $Or\theta\varphi$ (Fig. 1(a)). The cross section of the sphere at x_3=0 and the parameters characterizing the structural geometry of this sphere are shown in Fig. 1(b). The outer radius of the sphere (the outer and inner radius of the middle layer) will be denoted through a (r_1 and r_2, respectively). The values related to the outer, middle layers and core will be indicated by the upper indices (1), (2) and (3), respectively.

Consider the field equations written in the spherical coordinate system shown in Fig. 1(a).

Equation of motion:

$$\frac{\partial\sigma_{rr}^{(k)}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta r}^{(k)}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\sigma_{\varphi r}^{(k)}}{\partial\varphi}+\frac{1}{r}\left(2\sigma_{rr}^{(k)}-\sigma_{\varphi\varphi}^{(k)}-\sigma_{\theta\theta}^{(k)}+\sigma_{\theta r}^{(k)}\cot\theta\right)=\rho\frac{\partial^2u_r^{(k)}}{\partial t^2},$$

$$\frac{\partial\sigma_{r\theta}^{(k)}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}^{(k)}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\sigma_{\varphi\theta}^{(k)}}{\partial\varphi}+\frac{1}{r}\left(3\sigma_{r\theta}^{(k)}+\left(\sigma_{\theta\theta}^{(k)}-\sigma_{\varphi\varphi}^{(k)}\right)\cot\theta\right)=\rho\frac{\partial^2u_\theta^{(k)}}{\partial t^2},$$

$$\frac{\partial\sigma_{r\varphi}^{(k)}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\varphi}^{(k)}}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial\sigma_{\varphi\varphi}^{(k)}}{\partial\varphi}+\frac{1}{r}\left(2\sigma_{r\varphi}^{(k)}+\sigma_{\varphi r}^{(k)}+\left(\sigma_{\theta\varphi}^{(k)}+\sigma_{\varphi\theta}^{(k)}\right)\cot\theta\right)=\rho\frac{\partial^2u_\varphi^{(k)}}{\partial t^2}. \quad (1)$$

Constitutive equations:

$$\sigma_{rr}^{(k)}=\lambda^{(k)}(\varepsilon_{rr}^{(k)}+\varepsilon_{\theta\theta}^{(k)}+\varepsilon_{\varphi\varphi}^{(k)})+2\mu^{(k)}\varepsilon_{rr}^{(k)}\ ,\ \sigma_{\theta\theta}^{(k)}=\lambda^{(k)}(\varepsilon_{rr}^{(k)}+\varepsilon_{\theta\theta}^{(k)}+\varepsilon_{\varphi\varphi}^{(k)})+2\mu^{(k)}\varepsilon_{\theta\theta}^{(k)},$$

$$\sigma_{\varphi\varphi}^{(k)}=\lambda^{(k)}(\varepsilon_{rr}^{(k)}+\varepsilon_{\theta\theta}^{(k)}+\varepsilon_{\varphi\varphi}^{(k)})+2\mu^{(k)}\varepsilon_{\varphi\varphi}^{(k)},\ \sigma_{r\theta}^{(k)}=2\mu^{(k)}\varepsilon_{r\theta}^{(k)},$$

$$\sigma_{\theta\varphi}^{(k)}=2\mu^{(k)}\varepsilon_{\theta\varphi}^{(k)},\ \sigma_{r\varphi}^{(k)}=2\mu^{(k)}\varepsilon_{r\varphi}^{(k)}. \quad (2)$$

Strain-displacement relations:

$$\varepsilon_{rr}^{(k)}=\frac{\partial u_r^{(k)}}{\partial r}\ ,\ \varepsilon_{\theta\theta}^{(k)}=\frac{1}{r}\frac{\partial u_\theta^{(k)}}{\partial\theta}+\frac{1}{r}u_r^{(k)};\quad \varepsilon_{\varphi\varphi}^{(k)}=\frac{1}{r\sin\theta}\frac{\partial u_\varphi^{(k)}}{\partial\varphi}+\frac{1}{r}u_r^{(k)}+\frac{1}{r}u_\theta^{(k)}\cot\theta,$$

$$\varepsilon_{r\theta}^{(k)} = \frac{1}{2}\left(\frac{\partial u_{\theta}^{(k)}}{\partial r} + \frac{1}{r}\frac{\partial u_r^{(k)}}{\partial \theta} - \frac{u_{\theta}^{(k)}}{r}\right); \quad \varepsilon_{\theta\varphi}^{(k)} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_{\varphi}^{(k)}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_{\theta}^{(k)}}{\partial \varphi} - \frac{u_{\varphi}^{(k)}}{r}\cot\theta\right),$$

$$\varepsilon_{r\varphi}^{(k)} = \frac{1}{2}\left(\frac{\partial u_{\varphi}^{(k)}}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial u_r^{(k)}}{\partial \theta} - \frac{u_{\varphi}^{(k)}}{r}\right). \tag{3}$$

In (1)-(3) conventional notation is used and it is assumed that k=1, 2 and 3.

We assume that the materials of the core and outer layer are homogeneous isotropic, but the material of the middle layer is FG and isotropic, i.e., we assume that

$$\lambda^{(1)} = const_{\lambda}^{(1)}, \ \mu^{(1)} = const_{\mu}^{(1)}, \rho^{(1)} = const_{\rho}^{(1)}, \lambda^{(3)} = const_{\lambda}^{(3)},$$
$$\mu^{(3)} = const_{\mu}^{(3)}, \ \rho^{(3)} = const_{\rho}^{(3)},$$
$$\lambda^{(2)} = \lambda^{(2)}(r), \ \mu^{(2)} = \mu^{(2)}(r), \ \rho^{(2)} = \rho^{(2)}(r). \tag{4}$$

Specializations of the functions $\lambda^{(2)}(r)$, $\mu^{(2)}(r)$ and $\rho^{(2)}(r)$ in (4) will be given below under consideration of the numerical results.

This completes the consideration of the field equation and relations which are used in the present investigations. Now we consider formulation of the boundary and contact conditions. According to the nature of the problem under consideration, we assume that on the outer free surface of the sphere, i.e., at r=a (Fig. 1(b)), the following boundary conditions are satisfied

$$\sigma_{rr}^{(1)}\Big|_{r=a} = 0, \ \sigma_{r\theta}^{(1)}\Big|_{r=a} = 0, \ \sigma_{r\varphi}^{(1)}\Big|_{r=a} = 0. \tag{5}$$

Moreover, we assume that on the interface surfaces, i.e., at r=r_1 and r=r_2 (Fig. 1(b)), of the constituents, the following perfect contact conditions are satisfied

$$\sigma_{rr}^{(2)}\Big|_{r=r_1} = \sigma_{rr}^{(1)}\Big|_{r=r_1}, \ \sigma_{r\theta}^{(2)}\Big|_{r=r_1} = \sigma_{r\theta}^{(1)}\Big|_{r=r_1}, \ \sigma_{r\varphi}^{(2)}\Big|_{r=r_1} = \sigma_{r\varphi}^{(1)}\Big|_{r=r_1},$$
$$u_r^{(2)}\Big|_{r=r_1} = u_r^{(1)}\Big|_{r=r_1}, \ u_{\theta}^{(2)}\Big|_{r=r_1} = u_{\theta}^{(1)}\Big|_{r=r_1}, \ u_{\varphi}^{(2)}\Big|_{r=r_1} = u_{\varphi}^{(1)}\Big|_{r=r_1},$$
$$\sigma_{rr}^{(3)}\Big|_{r=r_2} = \sigma_{rr}^{(2)}\Big|_{r=r_2}, \ \sigma_{r\theta}^{(3)}\Big|_{r=r_2} = \sigma_{r\theta}^{(2)}\Big|_{r=r_2}, \ \sigma_{r\varphi}^{(3)}\Big|_{r=r_2} = \sigma_{r\varphi}^{(2)}\Big|_{r=r_2},$$
$$u_r^{(3)}\Big|_{r=r_2} = u_r^{(2)}\Big|_{r=r_2}, \ u_{\theta}^{(3)}\Big|_{r=r_2} = u_{\theta}^{(2)}\Big|_{r=r_2}, \ u_{\varphi}^{(3)}\Big|_{r=r_2} = u_{\varphi}^{(2)}\Big|_{r=r_2}. \tag{6}$$

Also we assume that all quantities related to the core satisfy the boundedness condition at the centre of the core, i.e., at r=0.

This completes formulation of the problem on the natural vibration of the three-layered hollow sphere with middle layer made of FGM.

3. Method of solution

Solution to the system of Eqs. (1)-(3) for the core and outer layer of the sphere can be found in the classical works such as those described in the monographs by Eringen and Suhubi (1975), using the following classical Lame (or Helmholtz) decomposition

$$u_r^{(k)} = \frac{\partial \phi^{(k)}}{\partial r} + \frac{\partial^2 (r\chi^{(k)})}{\partial r^2} - r\nabla^2 \chi^{(k)} \quad , \quad u_\theta^{(k)} = \frac{1}{r}\frac{\partial \phi^{(k)}}{\partial \theta} + \frac{1}{\sin\theta}\frac{\partial \psi^{(k)}}{\partial \varphi} + \frac{1}{r}\frac{\partial^2 (r\chi^{(k)})}{\partial \theta \partial r}$$

$$u_\varphi^{(k)} = \frac{1}{r\sin\theta}\frac{\partial \phi^{(k)}}{\partial \varphi} - \frac{\partial \psi^{(k)}}{\partial \theta} + \frac{1}{\mathrm{r}\sin\theta}\frac{\partial^2 (r\chi^{(k)})}{\partial \varphi \partial r}, \quad k = 1,3. \tag{7}$$

The functions $\phi^{(k)}$, $\chi^{(k)}$ and $\psi^{(k)}$ are solutions to the following equations

$$\nabla^2 \phi^{(k)} - \frac{1}{(c_1^{(k)})^2}\frac{\partial^2 \phi^{(k)}}{\partial t^2} = 0 \, , \nabla^2 \chi^{(k)} - \frac{1}{(c_2^{(k)})^2}\frac{\partial^2 \chi^{(k)}}{\partial t^2} = 0 \, ,$$

$$\nabla^2 \psi^{(k)} - \frac{1}{(c_2^{(k)})^2}\frac{\partial^2 \psi^{(k)}}{\partial t^2} = 0 \, . \tag{8}$$

where

$$c_1^{(k)} = \sqrt{(\lambda^{(k)} + 2\mu^{(k)})/\rho^{(k)}} \, , \; c_2^{(k)} = \sqrt{\mu^{(k)}/\rho^{(k)}} \, ,$$

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \varphi^2} \, . \tag{9}$$

Solutions of the Eqs. (8) and (9) for the case under consideration are found as follows

$$\phi^{(1)}(r,\theta,\varphi,t) = \left[A^{(1)} j_n(\alpha^{(1)} r) + B^{(1)} y_n(\alpha^{(1)} r)\right] P_n^m(\cos\theta)\cos m\varphi e^{i\omega t} \, ,$$
$$\psi^{(1)}(r,\theta,\varphi,t) = \left[C^{(1)} j_n(\beta^{(1)} r) + D^{(1)} y_n(\beta^{(1)} r)\right] P_n^m(\cos\theta)\sin m\varphi e^{i\omega t}$$
$$\chi^{(1)}(r,\theta,\varphi,t) = \left[E^{(1)} j_n(\beta^{(1)} r) + F^{(1)} y_n(\beta^{(1)} r)\right] P_n^m(\cos\theta)\cos m\varphi e^{i\omega t} \, ,$$
$$\phi^{(3)}(r,\theta,\varphi,t) = A^{(3)} j_n(\alpha^{(3)} r) P_n^m(\cos\theta)\cos m\varphi e^{i\omega t}$$
$$\psi^{(3)}(r,\theta,\varphi,t) = C^{(3)} j_n(\beta^{(3)} r) P_n^m(\cos\theta)\sin m\varphi e^{i\omega t}$$
$$\chi^{(3)}(r,\theta,\varphi,t) = E^{(3)} j_n(\beta^{(3)} r) P_n^m(\cos\theta)\cos m\varphi e^{i\omega t}$$
$$\alpha^{(k)} = \omega/c_1^{(k)} \, , \; \beta^{(k)} = \omega/c_2^{(k)} \, . \tag{10}$$

In (10), $j_n(cr)$ and $y_n(cr)$ are spherical Bessel functions of the first and second kind and

$$j_n(cr) = \left(\frac{\pi}{2cr}\right)^{\frac{1}{2}} J_{n+1/2}(cr) \, , \; y_n(cr) = \left(\frac{\pi}{2cr}\right)^{\frac{1}{2}} Y_{n+1/2}(cr) \tag{11}$$

where $J_{n+1/2}(cr)$ and $Y_{n+1/2}(cr)$ are the Bessel functions of the first and the second kind with non-integer order, respectively. Moreover, $P_n^m(\cos\theta)$ in the expression (10) denotes the associated

Legendre functions with m-th order and with n-th harmonic.

Thus, using the relations (11), (10) and (7) we obtain expressions for the displacements and, after substituting these expressions into the Eqs. (3) and (2), we obtain expressions for the components of the stress tensor. For simplification of writing the obtained expressions, we introduce two sets of complete orthogonal functions in $[0, \pi]$ determined as follows

$$X_{nm}(\theta) = P_n^m(\cos\theta) \quad , \quad Y_{nm}(\theta) = n\cot\theta P_n^m(\cos\theta) - \frac{n+m}{\sin\theta} P_{n-1}^m(\cos\theta) . \tag{12}$$

Thus, using the notation (12) we can write the following expressions for the sought values

$$u_r^{(1)} = \frac{1}{r}\left\{A^{(1)}u_{11}^{(1)} + B^{(1)}u_{12}^{(1)} + E^{(1)}u_{31}^{(1)} + F^{(1)}u_{32}^{(1)}\right\} X_{nm}(\theta)\cos m\varphi e^{i\omega t},$$

$$u_r^{(3)} = \frac{1}{r}\left\{A^{(3)}u_{11}^{(3)} + E^{(3)}u_{31}^{(3)}\right\} X_{nm}(\theta)\cos m\varphi e^{i\omega t}$$

$$u_\theta^{(1)} = \frac{1}{r}\left\{\left[A^{(1)}v_{11}^{(1)} + B^{(1)}v_{12}^{(1)} + E^{(1)}v_{31}^{(1)} + F^{(1)}v_{32}^{(1)}\right]Y_{nm}(\theta) + \right.$$

$$\left.(C^{(1)}v_{21}^{(1)} + D^{(1)}v_{22}^{(1)})\frac{m}{\sin\theta}X_{nm}(\theta)\right\}\cos m\varphi e^{i\omega t},$$

$$u_\theta^{(3)} = \frac{1}{r}\left\{\left[A^{(3)}v_{11}^{(3)} + E^{(3)}v_{31}^{(3)}\right]Y_{nm}(\theta) + C^{(3)}v_{21}^{(3)}\frac{m}{\sin\theta}X_{nm}(\theta)\right\}\cos m\varphi e^{i\omega t},$$

$$u_\varphi^{(1)} = \frac{1}{r}\left\{\left[A^{(1)}v_{11}^{(1)} + B^{(1)}v_{12}^{(1)} + E^{(1)}v_{31}^{(1)} + F^{(1)}v_{32}^{(1)}\right]\frac{-m}{\sin\theta}X_{nm}(\theta) + \right.$$

$$\left.(-C^{(1)}v_{21}^{(1)} - D^{(1)}v_{22}^{(1)})Y_{nm}(\theta)\right\}\sin m\varphi e^{i\omega t},$$

$$u_\varphi^{(3)} = \frac{1}{r}\left\{\left[A^{(3)}v_{11}^{(3)} + E^{(3)}v_{31}^{(3)}\right]\frac{-m}{\sin\theta}X_{nm}(\theta) - C^{(3)}v_{21}^{(3)}Y_{nm}(\theta)\right\}\sin m\varphi e^{i\omega t},$$

$$\sigma_{rr}^{(1)} = \frac{2\mu^{(1)}}{r^2}\left[A^{(1)}T_{111}^{(1)} + B^{(1)}T_{112}^{(1)} + E^{(1)}T_{131}^{(1)} + F^{(1)}T_{132}^{(1)}\right]X_{nm}(\theta)\cos m\varphi e^{i\omega t},$$

$$\sigma_{rr}^{(3)} = \frac{2\mu^{(3)}}{r^2}\left[A^{(3)}T_{111}^{(3)} + E^{(3)}T_{131}^{(3)}\right]X_{nm}(\theta)\cos m\varphi e^{i\omega t}$$

$$\sigma_{r\theta}^{(1)} = \frac{2\mu^{(1)}}{r^2}\left\{\left[A^{(1)}T_{411}^{(1)} + B^{(1)}T_{412}^{(1)} + E^{(1)}T_{431}^{(1)} + F^{(1)}T_{432}^{(1)}\right]Y_{nm}(\theta) + \right.$$

$$\left.\left[-C^{(1)}T_{421}^{(1)} - D^{(1)}T_{422}^{(1)}\right]\frac{m}{\sin\theta}X_{nm}(\theta)\right\}\cos m\varphi e^{i\omega t},$$

$$\sigma_{r\theta}^{(3)} = \frac{2\mu^{(3)}}{r^2}\left\{\left[A^{(3)}T_{411}^{(3)} + E^{(3)}T_{431}^{(3)}\right]Y_{nm}(\theta) - C^{(3)}T_{421}^{(3)}\frac{m}{\sin\theta}X_{nm}(\theta)\right\}\cos m\varphi e^{i\omega t},$$

$$\sigma_{r\varphi}^{(1)}=\frac{2\mu^{(1)}}{r^2}\Big\{\Big[A^{(1)}T_{411}^{(1)}+B^{(1)}T_{412}^{(1)}+E^{(1)}T_{431}^{(1)}+F^{(1)}T_{432}^{(1)}\Big]\frac{m}{\sin\theta}X_{nm}(\theta)+$$

$$\Big[-C^{(1)}T_{421}^{(1)}-D^{(1)}T_{422}^{(1)}\Big]Y_{nm}(\theta)\Big\}\cos m\varphi e^{i\omega t},$$

$$\sigma_{r\varphi}^{(3)}=\frac{2\mu^{(3)}}{r^2}\Big\{\Big[A^{(3)}T_{411}^{(3)}+E^{(3)}T_{431}^{(3)}\Big]\frac{m}{\sin\theta}X_{nm}(\theta)-C^{(3)}T_{421}^{(1)}Y_{nm}(\theta)\Big\}\cos m\varphi e^{i\omega t}, \tag{13}$$

where

$$u_{11}^{(1)}=nj_n(\alpha^{(1)}r)-\alpha^{(1)}rj_{n+1}(\alpha^{(1)}r),\ \ u_{12}^{(1)}=ny_n(\alpha^{(1)}r)-\alpha^{(1)}ry_{n+1}(\alpha^{(1)}r),$$

$$u_{31}^{(1)}=n(n+1)j_n(\beta^{(1)}r),\ \ u_{32}^{(1)}=n(n+1)\mathrm{y}_n(\beta^{(1)}r),$$

$$v_{11}^{(1)}=j_n(\alpha^{(1)}r),\ \ v_{12}^{(1)}=y_n(\alpha^{(1)}r), v_{21}^{(1)}=j_n(\beta^{(1)}r),\ \ v_{22}^{(1)}=y_n(\beta^{(1)}r),$$

$$v_{31}^{(1)}=(n+1)j_n(\beta^{(1)}r)-\beta^{(1)}rj_{n+1}(\beta^{(1)}r),$$

$$T_{111}^{(1)}=(n^2-n-\frac{1}{2}(\beta^{(1)})^2r^2)j_n(\alpha^{(1)}r)+2\alpha^{(1)}rj_{n+1}(\alpha^{(1)}r),$$

$$T_{112}^{(1)}=(n^2-n-\frac{1}{2}(\alpha^{(1)})^2r^2)y_n(\alpha^{(1)}r)+2\alpha^{(1)}ry_{n+1}(\alpha^{(1)}r)$$

$$T_{131}^{(1)}=n(n+1)\Big[(n-1)j_n(\beta^{(1)}r)-\beta^{(1)}rj_{n+1}(\beta^{(1)}r)\Big],$$

$$T_{132}^{(1)}=n(n+1)\Big[(n-1)y_n(\beta^{(1)}r)-\beta^{(1)}ry_{n+1}(\beta^{(1)}r)\Big]$$

$$T_{411}^{(1)}=(n-1)j_n(\alpha^{(1)}r)-\alpha^{(1)}rj_{n+1}(\alpha^{(1)}r),\ \ T_{412}^{(1)}=(n-1)y_n(\alpha^{(1)}r)-\alpha^{(1)}ry_{n+1}(\alpha^{(1)}r),$$

$$T_{421}^{(1)}=\frac{1}{2}r\Big[(n-1)j_n(\beta^{(1)}r)-\beta^{(1)}rj_{n+1}(\beta^{(1)}r)\Big],$$

$$T_{422}^{(1)}=\frac{1}{2}r\Big[(n-1)y_n(\beta^{(1)}r)-\beta^{(1)}ry_{n+1}(\beta^{(1)}r)\Big],$$

$$T_{431}^{(1)}=(n^2-1-\frac{1}{2}(\beta^{(1)})^2r^2)j_n(\beta^{(1)}r)+\beta^{(1)}rj_{n+1}(\beta^{(1)}r),$$

$$T_{432}^{(1)}=(n^2-1-\frac{1}{2}(\beta^{(1)})^2r^2)y_n(\beta^{(1)}r)+\beta^{(1)}ry_{n+1}(\beta^{(1)}r),$$

$$u_{11}^{(3)}=nj_n(\alpha^{(3)}r)-\alpha^{(3)}rj_{n+1}(\alpha^{(3)}r),\ \ u_{31}^{(3)}=n(n+1)j_n(\beta^{(3)}r),\quad ,$$

$$v_{21}^{(3)}=j_n(\beta^{(3)}r),\ \ v_{31}^{(3)}=(n+1)j_n(\beta^{(3)}r)-\beta^{(3)}rj_{n+1}(\beta^{(3)}r),$$

$$T_{111}^{(3)}=(n^2-n-\frac{1}{2}(\beta^{(3)})^2r^2)j_n(\alpha^{(3)}r)+2\alpha^{(3)}rj_{n+1}(\alpha^{(3)}r),$$

$$T_{131}^{(3)} = n(n+1)\left[(n-1)j_n(\beta^{(3)}r) - \beta^{(3)}rj_{n+1}(\beta^{(3)}r)\right],$$

$$T_{411}^{(3)} = (n-1)j_n(\alpha^{(3)}r) - \alpha^{(3)}rj_{n+1}(\alpha^{(3)}r), \quad T_{421}^{(3)} = \frac{1}{2}r\left[(n-1)j_n(\beta^{(3)}r) - \beta^{(3)}rj_{n+1}(\beta^{(3)}r)\right],$$

$$T_{431}^{(3)} = (n^2 - 1 - \frac{1}{2}(\beta^{(3)})^2 r^2)j_n(\beta^{(3)}r) + \beta^{(3)}rj_{n+1}(\beta^{(3)}r). \tag{14}$$

In (14) $\alpha^{(k)} = \omega/c_1^{(k)}$, $\beta^{(k)} = \omega/c_2^{(k)}$, $c_1^{(k)} = \sqrt{(\lambda^{(k)} + 2\mu^{(k)})/\rho^{(k)}}$, $c_2^{(2)} = \sqrt{\mu^{(2)}/\rho^{(2)}}$ (k=1,3). Note that in (13) the expressions for the stresses which enter the boundary and contact conditions have been written.

Thus, substituting the expressions (13) into the boundary (5) and contact conditions (6), according to (12), we obtain two uncoupled systems of algebraic equations. The first of these systems contains the unknown constants $A^{(1)}$, $B^{(1)}$, $E^{(1)}$, $F^{(1)}$, $A^{(3)}$ and $E^{(3)}$, and the second contains the unknown constants $C^{(1)}$, $D^{(1)}$ and $C^{(3)}$. According to the expressions obtained for the stresses $\sigma_{r\theta}^{(k)}$ and $\sigma_{r\varphi}^{(k)}$ and which are given in (13), the aforementioned equations obtained from the boundary and contact conditions with respect to these stresses coincide with each other. Additionally, according to the expressions in (13) obtained for the displacements $u_\theta^{(k)}$ and $u_\varphi^{(k)}$, the equations obtained from the contact conditions with respect to these displacements also coincide with each other. Consequently, for determination of the foregoing unknown constants, i.e., for obtaining the two sets of systems of algebraic equations it is enough to use only the contact and boundary conditions written with respect to the stresses $\sigma_{rr}^{(k)}$ and $\sigma_{r\theta}^{(k)}$ (or $\sigma_{r\varphi}^{(k)}$), and displacements $u_r^{(k)}$ and $u_\theta^{(k)}$ (or $u_\varphi^{(k)}$).

Now we consider the solution of the Eqs. (1), (2) and (3) for the middle layer, the material of which is FG. It is evident that the solution procedure considered above does not apply directly for solution to these equations and therefore, according to Akbarov (2006, 2015), we act as follows. First, the middle layer with thickness $h_m=r_1-r_2$ is divided into M number of sublayers with thickness $h'_m=r_1-r_2/M$ and it is assumed that within the scope of each p-th ($1 \le p \le M$) sublayer, the material is homogeneous and the Lame constants and density of this material are determined as follows

$$\mu^{(2)p} = \mu^{(2)}(r)\Big|_{r=r_1+(p-1/2)h'_m}, \quad \lambda^{(2)p} = \lambda^{(2)}(r)\Big|_{r=r_1+(p-1/2)h'_m},$$

$$\rho^{(2)p} = \rho^{(2)}(r)\Big|_{r=r_1+(p-1/2)h'_m}. \tag{15}$$

In this way, the Eqs. (1), (2) and (3) for the middle layer of the sphere, which are equations with variable coefficients, are reduced to the series of the same equations with constant coefficients determined according to the relations in (15). Then the foregoing solution procedure is applied for determination of the solution to these equations as a result of which we obtain the expressions given in (13) and (14) with the obvious corresponding changes. Moreover we assume that between the sublayers, i.e., on the interface surfaces of the layers, perfect contact conditions are satisfied. We rewrite the contact conditions in (6) taking into consideration the aforementioned sublayers.

$$\sigma_{rr}^{(3)}\Big|_{r=r_2} = \sigma_{rr}^{(2)1}\Big|_{r=r_2}, \quad \sigma_{r\theta}^{(3)}\Big|_{r=r_2} = \sigma_{r\theta}^{(2)1}\Big|_{r=r_2}, \quad \sigma_{r\varphi}^{(3)}\Big|_{r=r_2} = \sigma_{r\varphi}^{(2)1}\Big|_{r=r_2},$$

$$u_r^{(3)}\Big|_{r=r_2} = u_r^{(2)1}\Big|_{r=r_2}, \quad u_\theta^{(3)}\Big|_{r=r_2} = u_\theta^{(2)1}\Big|_{r=r_2}, \quad u_\varphi^{(3)}\Big|_{r=r_2} = u_\varphi^{(2)1}\Big|_{r=r_2},$$

$$\sigma_{rr}^{(2)1}\Big|_{r=r_2+h'_m} = \sigma_{rr}^{(2)2}\Big|_{r=r_2+h'_m}, \quad \sigma_{r\theta}^{(2)1}\Big|_{r=r_2+h'_m} = \sigma_{r\theta}^{(2)1}\Big|_{r=r_2+h'_m}, \quad \sigma_{r\varphi}^{(2)1}\Big|_{r=r_2+h'_m} = \sigma_{r\varphi}^{(2)2}\Big|_{r=r_2+h'},$$

$$u_r^{(2)1}\Big|_{r=r_2+h'_m} = u_r^{(2)2}\Big|_{r=r_2+h'_m}, \quad u_\theta^{(2)1}\Big|_{r=r_2+h'_m} = u_\theta^{(2)2}\Big|_{r=r_2+h'_m}, \quad u_\varphi^{(2)1}\Big|_{r=r_2+h'_m} = u_\varphi^{(2)2}\Big|_{r=r_2+h'_m},$$

....

$$\sigma_{rr}^{(2)p-1}\Big|_{r=r_2+ph'_m} = \sigma_{rr}^{(2)p}\Big|_{r=r_2+ph'_m}, \quad \sigma_{r\theta}^{(2)p-1}\Big|_{r=r_2+ph'_m} = \sigma_{r\theta}^{(2)p}\Big|_{r=r_2+ph'_m},$$

$$\sigma_{r\varphi}^{(2)p-1}\Big|_{r=r_2+ph'_m} = \sigma_{r\varphi}^{(2)p}\Big|_{r=r_2+ph'}, \quad u_r^{(2)p-1}\Big|_{r=r_2+ph'_m} = u_r^{(2)p}\Big|_{r=r_2+ph'_m},$$

$$u_\theta^{(2)p-1}\Big|_{r=r_2+ph'_m} = u_\theta^{(2)p}\Big|_{r=r_2+ph'_m}, \quad u_\varphi^{(2)p-1}\Big|_{r=r_2+ph'_m} = u_\varphi^{(2)p}\Big|_{r=r_2+ph'_m},$$

....

$$\sigma_{rr}^{(2)M}\Big|_{r=r_1} = \sigma_{rr}^{(1)}\Big|_{r=r_1}, \quad \sigma_{r\theta}^{(2)M}\Big|_{r=r_1} = \sigma_{r\theta}^{(1)}\Big|_{r=r_1}, \quad \sigma_{r\varphi}^{(2)M}\Big|_{r=r_1} = \sigma_{r\varphi}^{(1)}\Big|_{r=r_1},$$

$$u_r^{(2)M}\Big|_{r=r_1} = u_r^{(1)}\Big|_{r=r_1}, \quad u_\theta^{(2)M}\Big|_{r=r_1} = u_\theta^{(1)}\Big|_{r=r_1}, \quad u_\varphi^{(2)M}\Big|_{r=r_1} = u_\varphi^{(1)}\Big|_{r=r_1}, \tag{16}$$

where

$$u_r^{(2)p} = \frac{1}{r}\left\{A^{(2)p}u_{11}^{(2)p} + B^{(2)p}u_{12}^{(2)p} + E^{(2)p}u_{31}^{(2)} + F^{(2)}u_{32}^{(2)p}\right\} X_{nm}(\theta)\cos m\varphi e^{i\omega t},$$

$$u_\theta^{(2)p} = \frac{1}{r}\left\{\left[A^{(2)p}v_{11}^{(2)p} + B^{(2)p}v_{12}^{(2)p} + E^{(2)p}v_{31}^{(2)p} + F^{(2)p}v_{32}^{(2)p}\right]Y_{nm}(\theta) + \right.$$

$$\left.(C^{(2)p}v_{21}^{(2)p} + D^{(2)p}v_{22}^{(2)p})\frac{m}{\sin\theta}X_{nm}(\theta)\right\}\cos m\varphi e^{i\omega t},$$

$$u_\varphi^{(2)p} = \frac{1}{r}\left\{\left[A^{(2)p}v_{11}^{(2)p} + B^{(2)p}v_{12}^{(2)p} + E^{(2)p}v_{31}^{(2)p} + F^{(2)p}v_{32}^{(2)p}\right]\frac{-m}{\sin\theta}X_{nm}(\theta) + \right.$$

$$\left.(-C^{(2)p}v_{21}^{(2)p} - D^{(2)p}v_{22}^{(2)p})Y_{nm}(\theta)\right\}\sin m\varphi e^{i\omega t},$$

$$\sigma_{rr}^{(2)p} = \frac{2\mu^{(2)p}}{r^2}\left[A^{(2)p}T_{111}^{(2)p} + B^{(2)p}T_{112}^{(2)p} + E^{(2)p}T_{131}^{(2)p} + F^{(2)p}\,T_{132}^{(2)p}\right]X_{nm}(\theta)\cos m\varphi e^{i\omega t}$$

$$\sigma_{r\theta}^{(2)p} = \frac{2\mu^{(2)p}}{r^2}\left\{\left[A^{(2)p}T_{411}^{(2)p} + B^{(2)p}T_{412}^{(2)p} + E^{(2)p}T_{431}^{(2)p} + F^{(2)p}\,T_{432}^{(2)p}\right]Y_{nm}(\theta) + \right.$$

$$\left[-C^{(2)p}T_{421}^{(2)p}-D^{(2)p}T_{422}^{(2)p}\right]\frac{m}{\sin\theta}X_{nm}(\theta)\Big\}\cos m\varphi e^{i\omega t},$$

$$\sigma_{r\varphi}^{(2)p}=\frac{2\mu^{(2)p}}{r^2}\Big\{\left[A^{(2)p}T_{411}^{(2)p}+B^{(2)p}T_{412}^{(2)p}+E^{(2)p}T_{431}^{(2)p}+F^{(2)p}T_{432}^{(2)p}\right]\frac{m}{\sin\theta}X_{nm}(\theta)+$$

$$\left[-C^{(2)p}T_{421}^{(2)p}-D^{(2)p}T_{422}^{(2)p}\right]Y_{nm}(\theta)\Big\}\cos m\varphi e^{i\omega t}. \tag{17}$$

In (17) the following notation is used.

$$u_{11}^{(2)}=nj_n(\alpha^{(2)p}r)-\alpha^{(2)p}rj_{n+1}(\alpha^{(2)p}r),\ \ u_{12}^{(2)p}=ny_n(\alpha^{(2)p}r)-\alpha^{(2)p}ry_{n+1}(\alpha^{(2)p}r),$$

$$u_{31}^{(2)p}=n(n+1)j_n(\beta^{(2)p}r),\ \ u_{32}^{(2)p}=n(n+1)\,\mathrm{y}_n(\beta^{(2)p}r),$$

$$v_{11}^{(2)}=j_n(\alpha^{(2)p}r),\ \ v_{12}^{(2)}=y_n(\alpha^{(2)p}r),\ \ v_{21}^{(2)p}=j_n(\beta^{(2)p}r),\ \ v_{22}^{(2)p}=y_n(\beta^{(2)p}r),$$

$$v_{31}^{(2)}=(n+1)j_n(\beta^{(2)p}r)-\beta^{(2)p}rj_{n+1}(\beta^{(2)p}r),$$

$$v_{32}^{(2)p}=(n+1)y_n(\beta^{(2)p}r)-\beta^{(2)p}ry_{n+1}(\beta^{(2)p}r),$$

$$T_{111}^{(2)p}=(n^2-n-\frac{1}{2}(\beta^{(2)p})^2r^2)j_n(\alpha^{(2)p}r)+2\alpha^{(2)p}rj_{n+1}(\alpha^{(2)p}r),$$

$$T_{112}^{(2)p}=(n^2-n-\frac{1}{2}(\beta^{(2)p})^2r^2)y_n(\alpha^{(2)p}r)+2\alpha^{(2)p}ry_{n+1}(\alpha^{(2)p}r)$$

$$T_{131}^{(2)p}=n(n+1)\left[(n-1)j_n(\beta^{(2)p}r)-\beta^{(2)p}rj_{n+1}(\beta^{(2)p}r)\right],$$

$$T_{132}^{(2)p}=n(n+1)\left[(n-1)y_n(\beta^{(2)p}r)-\beta^{(2)p}ry_{n+1}(\beta^{(2)p}r)\right]$$

$$T_{411}^{(2)p}=(n-1)j_n(\alpha^{(2)p}r)-\alpha^{(2)p}rj_{n+1}(\alpha^{(2)p}r),$$

$$T_{412}^{(2)p}=(n-1)y_n(\alpha^{(2)p}r)-\alpha^{(2)p}ry_{n+1}(\alpha^{(2)p}r),$$

$$T_{421}^{(2)p}=\frac{1}{2}r\left[(n-1)j_n(\beta^{(2)p}r)-\beta^{(2)p}rj_{n+1}(\beta^{(2)p}r)\right],$$

$$T_{422}^{(2)p}=\frac{1}{2}r\left[(n-1)y_n(\beta^{(2)p}r)-\beta^{(2)p}ry_{n+1}(\beta^{(2)p}r)\right],$$

$$T_{431}^{(2)p}=(n^2-1-\frac{1}{2}(\beta^{(2)p})^2r^2)j_n(\beta^{(2)p}r)+\beta^{(2)p}rj_{n+1}(\beta^{(2)p}r),$$

$$T_{432}^{(2)p}=(n^2-1-\frac{1}{2}(\beta^{(2)p})^2r^2)y_n(\beta^{(2)p}r)+\beta^{(2)p}ry_{n+1}(\beta^{(2)p}r),$$

$$\alpha^{(k)p}=\omega\big/c_1^{(k)p},\quad \beta^{(k)p}=\omega\big/c_2^{(k)p},$$

$$c_1^{(k)p} = \sqrt{(\lambda^{(k)p} + 2\mu^{(k)p})/\rho^{(k)p}}\ ,\ \ c_2^{(2)p} = \sqrt{\mu^{(2)p}/\rho^{(2)p}}\ . \tag{18}$$

Thus, using (13) and (18) we obtain two uncoupled systems of algebraic equations from the boundary (5) and contact (16) conditions. The first (second) system contains the unknowns $A^{(1)}$, $B^{(1)}$, $E^{(1)}$, $A^{(1)}$, $A^{(2)1}$, $B^{(2)1}$, $E^{(2)1}$, $F^{(2)1}$,..., $A^{(2)M}$, $B^{(2)M}$, $E^{(2)M}$, $F^{(2)M}$, $A^{(3)}$ and $E^{(3)}$ ($C^{(1)}$, $D^{(1)}$, $C^{(2)1}$, $D^{(2)1}$,..., $C^{(3)M}$, $D^{(2)M}$ and $C^{(3)}$). Equating to zero the determinant of the coefficient matrix of the first (second) group of equations, we obtain the following equations for determination of the frequency of the natural vibration

$$\det\left(\gamma_{q_1q_2}\right) = 0 \qquad q_1; q_2 = 1,2,...,4M+6 \ \ \text{(for the spheroidal vibration)}, \tag{19}$$

$$\det\left(\delta_{p_1p_2}\right) = 0\,,\ \ p_1; p_2 = 1,2,...,2M+3 \quad \text{(for the torsional vibration)}. \tag{20}$$

Note that the meaning of the spheroidal and torsional vibrations is well-known and can be found in many monographs related to elastodynamics (see, for instance, the monograph by Eringen and Suhubi (1975)). Therefore, here we do not consider determination of these vibration modes, although this determination follows from the foregoing discussions and from the expressions of the displacements given in (13) and (17). Moreover, we note that the explicit expressions of the components γ_{q1q2} in (19) and of the components δ_{p1p2} in (20) can be easily determined from the expressions (13), (14), (17) and (18). The number M in the Eqs. (19) and (20) i.e., the number of sublayers, (the summation of which gives the middle layer of the sphere) will be determined in the numerical solution procedure of these equations from the convergence requirement of the numerical results.

This completes the consideration of the solution method

4. Numerical results and discussions

Results, which will be discussed below, are obtained from the numerical solution to the Eqs. (19) and (20) which is made by employing the well-know “bi-section” method. Before the solution to the procedure, the functions $\lambda^{(2)}(r)$, $\mu^{(2)}(r)$ and $\rho^{(2)}(r)$ which characterize the functionally graded property of the middle-layer material of the sphere are selected. In the present investigations these functions are selected as follows

$$E^{(2)}(r) = E_0^{(2)}(1+\eta_1(a_1r+b_1)^{n_1})^{m_1}\ \ ,\ \ \nu^{(2)}(r) = \nu_0^{(2)}(1+\eta_2(a_2r+b_2)^{n_2})^{m_2}\,,$$

$$\rho^{(2)}(r) = \rho_0^{(2)}(1+\eta_3(a_3r+b_3)^{n_3})^{m_3}\,,\ \ \lambda^{(2)}(r) = \frac{E^{(2)}(r)\nu^{(2)}(r)}{(1+\nu^{(2)}(r))(1-2\nu^{(2)}(r))}\,,$$

$$\mu^{(2)}(r) = \frac{E^{(2)}(r)}{2\left(1+\nu^{(2)}(r)\right)}\,. \tag{21}$$

Here a_k, b_k, n_k, m_k and η_k (k=1,2,3) are real numbers, for which the meaning of each notation is obvious.

Table 1 Natural frequencies $\Omega = \omega a\sqrt{\rho^{(1)} / \mu^{(1)}}$ of the torsional vibration mode obtained in the present paper (upper number) and in the paper by Sato and Usami (1962a) (lower number)

n	$\Omega\left(=\omega a\sqrt{\rho^{(1)}/\mu^{(1)}}\right)$				
	1	**2**	**3**	**4**	**5**
0	4.274 -	7.596 -	10.812 -	13.995 -	17.162 -
1	5.763 5.763	9.095 9.095	12.322 12.322	15.514 15.514	18.689 -
2	2.501 2.501	7.136 7.136	10.514 10.514	13.771 13.771	16.983 16.983
3	3.864 3.865	8.444 8.444	11.881 11.881	15.175 15.175	18.412 18.412
4	5.094 5.095	9.712 9.712	13.210 13.210	16.544 16.544	19.809 19.809
5	6.265 6.266	10.950 10.950	14.510 14.510	17.885 17.885	21.180 21.180

The main aim of the present numerical investigations is to determine how the functionally graded properties of the middle-layer material of the three-layered solid sphere acts on the natural frequencies of the spheroidal and torsional vibration of this solid sphere. Before consideration of the main numerical results, for testing of the used calculation algorithm and PC programs, which are composed by the authors and realized in MATLAB, we consider the numerical results obtained for the case where the material of the sphere is homogeneous, i.e., we assume that $\eta_1=\eta_2=\eta_3=0$ in (21) and $E^{(2)}/E^{(1)}=\rho^{(2)}/\rho^{(1)}=1$, $E^{(3)}/E^{(1)}=\rho^{(3)}/\rho^{(1)}=1$, $\nu^{(1)}=\nu^{(2)}=\nu^{(3)}=0.25$. Introduce the dimensionless frequency

$$\Omega = \omega a\sqrt{\rho^{(1)} / \mu^{(1)}} \tag{22}$$

and consider the results given in Tables 1 and 2 which show the first five values of Ω obtained for the first six harmonic numbers (n=0,1,...,5) for the torsional and spheroidal vibration modes, respectively.

Note that in these tables, the corresponding results obtained in the papers by Sato and Usami (1962a) (Table 1) and Sato *et al*. (1962) (Table 2) are also presented (lower numbers). The tables show that the results obtained by employing the present algorithm coincide almost completely with the corresponding ones obtained in the papers by Sato and Usami (1962a) and Sato *et al*. (1962). Note that in these tables the sign "-" means that the corresponding value of the natural frequency Ω is not considered in the corresponding references. Moreover, note that in the Table 2 for the zeroth harmonic, i.e., under spheroidal vibration mode for n=0 it is obtained three type results. This situation is explained with the following consideration.

By direct verification it is established that in the spheroidal vibration mode under n=0 the frequency Eq. (19) for the homogeneous solid sphere becomes as follows

$$T_{111}^{(3)}\Big|_{r=a} \times T_{431}^{(3)}\Big|_{r=a} = 0\,, \tag{23}$$

Table 2 Natural frequencies $\Omega = \omega a \sqrt{\rho^{(1)} / \mu^{(1)}}$ of the spheroidal vibration mode obtained in the present paper (upper number) and in the paper by Sato *et al*. (1962) (lower number)

n	$\Omega\left(=\omega a\sqrt{\rho^{(1)}/\mu^{(1)}}\right)$				
	1	**2**	**3**	**4**	**5**
0*	2.458 -	4.439 4.440	5.959 -	9.210 -	10.493 10.494
0**	4.439	10.493	16.073	21.579	27.058
0***	2.458	5.959	9.210	12.406	15.580
1	3.424 3.424	6.771 6.771	7.745 7.744	10.695 10.695	13.019 -
2	2.639 2.640	4.865 4.865	8.329 8.329	9.780 9.780	12.157 12.157
3	3.916 3.916	6.454 6.454	9.704 9.705	11.747 11.748	13.638 13.639
4	5.009 5.009	8.061 8.062	11.039 11.039	13.553 13.553	15.183 15.201
5	6.032 6.033	9.635 9.636	12.368 12.368	15.179 15.179	16.817 16.818

from which follows following two equations

$$T_{111}^{(3)}\Big|_{r=a} = 0 , \tag{24}$$

and

$$T_{431}^{(3)}\Big|_{r=a} = 0 . \tag{25}$$

Under direct solution of the Eq. (23) we obtain the roots given Table 2 in the line indicated by the 0*. However, by the solution to the Eqs. (24) and (25) we obtain the roots given in the Table 2 in the lines indicated by the 0** and 0*** respectively. However, in the paper by Sato *et al*. (1962), as well as, in the monograph by Eringen and Shubi (1975), it is given only the roots shown in the line $0***$ of the Table 2. We believe that this moment was missed in the studies by Sato *et al*. (1962). The other results which are considered in the Tables 1 and 2 almost completely consider with the corresponding ones obtained in the papers by Sato and Usami (1962a) and Sato *et al*. (1962). This confirms the validity of the algorithm and programs used in the present investigation.

Now we attempt to explain the effect of an increase (or a decrease) in the values of the modulus of elasticity under fixed values of the materials' densities, as well as the effect of an increase (or a decrease) in the values of the materials' densities under fixed values of the modulus of elasticity in the inward radial direction on the values of the natural frequencies. Consider the case where r_1/a=0.9, r_2/a=0.4 and $\nu^{(1)}=\nu^{(2)}=\nu^{(3)}=0.3$ where $\nu^{(k)}$ is Poisson's ratio of the k-th material. We analyze the results given in Tables 3 and 4 which illustrate the influence of the change of the modulus of elasticity and the change of the densities in the aforementioned case, respectively, on the values of Ω (22) for the torsional and spheroidal vibration modes. These results are obtained for the first six

Table 3 The influence of the change of the modulus of elasticity in the inward radial direction on the values of the natural frequencies $\Omega = \omega a\sqrt{\rho^{(1)} / \mu^{(1)}}$ obtained for the three-layered piecewise homogeneous solid sphere

$\frac{E^{(2)}}{E^{(1)}}$	$\frac{E^{(3)}}{E^{(1)}}$	n	**Torsional vibration**			**Spheroidal vibration**		
			1	**2**	**3**	**1**	**2**	**3**
0.5	0.3	0	2.8835	4.9135	7.0545	1.7025	3.5075	3.9195
		1	3.9535	5.9115	8.0745	2.6785	4.7325	5.6415
		2	2.0075	4.9975	6.9235	1.8215	3.8955	5.8745
		3	3.1405	6.0345	7.9525	2.7865	5.1655	6.9645
		4	4.1785	7.0505	9.0155	3.6425	6.3635	8.1345
		5	5.1785	8.0375	10.108	4.4465	7.4915	9.3515
0.7	0.5	0	3.4865	5.9845	8.5895	2.0715	4.1235	4.7405
		1	4.7365	7.2055	9.7945	3.0495	5.7155	6.6945
		2	2.2345	5.9335	8.4165	2.1575	4.3785	6.9605
		3	3.4795	7.1015	9.6285	3.2775	5.8355	8.1215
		4	4.6145	8.2375	10.850	4.2565	7.2665	9.3555
		5	5.7035	9.3385	12.071	5.1705	8.6275	10.678
0.9	0.7	0	3.9715	6.8565	9.8205	2.3695	4.6485	5.4345
		1	5.3725	8.2495	11.185	3.3665	6.5215	7.5935
		2	2.4175	6.6995	9.6135	2.4465	4.7725	7.8785
		3	3.7475	7.9845	10.963	3.6985	6.3375	9.1335
		4	4.9505	9.2285	12.308	4.7835	7.9045	10.439
		5	5.7915	9.4215	11.8145	6.0995	10.4345	13.6365
1	1	0	4.2745	7.5965	10.812	2.4585	4.9955	5.9595
		1	5.7635	9.0955	12.322	3.5295	7.1015	8.0775
		2	2.5015	7.1365	10.514	2.6465	5.0065	8.5275
		3	3.8645	8.4445	11.881	3.9375	6.6085	9.8825
		4	5.0945	9.7125	13.210	5.0455	8.2145	11.224
		5	6.2655	10.950	14.510	6.0835	9.7775	12.574
3	5	0	7.0985	13.440	19.142	3.3595	8.7125	11.044
		1	9.6575	15.712	21.634	5.6615	12.337	13.781
		2	3.4765	11.801	17.820	4.7495	7.6265	14.807
		3	5.2075	13.778	19.780	6.8195	9.4285	16.891
		4	6.6775	15.698	21.665	8.4935	11.257	18.608
		5	8.0085	17.592	23.530	10.024	13.028	20.239
5	7	0	8,8645	16,168	22,818	4.9325	10.935	13.171
		1	11,898	18,946	25,817	7.0545	14.986	16.952
		2	4,0085	14,505	21,524	5.9675	9.1185	17.543
		3	5,9055	16,926	23,942	8.6435	10.942	19.987
		4	7,4555	19,258	26,270	10.738	12.721	22.221
		5	8,8155	21,519	28,554	12.560	14.416	24.326
7	9	0	10.203	18.180	25.565	6.0755	12.708	14.511
		1	13.586	21.306	28.961	8.1385	16.601	19.265
		2	4.3885	16.512	24.222	6.9515	10.203	19.367
		3	6.3885	19.221	26.973	10.103	11.924	22.241
		4	7.9755	21.796	29.628	12.506	13.585	24.896
		5	9.3395	24.241	32.233	14.395	15.305	27.246

Table 4 The influence of the change of the material densities in the inward radial direction on the values of the natural frequencies $\Omega = \omega a\sqrt{\rho^{(1)} / \mu^{(1)}}$ obtained for the three-layered piecewise homogeneous solid sphere

$\frac{\rho^{(2)}}{\rho^{(1)}}$	$\frac{\rho^{(3)}}{\rho^{(1)}}$	n	**Torsional vibration**			**Spheroidal vibration**		
			1	**2**	**3**	**1**	**2**	**3**
0.5	0.3	0	6.4345	11.744	16.016	3.9145	6.2375	8.6875
		1	7.9535	13.798	18.962	4.4715	10.317	10.854
		2	2.7715	9.5855	15.328	3.3375	5.9175	12.060
		3	4.2295	11.295	16.853	4.9045	7.5785	13.533
		4	5.5165	13.027	18.461	6.1825	9.2955	15.050
		5	6.7225	14.752	20.138	7.3295	11.006	16.626
0.7	0.5	0	5.4685	9.6205	13.341	3.2245	5.6545	7.3205
		1	6.8985	11.510	15.479	4.0455	8.8525	9.4025
		2	2.6775	8.3305	12.962	3.0095	5.5165	10.365
		3	4.1105	9.7885	14.334	4.4475	7.1705	11.718
		4	5.3875	11.250	15.734	5.6515	8.8635	13.081
		5	6.5915	12.702	17.173	6.7575	10.538	14.479
0.9	0.7	0	4.8635	8.4435	11.788	2.8155	5.1995	6.4905
		1	6.2195	10.145	13.528	3.7265	7.8775	8.4915
		2	2.5595	7.5285	11.505	2.7585	5.1825	9.2765
		3	3.9495	8.8385	12.763	4.0915	6.7975	10.554
		4	5.1995	10.141	14.021	5.2285	8.4335	11.835
		5	6.3845	11.430	15.300	6.2885	10.037	13.148
1	1	0	4.2745	7.5965	10.812	2.4585	4.9955	5.9595
		1	5.7635	9.0955	12.322	3.5295	7.1015	8.0775
		2	2.5015	7.1365	10.514	2.6465	5.0065	8.5275
		3	3.8645	8.4445	11.881	3.9375	6.6085	9.8825
		4	5.0945	9.7125	13.210	5.0455	8.2145	11.224
		5	6.2655	10.950	14.510	6.0835	9.7775	12.574
3	5	0	2.0555	4.2975	5.7945	1.2135	3.1335	3.2605
		1	3.0495	5.0605	6.7775	2.1395	3.9085	4.7735
		2	1.6765	4.0605	5.7595	1.6495	3.2555	4.7725
		3	2.6085	5.0485	6.5155	2.4945	4.3555	5.7425
		4	3.4585	5.9855	7.3625	3.2525	5.3845	6.7695
		5	4.2725	6.8635	8.2955	3.9855	6.3365	7.8025
5	7	0	1.7725	3.4905	4.8075	1.0185	2.4875	2.6715
		1	2.5545	4.1175	5.5985	1.7515	3.2245	3.8305
		2	1.3145	3.3245	4.7185	1.3085	2.6395	3.9495
		3	2.0435	4.0605	5.3655	1.9755	3.4995	4.7305
		4	2.7065	4.7555	6.0655	2.5805	4.2885	5.5315
		5	3.3405	5.4165	6.8035	3.1705	5.0125	6.3205
7	9	0	1.5735	3.0125	4.1935	0.8955	2.1245	2.3175
		1	2.2335	3.5625	4.8615	1.5175	2.8085	3.2855
		2	1.1105	2.8715	4.0975	1.1175	2.2755	3.4415
		3	1.7255	3.4775	4.6635	1.6865	3.0025	4.1075
		4	2.2845	4.0525	5.2625	2.2045	3.6655	4.7825
		5	2.8185	4.6025	5.8765	2.7105	4.2725	5.4395

Table 5 Convergence of the numerical results with respect to the number M of the sublayers obtained for the natural frequencies $\Omega = \omega a\sqrt{\rho^{(1)} / \mu^{(1)}}$ obtained for the three-layered solid sphere with middle layer made of FGM

n	M	**Torsional vibration**			**Spheroidal vibration**		
		1	2	3	1	2	3
4	1	7.8615	21.2615	29.0515	12.1745	13.4075	24.6505
	3	7.3165	19.8685	28.7605	11.7955	12.4615	23.2405
	5	7.2465	19.7885	28.6555	11.7725	12.3155	23.1385
	7	7.2255	19.7665	28.6255	11.7625	12.2735	23.1065
	9	7.2175	19.7575	28.6135	11.7575	12.2565	23.0925
	11	7.2125	19.7535	28.6075	11.7555	12.2475	23.0845
	13	7.2105	19.7505	28.6035	11.7545	12.2425	23.0805
	15	7.2085	19.7485	28.6015	11.7535	12.2395	23.0775
	17	7.2075	19.7475	28.5995	11.7525	12.2375	23.0765
	19	7.2065	19.7465	28.5985	11.7525	12.2365	23.0745
	21	7.2065	19.7465	28.5985	11.7515	12.2345	23.0735
5	1	9.2245	23.6535	31.4925	14.0325	15.0885	26.7905
	3	8.6275	21.8155	30.9485	13.4155	14.0165	25.3325
	5	8.5425	21.7095	30.8325	13.4185	13.8205	25.1575
	7	8.5175	21.6805	30.7985	13.4195	13.7585	25.1115
	9	8.5065	21.6695	30.7845	13.4195	13.7325	25.0915
	11	8.5005	21.6635	30.7765	13.4195	13.7185	25.0815
	13	8.4975	21.6595	30.7725	13.4195	13.7105	25.0765
	15	8.4955	21.6575	30.7705	13.4195	13.7055	25.0725
	17	8.4945	21.6565	30.7685	13.4195	13.7025	25.0705
	19	8.4935	21.6555	30.7675	13.4195	13.7005	25.0685
	21	8.4925	21.6545	30.7665	13.4195	13.6985	25.0675

harmonics and the first three roots are presented for each harmonic.

Thus, it follows from these results that an increase (a decrease) in the values of the modulus of elasticity in the inward radial direction causes an increase (a decrease) in the values of the natural frequency Ω. Also, these results show that an increase (a decrease) in the values of the densities in the inward radial direction causes a decrease (an increase) in the values of Ω. This conclusion gives some orientation for estimation and explanation of the numerical results which are obtained for the case where the material of the middle layer of the sphere is FG. It should be noted that the results obtained in the cases where the modulus of elasticity and densities are changed simultaneously can also be explained and estimated according to the foregoing conclusions.

Thus, we consider the numerical results related to the case where the material of the middle layer of the sphere is FG. Assume that, as in the foregoing case, r_1/a=0.9, r_2/a=0.4 and $\nu^{(1)}=\nu^{(2)}=\nu^{(3)}$=0.3. First, we illustrate some fragments of the results which show convergence with respect to the number M of the sublayers into which the middle layer is divided. These fragments are given in Table 5 for the torsional and spherical vibration modes in the case where $a_1=-14/a$, b_1=14.6, η_1=1, η_2=0, η_3=0, $E^{(3)}/E^{(1)}$=12 and $n_1=m_1$=1. Consequently, we assume that the density and

Poisson's ratio of the material of the middle layer are constants, but the modulus of elasticity changes linearly through the thickness, so that, the ratio $E^{(2)}(r)/E_0^{(2)}$ increases from $(E^{(2)}(r)/E_0^{(2)})\Big|_{r=0.9a}=3$ until $(E^{(2)}(r)/E_0^{(2)})\Big|_{r=0.4a}=10$ in the inward radial direction. Analyses of the results given in Table 5 show the high effectiveness of the approach used in the convergence sense with respect to the sublayers' number M. Taking these and many other results, which are not given here, into consideration allows us to conclude that it is enough to take M=21 to obtain numerical results with very high accuracy. Taking this conclusion into account we assume that M=21 under obtaining all numerical results which will be discussed below.

Now we consider the numerical results related namely to the case where the middle layer material is FG and assume that $\nu^{(1)}=\nu^{(2)}=\nu^{(3)}=0.3$, i.e., assume that $\eta_2=0$. Moreover, as above, assume that $r_1/a=0.9$, $r_2/a=0.4$, $E^{(3)}/E^{(1)}=9$ and suppose that $m_1=m_3=1$. We determine the constants a_1 and b_1 (a_3 and b_3) from the way that the ratio $E^{(2)}(r)/E^{(1)}$ (the ratio $\rho^{2)}(r)/\rho^{(1)}$) increases from $(E^{(2)}(r)/E^{(1)})\Big|_{r=0.9a}=3$ (from $(\rho^{(2)}(r)/\rho^{(1)})\Big|_{r=0.9a}=3$) until $(E^{(2)}(r)/E^{(1)})\Big|_{r=0.4a}=7$ (until $(\rho^{(2)}(r)/\rho^{(1)})\Big|_{r=0.4a}=7$). So that, for determination of the constants a_k and b_k (k=1,3) we obtain the following expressions

$$E^{(2)}(r)=E_0^{(2)}(1+(a_1r+b_1)^{n_1}),\quad \rho^{(2)}(r)=\rho_0^{(2)}(1+(a_3r+b_3)^{n_3}),$$

$$a_k=2\Big[(2)^{1/n_k}-(6)^{1/n_k}\Big]a,\quad b_k=\Big[1.8\times(6)^{1/n_k}-0.8\times(2)^{1/n_k}\Big]a\ . \tag{26}$$

Table 6 The influence of an increase of the modulus of elasticity in the inward radial direction under constant material densities on the values of the natural frequencies obtained for the three-layered solid sphere with middle layer made of FGM

n	n_1	**Torsional vibration**			**Spheroidal vibration**		
		1	**2**	**3**	**1**	**2**	**3**
	0	6.7865	14.114	20.759	1.3845	9.0525	11.966
	0.2	9.8645	17.676	24.970	5.8655	12.229	14.067
	0.5	9.4305	17.079	24.237	5.5485	11.672	13.532
0	1	8.8975	16.430	23.402	5.0725	11.037	13.001
	1.5	8.5315	16.035	22.872	4.6815	10.627	12.738
	2	8.2685	15.770	22.512	4.3645	10.346	12.600
	3	7.9205	15.433	22.062	3.8895	9.9925	12.473
	0	9.8235	16.330	22.944	5.7165	12.672	14.007
	0.2	13.110	20.726	28.263	8.7835	17.598	20.941
	0.5	12.519	20.027	27.400	8.0695	16.919	19.625
1	1	11.835	19.246	26.410	7.3165	16.086	17.979
	1.5	11.395	18.749	25.779	6.8795	15.566	16.902
	2	11.097	18.404	25.349	6.6085	15.217	16.185
	3	10.725	17.949	24.803	6.0715	14.004	14.827

Table 6 Continued

2	0	3.4895	11.985	18.379	5.0145	7.8885	15.861
	0.2	4.1955	15.884	23.547	6.7285	9.6555	18.794
	0.5	3.9895	15.121	22.725	6.4535	9.1075	18.112
	1	3.7955	14.262	21.785	6.1215	8.6365	17.374
	1.5	3.6945	13.726	21.176	5.8995	8.4155	16.942
	2	3.6375	13.371	20.748	5.7475	8.2965	16.670
	3	3.5775	12.939	20.184	5.5545	8.1755	16.363
3	0	5.2115	13.893	20.203	6.9755	9.5125	17.786
	0.2	6.1125	18.423	26.187	9.7385	11.327	21.407
	0.5	5.8225	17.472	25.220	9.2735	10.722	20.507
	1	5.5575	16.426	24.102	8.7025	10.221	19.611
	1.5	5.4285	15.789	23.369	8.3275	9.9965	19.124
	2	5.3585	15.374	22.855	8.0725	9.8785	18.838
	3	5.2905	14.881	22.184	7.7555	9.7615	18.536
4	0	6.6795	15.758	21.949	8.5645	11.280	19.169
	0.2	7.6515	20.816	28.717	12.009	12.980	23.816
	0.5	7.3135	19.672	27.590	11.390	12.336	22.688
	1	7.0135	18.444	26.276	10.629	11.816	21.623
	1.5	6.8745	17.716	25.417	10.129	11.604	21.066
	2	6.8035	17.251	24.818	9.7925	11.504	20.742
	3	6.7385	16.712	24.046	9.3855	11.417	20.392
5	0	8.0095	17.619	23.704	10.051	13.035	20.559
	0.2	8.9905	23.079	31.183	13.816	14.668	26.016
	0.5	8.6275	21.749	29.878	13.090	13.984	24.720
	1	8.3155	20.360	28.359	12.186	13.458	23.512
	1.5	8.1775	19.557	27.374	11.600	13.263	22.882
	2	8.1095	19.056	26.695	11.217	13.179	22.507
	3	8.0515	18.493	25.833	10.773	13.115	22.078

Under the foregoing selection of functions which enter into relation (21) the changing (i.e., the increasing) character of the modulus of elasticity (or of the material density) depends only on the constant n_k. For illustration of the dependence of this character on the constant n_k, the graphs of the function $E^{(2)}(r)/E^{(1)}$ constructed for various values of this constant are given in Fig. 2. Note that, according to the foregoing assumptions, the same graphs are also obtained for the function $\rho^{(2)}(r)/\rho^{(1)}$. It follows from these graphs that the values of the integrals

$$S_E = \int_{0.4}^{0.9} E^{(2)}(r) / E^{(1)} dr \quad \text{and} \quad S_\rho = \int_{0.4}^{0.9} \rho^{(2)}(r) / \rho^{(1)} dr . \tag{27}$$

increase with a decrease of the constants $n_1(\geq 0)$ and $n_3(\geq 0)$, respectively. For instance, in the cases where $n_1(n_3)$=0.2, 0.5, 1.0, 1.5, 2.0 and 3.0, it is obtained that $S_E(S\rho)$=3.1672, 2.8333, 2.4999, 2.1666 and 1.9999, respectively. Consequently, the influence of the change character of the FGM in the inward radial direction can also be estimated through the values S_E and $S\rho$.

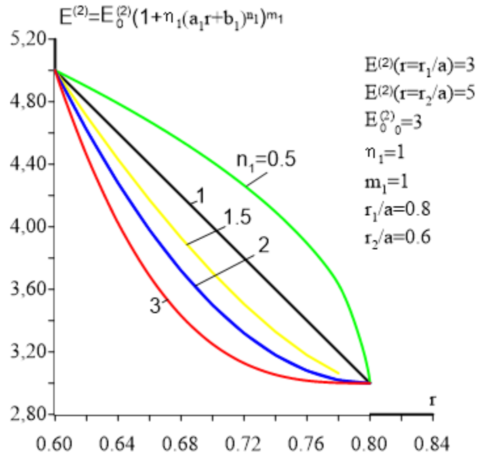


Fig. 2 Distribution of modulus of elasticity in the radial direction

Thus, we consider the results given in Tables 6 and 7 which show the influence of the change of the modulus of elasticity and of the material densities, respectively, under various n_1 ($=n_3$) on the values of the dimensionless natural frequency Ω (22) obtained for the torsional and spheroidal vibration modes. These results are presented for the first six harmonics and for the first three roots and, under obtaining the results given in Table 6 (in Table 7), it is assumed that $\eta_3=0$ ($\eta_1=0$). It follows from these tables that an increase in the values of the modulus of elasticity in the inward

Table 7 The influence of an increase of the material densities in the inward radial direction under constant modulus of elasticity on the values of the natural frequencies $\Omega = \omega a\sqrt{\rho^{(1)} / \mu^{(1)}}$ obtained for the three-layered solid sphere with middle layer made of FGM

n	n_3	Torsional vibration			Spheroidal vibration		
		1	**2**	**3**	**1**	**2**	**3**
	0	1.4445	3.6285	5.0015	0.9485	2.6995	3.0365
	0.2	1.6055	3.0905	4.2915	0.9085	2.2545	2.3635
	0.5	1.6345	3.1655	4.3955	0.9215	2.4015	2.4115
0	1	1.6505	3.2285	4.5015	0.9325	2.4575	2.5645
	1.5	1.6455	3.2595	4.5735	0.9375	2.4845	2.6675
	2	1.6335	3.2835	4.6355	0.9405	2.5045	2.7385
	3	1.6065	3.3245	4.7425	0.9425	2.5315	2.8265
	0	2.2925	4.5445	5.5505	1.8845	3.4075	4.1715
	0.2	2.2795	3.6695	4.9685	1.5905	2.9095	3.4145
	0.5	2.3225	3.7805	5.0775	1.6675	3.0145	3.5625
1	1	2.3505	3.8855	5.1865	1.7415	3.1145	3.7285
	1.5	2.3535	3.9485	5.2595	1.7815	3.1705	3.8315
	2	2.3475	4.0005	5.3195	1.8045	3.2065	3.8965
	3	2.3315	4.0905	5.4175	1.8275	3.2525	3.9715

Table 7 Continued

2	0	1.6725	3.2015	5.2585	1.6015	2.9595	4.1635
	0.2	1.2185	2.9415	4.2205	1.1855	2.4065	3.5695
	0.5	1.3365	3.0095	4.3545	1.2635	2.5445	3.7105
	1	1.4605	3.0635	4.4895	1.3505	2.6775	3.8515
	1.5	1.5325	3.0835	4.5775	1.4055	2.7505	3.9285
	2	1.5755	3.0905	4.6505	1.4435	2.7945	3.9755
	3	1.6205	3.0965	4.7765	1.4905	2.8405	4.0295
3	0	2.6055	4.1405	5.8985	2.4625	3.9725	5.0605
	0.2	1.8985	3.5835	4.7955	1.8005	3.1735	4.2755
	0.5	2.0885	3.6895	4.9425	1.9315	3.3475	4.4635
	1	2.2875	3.7815	5.0975	2.0785	3.5175	4.6555
	1.5	2.4025	3.8255	5.2025	2.1715	3.6125	4.7615
	2	2.4705	3.8515	5.2905	2.2355	3.6715	4.8245
	3	2.5395	3.8875	5.4405	2.3115	3.7395	4.8975
4	0	3.4565	5.0945	6.5495	3.2365	4.9505	5.9905
	0.2	2.5185	4.2015	5.4035	2.3665	3.8655	4.9855
	0.5	2.7755	4.3555	5.5615	2.5535	4.0695	5.2115
	1	3.0455	4.4975	5.7315	2.7625	4.2725	5.4435
	1.5	3.2015	4.5725	5.8485	2.8935	4.3895	5.5775
	2	3.2935	4.6215	5.9465	2.9795	4.4655	5.6615
	3	3.3835	4.6955	6.1125	3.0785	4.5585	5.7615
5	0	4.2725	6.0545	7.2305	3.9785	5.9265	6.8755
	0.2	3.1125	4.7985	6.0345	2.9235	4.5035	5.6715
	0.5	3.4345	5.0065	6.2085	3.1685	4.7425	5.9215
	1	3.7745	5.2065	6.3935	3.4395	4.9865	6.1815
	1.5	3.9715	5.3185	6.5195	3.6055	5.1325	6.3375
	2	4.0855	5.3965	6.6255	3.7115	5.2295	6.4405
	3	4.1945	5.5125	6.8025	3.8295	5.3585	6.5705

radial direction causes an increase in the values of the natural frequency Ω. According to Table 7, in general, a similar conclusion also occurs for the influence of the increase of the material density in the inward direction on the natural frequencies, i.e., the increase of the material density causes a decrease in the values of the natural frequency. However, this conclusion is violated in the zeroth harmonic for the first roots in the torsional mode.

As follows from comparison of the results obtained for the various values of the constants n_1 and n_3, the foregoing effects of the FGM of the middle layer on the natural frequencies become more considerable with the parameters S_E and S_ρ in (27) (or with decreasing of the constants $n_1(\geq 0)$ and $n_3(\geq 0)$). Note that the parameters S_E and S_ρ can be used as global characteristics of the FGM determined through the relations in (21).

Now we consider the numerical results which are obtained in the case where the modulus of elasticity and the density of the FGM of the middle layer increase simultaneously in the inward

Table 8 The influence of a simultaneous increase of the material densities and modulus of elasticity in the inward radial direction on the values of the natural frequencies $\Omega = \omega a\sqrt{\rho^{(1)}/\mu^{(1)}}$ obtained for the three-layered solid sphere with middle layer made of FGM

n	$n_1(=n_3)$	**Torsional vibration**			**Spheroidal vibration**		
		1	**2**	**3**	**1**	**2**	**3**
0	0	2,4955	9,2515	12,450	0.9885	6.9365	7.1785
	0.2	4,7185	9,3095	12,622	2.7285	6.7635	7.3465
	0.5	4,6295	9,1395	12,470	2.7435	6.9975	7.2335
	1	4,4365	8,9625	12,392	2.7045	7.0515	7.2145
	1.5	4,2435	8,9165	12,396	2.6285	6.9285	7.3095
	2	4,0725	8,9395	12,409	2.5405	6.8585	7.3425
	3	3,8095	9,0485	12,412	2.3725	6.8115	7.3345
1	0	5,5325	10,103	14,773	3.8165	7.6885	10.140
	0.2	6,8005	10,721	14,193	4.3335	8.6255	10.298
	0.5	6,6555	10,524	14,136	4.3175	8.6085	10.330
	1	6,4125	10,366	14,169	4.2585	8.5105	10.289
	1.5	6,2185	10,349	14,228	4.1915	8.4305	10.194
	2	6,0785	10,385	14,269	4.1325	8.3785	10.096
	3	5,9035	10,482	14,308	4.0465	8.3095	9.9595
2	0	3,0845	8,3415	11,274	3.8015	6.2835	9.2835
	0.2	3,1805	8,7115	12,152	3.6315	6.2995	10.228
	0.5	3,2125	8,5305	11,961	3.7765	6.2895	10.117
	1	3,2245	8,2865	11,839	3.9155	6.2425	9.9465
	1.5	3,2155	8,1435	11,834	3.9815	6.2025	9.8255
	2	3,2015	8,0695	11,856	4.0085	6.1795	9.7365
	3	3,1755	8,0255	11,892	4.0155	6.1685	9.6105
3	0	4,6715	10,459	13,079	5.3295	8.1475	11.757
	0.2	4,8625	10,375	13,653	5.3395	8.2755	11.989
	0.5	4,8865	10,166	13,509	5.5475	8.2455	11.829
	1	4,8745	9,9495	13,445	5.7285	8.1765	11.637
	1.5	4,7765	8,3665	13,010	5.7905	8.1335	11.524
	2	4,7555	8,3675	13,074	5.7965	8.1175	11.456
	3	4,7275	8,3735	13,107	5.7485	8.1215	11.392
4	0	6,0865	12,008	15,262	6.6155	9.9535	14.065
	0.2	6,1615	9,6635	13,988	6.7795	10.220	13.674
	0.5	6,1135	9,7425	14,186	7.0205	10.156	13.499
	1	6,3165	11,445	15,051	7.2035	10.051	13.333
	1.5	6,2655	11,430	15,083	7.2035	10.051	13.333
	2	6,2245	11,478	15,097	7.2045	9.9785	13.260
	3	6,1725	11,611	15,080	7.0985	9.9795	13.314
5	0	7,4145	13,371	17,164	7.8675	11.727	15.712
	0.2	7,7515	13,167	16,417	8.1325	12.098	15.196
	0.5	7,7305	12,957	16,475	8.3905	11.993	15.027
	1	7,6545	12,845	16,571	8.5525	11.853	14.903
	1.5	7,5875	12,884	16,631	8.5435	11.790	14.895
	2	7,5395	12,967	16,657	8.4755	11.771	14.945
	3	7,4845	13,129	16,668	8.3245	11.773	15.105

radial direction. Assume that the increase law of the modulus of elasticity remains as indicated above, i.e., the ratio $E^{(2)}(r)/E^{(1)}$ increases from $(E^{(2)}(r)/E^{(1)})\big|_{r=0.9a}=3$ until $(E^{(2)}(r)/E^{(1)})\big|_{r=0.4a}=7$. However, with respect to the material density we assume that the ratio $\rho^{(2)}(r)/\rho^{(1)}$ increases from $(\rho^{(2)}(r)/\rho^{(1)})\big|_{r=0.9a}=2$ until $(\rho^{(2)}(r)/\rho^{(1)})\big|_{r=0.4a}=5$ and $\rho^{(3)}/\rho^{(1)}=7$. The values of the other parameters remain as above. Thus, consider the values of the natural frequencies obtained for this case as given in Table 8. As in the foregoing tables, these results are obtained for the first six harmonics and for each harmonic the first three roots are found in the torsional and spheroidal vibration modes. Analysis of the data given in Table 8 shows that for the case under consideration, the natural frequencies increase with simultaneous increases in the parameters S_E and S_ρ in (24) (i.e., with simultaneous decreases in the constants $n_1(\geq 0)$ and $n_3(\geq 0)$). However, the magnitude of the increases is significantly less than that the magnitudes obtained in Table 6. It is obvious that this situation can be explained with the foregoing results, according to which, an increase of the material density in the inward radial direction causes a decrease, but the corresponding increase of the modulus of elasticity causes an increase in the values of the natural frequencies, and the simultaneous existence of the noted "decrease" and "increase" determines the character of the results given in Table 8.

This completes the consideration and analysis of the numerical results.

5. Conclusions

Thus, in the present paper, the natural vibration of the three-layered solid sphere with middle layer made of FGM is investigated by employing the exact three-dimensional equations and relations of elastodynamics in spherical coordinates. It is assumed that perfect contact conditions take place on the interface surfaces between the constituents of the sphere. The corresponding eigenvalue problem is solved by employing the discrete analytical method, according to which, the layer made of FGM is divided into a certain number of sublayers and within each sublayer the material is taken as homogeneous. Between the sublayers, complete contact conditions are satisfied and analytical solutions to the field equations are obtained for each layer separately. It is assumed that the material properties of the FGM change in the radial direction according to the power law. Numerical results on the dimensionless natural frequency Ω (22) obtained for certain concrete cases of this law are presented for the first six harmonics and for the first five or three roots of each harmonic for the torsional and spheroidal vibration modes. Analysis of these results allows us to make the following main concrete conclusions:

- An increase (a decrease) of the modulus of elasticity of the FGM in the inward radial direction causes an increase (a decrease) in the values of the natural frequency Ω;
- A decrease (an increase) of the density of the FGM in the inward radial direction, in general, causes an increase (a decrease) in the values of the natural frequency Ω;
- The influence of the character of the aforementioned change (increase or decrease) law on the values of Ω can be determined through the parameters S_E and S_ρ (27);
- An increase in the values of S_E, i.e. a decrease in the values of the constant n_1 which enter the relation (26), also, causes an increase in the values of Ω;
- A decrease in the values of S_ρ, i.e. an increase in the values of the constant n_3 which enter the relation (26), also, in general, causes an increase in the values of Ω;

- The character of the results obtained in the cases where the modulus of elasticity and the material density of the FGM are changed simultaneously, can be explained with the use of the foregoing conclusions.

The foregoing results and conclusions can also be recommended for engineers for increasing or decreasing the natural frequencies of inhomogeneous solid spheres. At the same time, the foregoing results allow for control of the natural frequencies of many-layered solid spheres through selection of the change law of the FGM.

Taking into consideration the significance of the obtained results for possible application, we believe that related investigations must be continued and these will be made in further works by the authors.

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