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Exact mathematical solution for free vibration of thick laminated plates

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Abstract. In this paper, the modified form of shear deformation plate theories is proposed. First, the displacement field geometry of classical and the first order shear deformation theories are compared with each other. Using this comparison shows that there is a kinematic relation among independent variables of the first order shear deformation theory. So, the modified forms of rotation functions in shear deformation theories are proposed. Governing equations for rectangular and circular thick laminated plates, having been analyzed numerically so far, are solved by method of separation of variables. Natural frequencies and mode shapes of the plate are determined. The results of the present method are compared with those of previously published papers with good agreement obtained. Efficiency, simplicity and excellent results of this method are extensible to a wide range of similar problems. Accurate solution for governing equations of thick composite plates has been made possible for the first time.

Keywords: classical theory; first order shear deformation; modified form; laminated plate; vibration; accurate solution

1. Introduction

Study of different types of continuous systems such as beams, plates and shells which are the mechanical models for several industrial devices, are subject of different scientific researches, due to their extensive applications. The behavior of mechanical structures reviewed during long periods and various methods have been developed for their analysis. For example, threads as the simplest form of one dimensional continuous system and membrane which have two dimensions are formulated simply. These equations have exact analytical solution and effect of initial conditions can be evaluated applying Fourier series. This type of work was conducted by Euler (1766) for the first time. Euler solved free vibrations in rectangular, triangular and circular membranes, but these solutions were not extendable to thick structures. After determining the behavior of beams, under pure bending, Bernoulli (1789) who was Euler's assistant, developed his analogy for the beams which only have bending rigidity. A real study of the free vibration of plates has been explained by Chladni (1802) in his book. In his experiment, by using powder distributed uniformly on the plate, he identified the place of accumulation of powder in which, after vibration,

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no displacement occurred on them. Also, in his method, similar frequency in each mode shape, the pattern of which was identifiable by help of powder, was determined. After different unsuccessful efforts being performed to obtain differential equation of the plates, finally Navier (1819) proposed differential equation of plate under a uniformly distributed transverse load. He offered the results of his discoveries about the plates, which were performed completely independently of the last efforts. The complete theory for thin plates was offered by Kirchhoff (1850, 1876), the German scientist; known as the classical plate theory (CPT). The most important effect of Kirchhoff's work in comparison to Navier's was explaining boundary shear forces of plate which were introduced as a twisting moments in plate boundary. In his work, different boundary conditions were expressed as a function of plate deformations and their derivatives in relation to independent variables of plate surface. Also, he understood that in the analysis of plates with large deflection, nonlinear terms are not negligible. Developing virtual deformation for analysis of plates and offering the frequency equation for plates are among his other works (Szilard 2004). In a model offered by Kirchhoff, deformation in all of the points on the plate was organized according to deformation in mid-plane. The most important assumption to achieve this purpose is that, the straight lines which initially are perpendicular to mid-plane, after deformation still remain straight and perpendicular. This issue will cause the shear stresses along thickness to be ignored. This assumption leads, in thick plates and high modes of vibration, an error to occur. For this reason, the classical plate theory forecasts the natural frequency more than real value and deflection of the plate less than real value. Therefore, Reissner (1944) proposed a shear deformation theory in which shear stresses have been obtained from three dimensional theory of elasticity. Mindlin (1951) proposed a new displacement field by considering the Taylor expansion in displacement of the plate about mid-plane. In his offered displacement field, relative in-plane deformations of plate with respect to mid-plane were proportional to rotation functions. Regarding independency of rotation functions in the method that later on was known as the first order shear deformation theory (FSDT), this theory had five degrees of freedom. Also, the number of its governing equations was five. Of course, before that people like Basset (1890), Hildebrand *et al.* (1949), Hencky (1947) had used this method (Nyfeh and Frank 2004). But ultimately, this theory was recorded as Mindlin theory. Shear stresses in Mindlin theory were constant across the plate thickness and their values were equal to those of mid-plane, while in real case, shear stress is a second order function of thickness, and its value at the bottom and top of its surface is zero. Therefore, this theory demanded applying a shear correction coefficient to modify. Yang *et al.* (YNS) (1966) generalized the theory of shear deformation for homogenous isotropic plates to the anisotropic laminated plates, which include shear deformation and rotary inertia. Regarding the importance of shear correction coefficient and since value of this coefficient in composite plates is a function of material property, type of layering and geometrical properties of the plate, obtaining this coefficient is a very complex task (Nyfeh and Frank 2004). Therefore, to calculate a logical and reasonable approximation, the higher order shear deformation theories (HSDT) which include more number of terms in Taylor expansion were developed. These theories were especially useful for the analysis of thick composite plates. The more number of terms in Taylor expansion is increased, accuracy of the problem would be more and as a result it will cause the more complex equations to be developed. Some of these researches in this context were conducted by different researches. Lo *et al.* (1977, 1977) extracted a theory in which transverse and in-plane deformations are second and third order function of plate thickness, respectively. Green and Naghdi (1981) proposed a dynamic theory for layered orthotropic plates. Some higher order shear deformation theories include displacement fields which are violating the equilibrium equations. Therefore, the forecast strain energy by them does not have enough accuracy. Beside, since in these theories shear stress is not zero in the top and bottom surfaces, they need coefficient to correct the shear which is unknown. Therefore, using such theories computationally is expensive. Some theories were proposed that did not have recent difficulties. Reddy (1984) proposed a third order shear deformation theory (TSDT) in which, shear stress is the second order function of thickness and in top and bottom surfaces, its value is zero. This theory includes the same five dependent variables in FSDT, but in-plane deformations are determined in such a way that boundary conditions of shear stresses are satisfied. So, there is no need to apply shear correction coefficient in TSDT. Zenkour (2004) proposed the displacement field of Reddy's theory as a new form. In his theory, shear stress is in the form of sinusoidal function of thickness which is equal to zero in top and bottom surfaces. Therefore, there is no need to apply shear correction coefficient. Most of recent theories include some assumptions applying which gives a displacement filed. The most general case of mechanical structures behavior analysis is three dimensional theory of elasticity in which there is no simplifying assumption. Srinivas *et al.* (1970) indicated that, in vibration analysis of thick laminated plates, ignoring shear deformation across plate thickness, leads to large errors. Therefore, Seide (1975), Srinivas and Rao (1970), Pagano (1970), Pagano and Hatfield (1972) offered three dimensional theories. These theories could cause a good approximation, in strain and stress fields of composite plates. Despite this, the most important issue in these theories was that increasing the number of layers causes more complex equations of motion. For this reason, lamination theories are commonly used where layers of composite plates are bonded to each other perfectly (Lekhnitski 1968, Vinson and Sierakowski 1986, Whitney 1987, Qatu 2004). Several other researches performed on plate theories, recently. Among them, Zenkour (2009) using a sinusoidal shear deformation theory and considering the interaction of plate foundation, showed a thermo-elastic bending analysis for FGM plates. Thai and Choi (2014) used the introduced refined theory by Shimpi (2002) to develop a finite element analysis for thick laminated plates. Their work results had good accuracy and were applicable for thin to very thick plates without shear locking, because the applied deformation field had similarity with CPT and at the same time, a third order function of thickness was used.

Despite all valuable researches and efforts which, during long times, are devoted to studying the behavior of plates with more efficient methods, still there is a gap for presence of an integrated, efficient, simple and comprehensive theory which is applicable for different problems, and also can give reasonable and acceptable results. The old methods, except simple and especial problems, result in numerical solutions. Numerical solution is time consuming and a hard task. Regarding vast application of plates, it seems having a theory which can lead to exact mathematical solution for a more extensive spectrum of problems can help the researchers.

This paper offered the modified form of shear deformation theories. In fact this theory will be introduced by approving a kinematic relation among independent variables of the plate, which describes its deformation and are equal to five in FSDT and TSDT. The rotation functions are shown to be proportional with the plate slope curves in both horizontal directions. Applying this relationship in governing equations, the main equation which indicates the transverse displacement of the plate decouples completely and will be solved by the method of separation of variables. The new theory is applied for linear vibration analysis of thick rectangular and circular laminated plates. The mode shapes for both cases are obtained and their corresponding natural frequencies are evaluated using boundary conditions analytically. Comparison of the results with those of published papers shows good agreements.

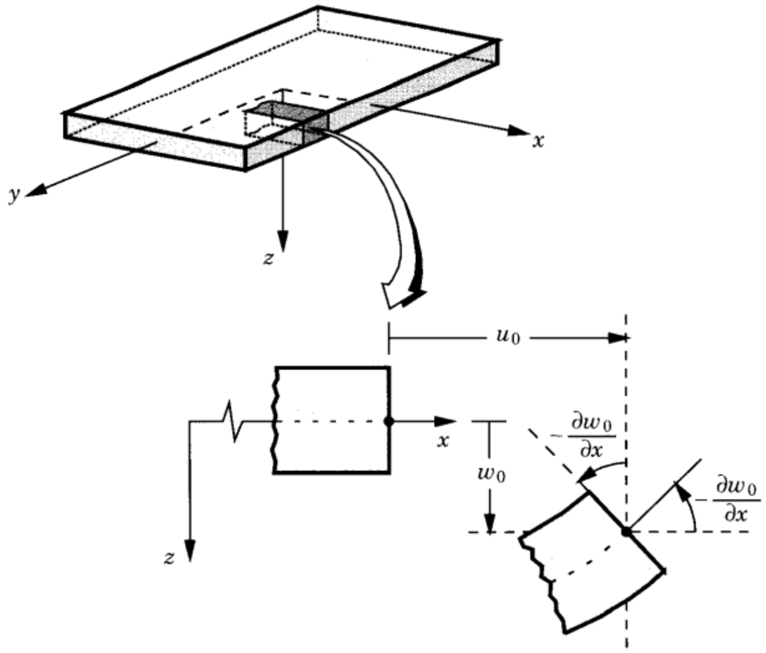


Fig. 1 Undeformed and deformed geometries of an edge of a rectangular plate in the CPT (Reddy 2004)

2. Displacement field of modified shear deformation theory

Consider a rectangular thin plate, shown in Fig. 1. Kirchhoff's assumptions for thin plates are: (1)-Displacement of mid-plane in comparison with plate thickness is insignificant, (2)-Straight lines that initially are perpendicular to mid-plane, after deformation, still remain straight and perpendicular to mid-plane, (3)-After pure bending is applied, the mid-plane of plate remains without strain and (4)-The stress component σ_{zz} which is perpendicular to mid-plane, is negligible in comparison with other stress components (Nyfeh and Frank 2004). From the second assumption, it is resulted that shear deformations are ignored. Under these assumptions, displacement field which describe the plate deformations is written as below (Washizu 1975)

$$
\begin{aligned}
u(x,y,z,t) &= u_0(x,y,t) - z\frac{\partial w_0(x,y,t)}{\partial x} \\
v(x,y,z,t) &= v_0(x,y,t) - z\frac{\partial w_0(x,y,t)}{\partial y} \\
w(x,y,z,t) &= w_0(x,y,t)
\end{aligned}
\tag{1}
$$

where u_0, v_0, and w_0 are the displacements of the mid-plane along x-, y-, and z- axes, respectively and also u, v, and w are the displacements of any arbitrary point of the plate along the above mentioned directions, respectively.

But in thick plates, the second Kirchhoff's assumption is not valid. In the moderately thick

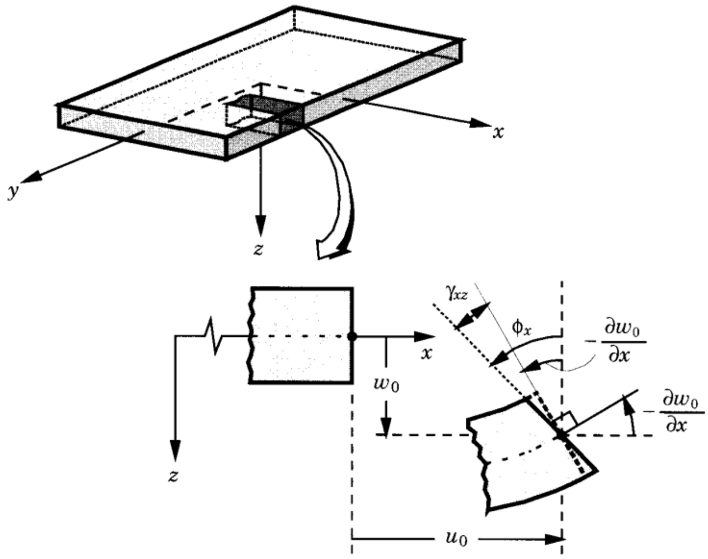


Fig. 2 Undeformed and deformed geometries of an edge of a rectangular plate in the FSDT (Reddy 2004)

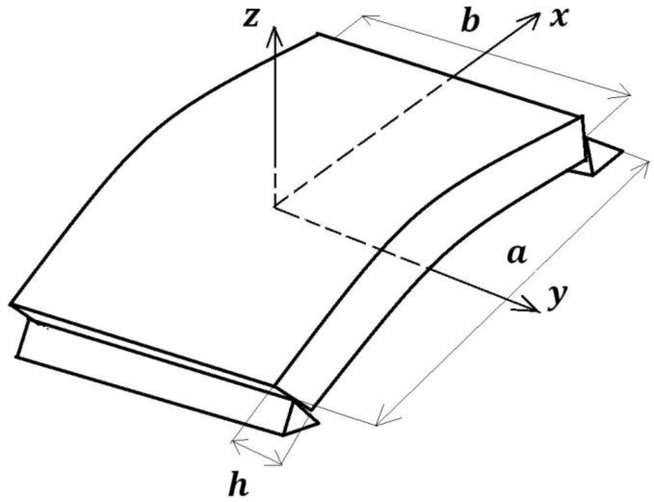


Fig. 3 A thick rectangular plate which vibrates in (1,0) mode

plates, displacement field of plate will be achieved by expansion about mid-plane. Fig. 2 shows a moderately thick plate deformation field, before and after deformation. Displacement field of a moderately thick plate FSDT is as follow (Reddy 2004)

$$
\begin{aligned}
u(x,y,z,t) &= u_0(x,y,t) + z\varphi_x(x,y,t) \\
v(x,y,z,t) &= v_0(x,y,t) + z\varphi_y(x,y,t) \\
w(x,y,z,t) &= w_0(x,y,t)
\end{aligned}
\tag{2}
$$

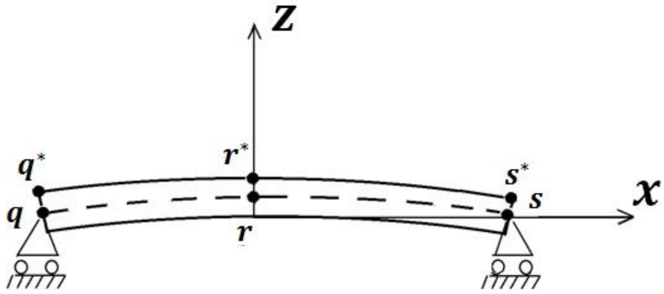
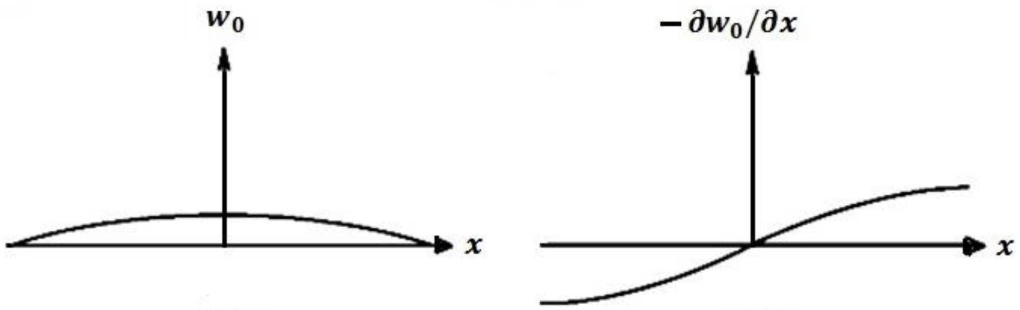


Fig. 4 Relative displacement of top surface points with respect to their corresponding pints in mid-plane in a rectangular plate



(a) (b)

Fig. 5 The deflection (a) and slope curve (b) of a rectangular plate which vibrating in (1, 0) mode

New variables φ_x and φ_y are the rotations of mid-plane, about y and x axis, respectively. In FSDT, these functions are considered as two fully independent functions of spatial variables and time. Governing equations of CPT are three partial differential equations and applying FSDT leads to five PDEs. In both theories, except in special cases, governing equations are very complicated and numerical approaches are needed to solve them. For this reason, approximately all the published related articles deal with numerical analysis. It is shown in the present paper that rotation functions in FSDT are not independent of the deflection of the plate.

Consider a rectangular plate with the dimensions of a, b and h in x, y and z directions, respectively, as is depicted in Fig. 3. This plate is vibrating in (1, 0) mode and its side view, perpendicular to plane of x-z, is shown in Fig. 4.

As can be observed in Fig. 4, bending the plate about y axis leads relative motion of different points of the top surface of the plate to have the following relationship with the mid-plane displacements

$$u_{q^*} < u_q \qquad\qquad u_{r^*} = u_r \qquad\qquad u_{s^*} > u_s \tag{3}$$

Considering simply supported boundary conditions on x axis, we assume the mode shape of mid-plane according to Fig. 4 is: $w_0=\cos(\pi x/a)$. So, the transverse displacement and slope curve of the plate are as depicted in Fig. 5.

Consider the first relation of CPT displacement field in Eq. (1). From Fig. 5 and the first relation of Eq. (1), it can be understood that relative displacements of different points on the top surface of the plate (Eq. (3)) are satisfied with CPT displacement field

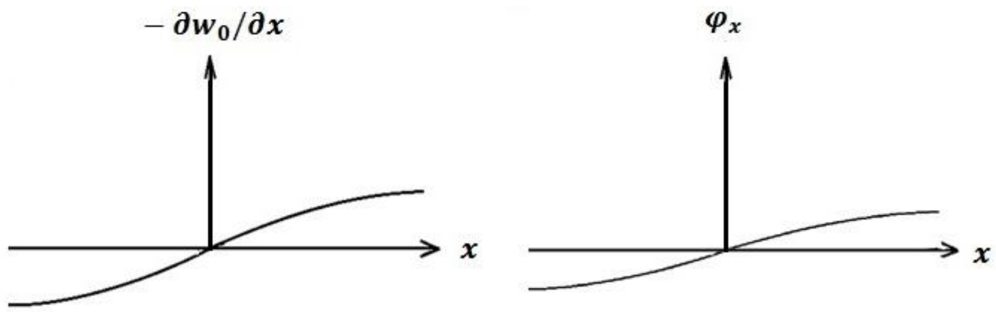
$$\begin{aligned} u\left(q^*\right) &= u_0\left(q\right) + \frac{h}{2}\left(-\frac{\partial w_0}{\partial x}\left(\frac{-a}{2}, y, t\right)\right) \\ u\left(r^*\right) &= u_0\left(r\right) + \frac{h}{2}\left(-\frac{\partial w_0}{\partial x}\left(0, y, t\right)\right) \\ u\left(s^*\right) &= u_0\left(s\right) + \frac{h}{2}\left(-\frac{\partial w_0}{\partial x}\left(\frac{a}{2}, y, t\right)\right) \end{aligned} \tag{4}$$

For a thick plate, the geometry constraint which is expressed in Eq. (3) and Fig. 4 is satisfied, too. That is, the relative deformation of points on top surface of the plate, with respect to corresponding mid-plane displacement must be satisfied in FSDT displacement. So

$$\begin{aligned} u\left(q^*\right) &= u_0\left(q\right) + \frac{h}{2}\varphi_x\left(\frac{-a}{2}, y, t\right) \\ u\left(r^*\right) &= u_0\left(r\right) + \frac{h}{2}\varphi_x\left(0, y, t\right) \\ u\left(s^*\right) &= u_0\left(s\right) + \frac{h}{2}\varphi_x\left(\frac{a}{2}, y, t\right) \end{aligned} \tag{5}$$

Regarding Eqs. (4)-(5) and considering that Eq. (5) must satisfy geometrical constraint which is expressed by Eq. (3) and Fig. 4, we conclude rotation φ_x cannot be independent of the slope curve of the plate. Therefore, the behavior of φ_x is predicted as shown in Fig. 6. As a result, from two unknown constants, the rotations are obtained as functions of the plate slopes

$$\begin{aligned} \varphi_x\left(x, y, t\right) &= k_1 \frac{\partial w_0\left(x, y, t\right)}{\partial x} \\ \varphi_y\left(x, y, t\right) &= k_2 \frac{\partial w_0\left(x, y, t\right)}{\partial y} \end{aligned} \tag{6}$$



(a) (b)

Fig. 6 The slope curve (a) and rotation (b) of a thick rectangular plate which vibrating in (1, 0) mode

According to Eq. (6), displacement fields of FSDT and TSDT (Reddy 2004) can be rearranged with three functions and two unknown constants in their modified form. Since the transverse shear strains are obtained from sum of rotation functions and plate slopes (Reddy 2004) therefore, the additional assumption of modified shear deformation theories is that the transverse shear strains in plates are proportional to the slope curves of the plate. Here, we introduce the new displacement field which is called modified first order shear deformation theory (MFSDT) as the following form

$$\begin{aligned} u(x,y,z,t) &= u_0(x,y,t) + zk_1 \frac{\partial w_0(x,y,t)}{\partial x} \\ v(x,y,z,t) &= v_0(x,y,t) + zk_2 \frac{\partial w_0(x,y,t)}{\partial y} \\ w(x,y,z,t) &= w_0(x,y,t) \end{aligned} \tag{7}$$

and the modified third order shear deformation theory of Reddy (MTSDT)

$$\begin{aligned} u(x,y,z,t) &= u_0(x,y,t) + zk_1 \frac{\partial w_0(x,y,t)}{\partial x} - \frac{4z^3}{3h^2}(k_1+1)\left(\frac{\partial w_0(x,y,t)}{\partial x}\right) \\ v(x,y,z,t) &= v_0(x,y,t) + zk_2 \frac{\partial w_0(x,y,t)}{\partial y} - \frac{4z^3}{3h^2}(k_2+1)\left(\frac{\partial w_0(x,y,t)}{\partial y}\right) \\ w(x,y,z,t) &= w_0(x,y,t) \end{aligned} \tag{8}$$

The constants i.e., k_1 and k_2 are the ratios of $\varphi_x/(\partial w_0/\partial x)$ and $\varphi_y/(\partial w_0/\partial y)$, respectively and are inserted in plate equations so that shear effects can be evaluated. So, we call them ratios of rotation. It can be seen that independent variables of MFSDT, describing the plate deformation, are three functions and two constants. These unknowns will be determined using PDEs of motion.

3. Physical interpretation of rotation ratios k_1 and k_2

To have a better understanding of rotation ratios, here we suggest a mental experiment on a thick plate. As already stated, the constants k_1 and k_2 are the ratio of rotations in MFSDT on the rotations of CPT. So, the physical significant of rotation ratios can be investigated by comparing two plates which are subjected to assumptions of CPT and MFSDT. The essential assumption of CPT is that the straight lines which are perpendicular to mid-plane before deformation remains straight and perpendicular after deformation. In the suggested mental experiment, the boundary conditions are selected in a way that, the mentioned assumption be satisfied for thick plates, too.

Consider the plate shown in Fig. 7(a), the boundary layers of which are bonded perfectly with two rigid sheets (rigid B.C.) and it is impossible for a relative deformation to occur between them. Now, we deform the rigid sheets so that they lie on the radial lines of a circle. Consequently, top and bottom planes of plate will constitute the circumferential lines of same circle with different radiuses (Fig. 7(b)). Thus, although plate is thick, the mentioned assumption of CPT is satisfied. In fact, the deformation of rigid sheets causes the stretch and compress stresses to be developed in top and bottom planes, respectively.

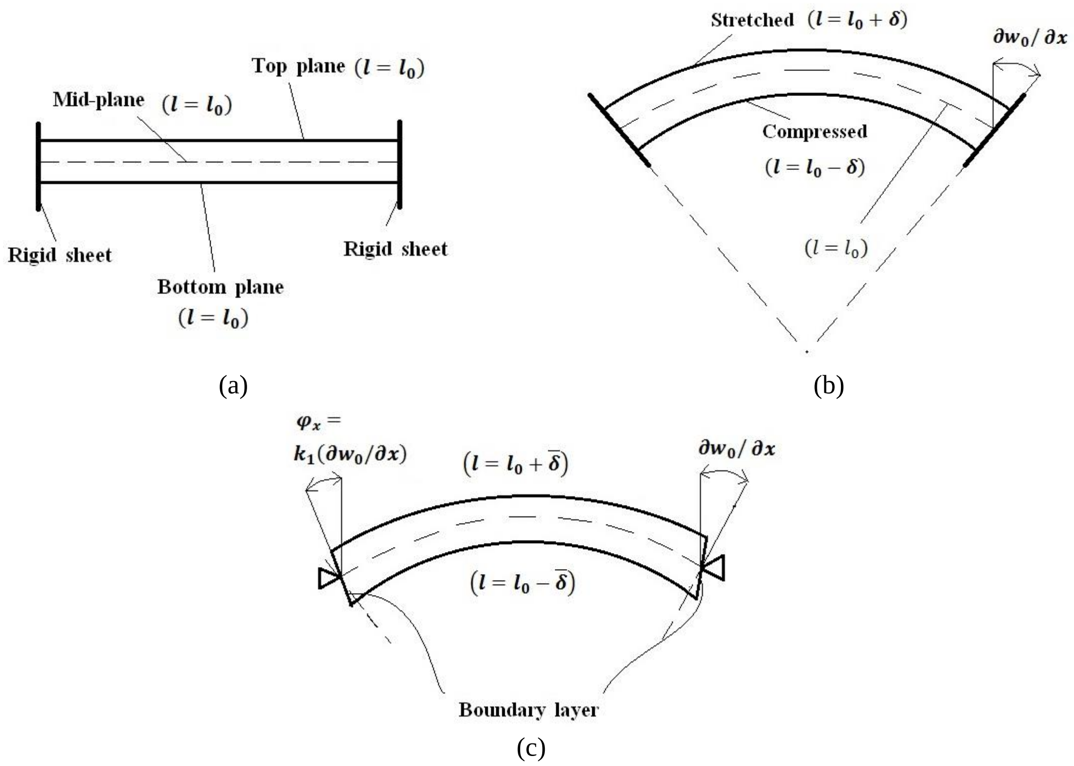


Fig. 7 Side view of a thick rectangular plate before deformation with rigid B.C. (a), after deformation with rigid B.C. (b) and after deformation with simply supported B.C. (c)

Now assume the rigid B.C. is replaced by simply supported B.C. and mid-plane still has saved its sectorial form. According to essential assumption of FSDT and MFSDT, now the straight lines are still straight but not perpendicular to mid-plane (Reddy 2004) (Fig. 7(c)). In this case, top plane is shortened to release its stretch stress and bottom plane becomes larger to release its compress stress as well as it is possible. So, the absolute value of deformation in bottom and top planes in simply supported B.C. case, are less than their corresponding value in rigid B.C. case: $\delta > \bar{\delta}$ (Fig. 7). Since the rigid B.C. is representative of CPT assumptions, we conclude the absolute value of rotation in a thick plate is less than the corresponding rotation of a thin plate: $|\varphi_x|<|\partial w_0/\partial x|$. This is while that, most of previously published books and article have mistaken in depicting FSDT deformation field (Fig. 2) and have considered: $|\varphi_x|>|\partial w_0/\partial x|$, the true form of FSDT and MFSDT deformation field is represented in Fig. 8.

Regarding the above context, the rotations of thick plates are a percentage of plate slopes. Absolute value of rotation ratios $|k_i|$, is representative of this percentage. So, the general form of rotation ratios should satisfy this relationship: $-1<k_i<0$. In thin plates and lower modes value of k_i tends to -1 and for thick plates and higher modes it approaches zero. As a consequence, when k_i approaches zero, shear strains have their maximum effect and equality of $k_i=-1$ is corresponded to removing shear effects.

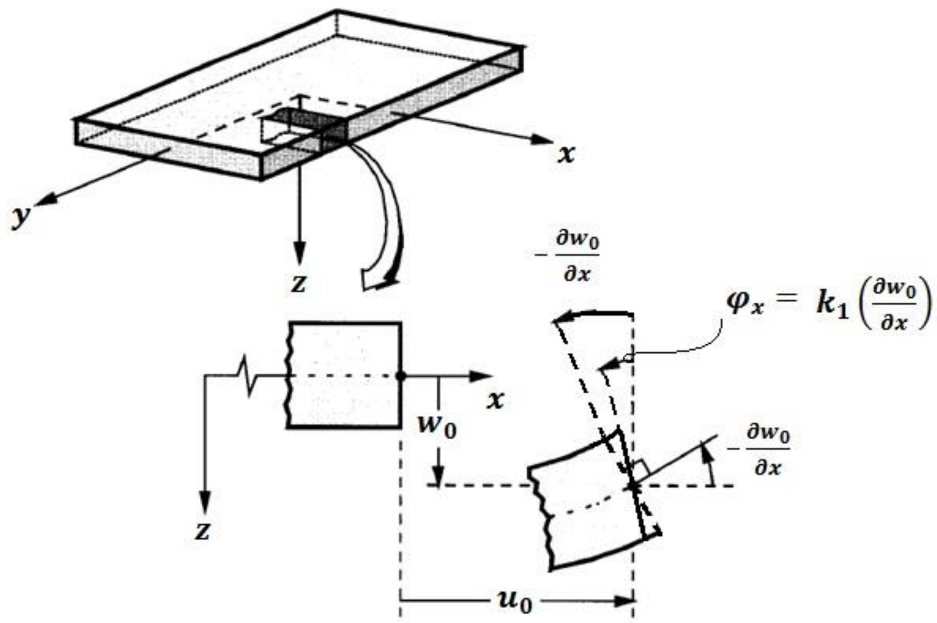


Fig. 8 Undeformed and deformed geometries of an edge in a rectangular plate in the MFSDT

4. Examples

4.1 Vibrations of cross ply laminated rectangular plate

Different researches have been devoted to studying the vibrations of rectangular plates. Chen and Lue (Chen and Lue 2005) conducted a semi analytical method which is combination of spatial approximate and DQM technique to review the vibrations of laminated rectangular plates. Viswanathan and Lee (2007) analyzed the rectangular composite plate by Spline method. Civalek (2008, 2009) showed the fundamental frequency of linearly varying thickness orthotropic plates, and in another work, using DSC method studied the vibration of thick symmetrically laminated plate applying FSDT. Shooshtari and Razavi (2010) studied the nonlinear vibrations of a FML rectangular plate. Neglecting the nonlinear terms and considering cross ply fibers and symmetric lamination for plate, we obtain the equations of motion in reference (Reddy 2004) which are reduced to

$$\begin{aligned}
&KA_{55}\left(w_{0,xx}+\varphi_{x,x}\right)+KA_{44}\left(w_{0,yy}+\varphi_{y,y}\right)=I_0 w_{0,tt}\\
&D_{11}\varphi_{x,xx}+D_{12}\varphi_{y,xy}+D_{66}\left(\varphi_{x,yy}+\varphi_{y,xy}\right)-KA_{55}\left(w_{0,x}+\varphi_x\right)=I_2\varphi_{x,tt}\\
&D_{22}\varphi_{y,yy}+D_{12}\varphi_{x,xy}+D_{66}\left(\varphi_{x,xy}+\varphi_{y,xx}\right)-KA_{44}\left(w_{0,y}+\varphi_y\right)=I_2\varphi_{y,tt}
\end{aligned} \tag{9}$$

where A_{ij} are extensional stiffness and the D_{ij} are bending stiffness (Reddy 2004). These equations have been obtained by using FSDT where the in-plane deformations (i.e., u_0 and v_0) and their corresponding governing equations are neglected to investigate only transverse vibrations.

In MFSDT, similar to FSDT, transverse shear stress is constant along thickness, while its real value is a function of thickness. For this reason, a shear correction coefficient K=5/6 is considered for all examples, except in thick plates ($h/a \geq 0.2$) which is considered to be: K=1. First equation is simply the Newton's second law in vertical direction and the other two are result of moments which are induced by forces along x and y axis, respectively about mid-plane. Applying MFSDT assumptions, beside spatial and time dependent functions, we can achieve the natural frequencies of the plate directly from boundary conditions. In order to study the transverse vibrations of plate, the first equation of Eq. (9) must be reviewed. For this, the rotations in this equation are replaced by their corresponding forms in MFSDT which are given in Eq. (6). This issue results in the decoupling of the deflection of the plate w_0 that is

$$KA_{55}\left(1+k_1\right)w_{0,xx}+KA_{44}\left(1+k_2\right)w_{0,yy}=I_0 w_{0,tt} \tag{10}$$

Since w_0 is a function of spatial variables and time, the method of separation of variables is used to solve Eq. (10). Based on this method, we have

$$w_0\,(x,y,t) = g\,(x)\,p\,(y)\,f\,(t) \tag{11}$$

where, $g(x)$, $p(y)$ and $f(t)$ are single variable functions of x, y and time, respectively. Substituting Eq. (11) into (10) and separating the time variable give

$$KA_{55}\left(\frac{g_{,xx}}{g}\left(1+k_1\right)\right)+KA_{44}\left(\frac{p_{,yy}}{p}\left(1+k_2\right)\right)=I_0\frac{f_{,tt}}{f}=-\mu^2 \tag{12}$$

The new parameter of $\mu^2=\lambda^2+\eta^2$ is defined to separate spatial variables from each other. Response of system is achieved as the following from solution of Eq. (12)

$$\begin{aligned}
f_{(t)} &= \sin\left(\frac{\mu\, t}{\sqrt{I_0}}+\psi\right)\\
g_{(x)} &= A_n.\sin\left(\frac{\eta\, x}{\sqrt{KA_{55}\left(1+k_1\right)}}\right)+B_n.\cos\left(\frac{\eta\, x}{\sqrt{KA_{55}\left(1+k_1\right)}}\right)\\
p_{(y)} &= C_m.\sin\left(\frac{\lambda\, y}{\sqrt{KA_{44}\left(1+k_2\right)}}\right)+D_m.\cos\left(\frac{\lambda\, y}{\sqrt{KA_{44}\left(1+k_2\right)}}\right)
\end{aligned} \tag{13}$$

where $\omega_n=\mu/\sqrt{I_0}$ is the natural frequency of the system. Obtaining frequency parameter of μ, gives the natural frequency of the system for all mode shapes of vibrations. Moreover, the mode shapes of the plate can be determined for all modes of vibration. The effect of each mode on the motion of plate can be easily determined using double Fourier series. The parameters m and n show the modes of vibration. Assuming that x-and y-axes lie on plate's length, for simply supported case the following two boundary conditions are used to obtain unknown constants

$$\begin{aligned}
w_0\left(0,y,t\right)=0 &\Rightarrow B_n=0\\
w_0\left(x,0,t\right)=0 &\Rightarrow D_m=0
\end{aligned} \tag{14}$$

Finally, six unknown constants of η, λ, k_1, k_2, ψ and $\phi_{(n,m)}=A_nC_m$ remain in the response of the system. ψ, is obtained from initial condition of speed. $\phi_{(n,m)}$, is obtained from the initial mode shape as a series, which, is the sum of all of the mode shapes with specified weights. The constants λ and η will be obtained from other boundary conditions of plate. But the constants k_1 and k_2 represent the rotation functions of plate. In the FSDT, these functions are obtained using two last PDEs of Eq. (9). In the MFSDT, the same PDEs are used and solved algebraically. To this end, Eq. (6) and (11) are substituted into two last PDEs of Eq. (9) and time variable is separated to obtain

$$\begin{aligned} &D_{11}\left(pg_{,xxx}\right)+\left(D_{12}\frac{k_2}{k_1}+D_{66}\frac{k_1+k_2}{k_1}\right)\left(g_{,x}p_{,yy}\right)+\left(\alpha\mu^2-KA_{55}\frac{1+k_1}{k_1}\right)\left(g_{,x}p\right)=0 \\ &D_{22}\left(gp_{,yyy}\right)+\left(D_{12}\frac{k_1}{k_2}+D_{66}\frac{k_1+k_2}{k_2}\right)\left(p_{,y}g_{,xx}\right)+\left(\alpha\mu^2-KA_{44}\frac{1+k_2}{k_2}\right)\left(p_{,y}g\right)=0 \end{aligned} \tag{15}$$

where $\alpha=I_2/I_0$. Eq. (15) includes functions $g(x)$ and $p(y)$, variables x and y, and constants, η, λ, k_1, and k_2. Since all the four constants are independent of plate area, an arbitrary point on the surface of plate can be selected. Selecting an arbitrary point in these equations is the only approach which has been used in the solution process of the present method. Therefore Eq. (15) is reduced to two algebraic equations with four unknown constants. On the other hand, the boundary conditions give equations that include these four unknown constants. So

$$\begin{aligned} &w_0(a,y,t)=0\Rightarrow\frac{\eta\, a}{\sqrt{KA_{55}(1+k_1)}}=n.\pi \\ &w_0(x,b,t)=0\Rightarrow\frac{\lambda\, b}{\sqrt{KA_{44}(1+k_2)}}=m.\pi \end{aligned} \tag{16}$$

Simultaneous solution of algebraic Eqs. (15)-(16) gives the unknown parameters. If the achieved unknowns from above algebraic equations are constant for each arbitrary x and y, it is concluded that the only used approach of present method is indeed an exact decision. As a result, in this case the obtained results will be the exact response of the governing equations.

The obtained mode shapes by the present method are the same as actual ones. The natural frequency which is obtained using MFSDT is compared with that of the published literature. The natural frequencies obtained using CPT are compared with available results as seen in Table 1. The natural frequencies of CPT are achieved using following equation (Chakraverty 2009)

$$\omega_{CPT}=\pi^2\left(\left(\frac{n}{a}\right)^2+\left(\frac{m}{b}\right)^2\right)\sqrt{\frac{D}{\rho h}} \tag{17}$$

Rotation ratios k_1 and k_2 are also determined for each mode shape as shown in this table. These parameters are considered as $k_1=k_2=-1$ in CPT for all conditions while, in FSDT, they are functions of plate area. In MFSDT, k_1 and k_2 are considered to be constant on the plate area and are only functions of mode shape, material property and geometry of the plate.

It is seen from Table 1 that the present method gives acceptable results for thin plate. However, the error of CPT increases in higher modes. The reason is that, since k_1 and k_2 parameters are considered to be -1 in CPT, it can be seen that in higher modes, because of increasing of transverse shear stresses, the absolute value of ratios of rotation (k_1 and k_2) are less than $|-1|$.

The dimensionless frequencies of a moderately thick isotropic square plate, which are obtained

Table 1 Natural frequency of thin isotropic plate
$\left(x/a = 0.5, y/b = 0.5, a = 0.2m, b = 0.5m, \rho = 2700\frac{kg}{m^3}, h = 0.005m, E = 71Gpa, v = \frac{1}{3} \right)$

Mode (*n*, *m*)	Method		Rotation ratio	
	CPT	MFSDT	k_1	k_2
(1,1)	2246.98	2243.91	-0.99786	-0.99786
(2,1)	8058.16	8018.88	-0.99237	-0.99237
(1,2)	3176.77	3170.63	-0.99697	-0.99697
(2,2)	8987.95	8939.13	-0.99150	-0.99256
(2,3)	10537.59	10470.61	-0.99006	-0.99006
(3,2)	18673.59	18464.76	-0.98255	-0.98255
(3,3)	20222.88	19978.78	-0.98114	-0.98114

Table 2 Dimensionless natural frequency for moderately thick isotropic rectangular plate
$\left(a = b, v = 0.3, \frac{h}{a} = 0.1 \right)$

Solution	(1,1)	(2,1)	(1,2)	(2,2)	(3,1)	(1,3)	(3,2)	(2,3)
Exact*	9.315	22.260	22.260	34.207	41.714	41.714	52.391	52.391
HSDT*	9.310	22.220	22.220	34.110	41.580	41.580	52.210	52.210
FSDT*	9.300	22.176	22.176	34.018	41.440	41.440	51.974	51.974
Shooshtari*	9.337	22.326	22.326	34.249	41.679	41.679	52.137	52.137
MFSDT	9.302	22.192	22.192	34.053	41.492	41.492	52.052	52.052
Error%	0.021	0.0721	0.0721	0.1028	0.1254	0.1254	0.1498	0.1498

*Shooshtari and Razavi (2010)

from Eq. (18), summarized in Table 2. The MFSDT results of this table are compared with those of FSDT for evaluating the error. The important note which here should be explained is that, FSDT cannot be known as the validity measurement factor of MFSDT and MFSDT contains an additional assumption in comparison with FSDT. Despite this, their results are usually confirmed with each other. It is observed that the results are in very good agreement with the published ones for higher modes as well as lower modes.

Table 3 shows the fundamental natural frequencies of a square laminated composite plate for different length-to-thickness ratios (*a/h*). These dimensionless natural frequencies are obtained by Eq. (19). Table 4 shows the dimensionless frequencies of different modes for the same plate of Table 3.

$$\Omega_2 = \omega_n \frac{a^2}{h}\sqrt{\frac{2\rho(1+v)}{E}} \tag{18}$$

$$\Omega_{orth} = \omega_n \frac{a^2}{h}\sqrt{\frac{\rho}{E_2}} \tag{19}$$

Table 3 Dimensionless fundamental natural frequency for rectangular laminated plate
$\left(\frac{E_1}{E_2}=40, G_{12}=G_{13}=0.6E_2, G_{23}=0.5E_2, v_{12}=0.5, (0°, 90°, 0°)\right)$

Solution	a/h			
	2	5	10	100
Xiang*	5.744	10.475	14.817	18.787
Reddy*	5.205	10.290	14.767	18.891
Shooshtari*	5.197	10.704	14.825	18.829
MFSDT	5.205	10.289	14.766	18.829

*Shooshtari and Razavi (2010)

Table 4 Dimensionless natural frequencies for moderately thick rectangular plate
$\left(\frac{a}{h}=10, (0°, 90°, 0°)\right)$

Solution	(1,1)	(1,2)	(2,1)
Omer civalek*	14.765	22.150	36.688
Khdeir*	14.766	22.158	36.900
Song xiang*	14.817	22.203	36.435
Shooshtari*	14.825	22.137	37.327
MFSDT	14.766	22.157	37.379

*Shooshtari and Razavi (2010)

4.2 Vibrations of cross ply laminated circular plate

Vibrations of circular plates have been studied in different researches. Rao and Prasad (1980) obtained the natural frequencies of circular plate based on FSDT. Hosseini-Hashemi *et al* (2010) performed an accurate analytical solution for free vibrations of thick plate using TSDT. Viswanathan *et al* (2009) studied the free vibration of laminated circular plates by applying FSDT. Mbakogu and Takagishi (1998) analyzed the free vibrations of polar orthotropic circular plate. Liew *et al* (1997) analyzed circular isotropic plate using DQM technique on governing equations which was obtained from FSDT. In reference (Viswanathan *et al* 2009), the governing equations for asymmetric orthotropic plate are given. Simplifying these equations for symmetric plate and ignoring in-plane inertias, we obtain the following equations

$$KA_{44}\left\{\frac{1}{r}\left(\varphi_{\theta,\theta}+\frac{1}{r}w_{,\theta\theta}\right)\right\}+KA_{55}\left\{\varphi_{r,r}+w_{,rr}+\frac{1}{r}\left(\varphi_r+w_{,r}\right)\right\}=I_0 w_{,tt}$$

$$D_{11}\left\{\varphi_{r,rr}+\frac{1}{r}\varphi_{r,r}\right\}+D_{12}\left\{\frac{1}{r}\varphi_{\theta,\theta r}\right\}+D_{22}\left\{-\frac{1}{r^2}\left(\varphi_{\theta,\theta}+\varphi_r\right)\right\}$$

$$+D_{66}\left\{\frac{1}{r^2}\left(\varphi_{r,\theta\theta}-\varphi_{\theta,\theta}\right)+\frac{1}{r}\varphi_{\theta,r\theta}\right\}-KA_{55}\left\{\varphi_r+w_{,r}\right\}=I_2\varphi_{r,tt}$$

$$D_{12}\left\{\frac{1}{r}\varphi_{r,r\theta}\right\}+D_{66}\left\{\frac{1}{r^2}\left(\varphi_{r,\theta}-\varphi_{\theta}\right)+\frac{1}{r}\left(\varphi_{r,\theta r}+\varphi_{\theta,r}\right)+\varphi_{\theta,rr}\right\}$$
$$+D_{22}\left\{\frac{1}{r^2}\left(\varphi_{\theta,\theta\theta}+\varphi_{r,\theta}\right)\right\}-KA_{44}\left\{\varphi_{\theta}+\frac{1}{r}w_{,\theta}\right\}=I_2\varphi_{\theta,tt} \tag{20}$$

In which the first equation describes the transverse motion and the other two equations are moments of in-plane stresses about mid-plane. To obtain the response of the circular plate using MFSDT, Eq. (6) is rewritten in the polar coordinates. So

$$\varphi_r\left(r,\theta,t\right)=k_1\frac{\partial w\left(r,\theta,t\right)}{\partial r}$$
$$\varphi_\theta\left(r,\theta,t\right)=\frac{k_2}{r}\frac{\partial w\left(r,\theta,t\right)}{\partial\theta} \tag{21}$$

Substituting the rotations from Eq. (21) into first equation of Eq. (20) results in the decoupling of w_0. Then, according to the method of separation of variables, w_0 is assumed to be

$$w_0\left(r,\theta,t\right)=g\left(r\right)p\left(\theta\right)f\left(t\right) \tag{22}$$

where, $g(r)$ and $p(\theta)$ are undetermined functions of r and θ, respectively. Substituting Eq. (22) into the decoupled differential equation, separation of the variables gives

$$KA_{44}\left(\frac{1+k_2}{r^2}\frac{p_{,\theta\theta}}{p}\right)+KA_{55}\left(1+k_1\right)\left(\frac{g_{,rr}}{g}+\frac{1}{r}\frac{g_{,r}}{g}\right)=I_0\frac{f_{,tt}}{f}=-\mu^2 \tag{23}$$

where, μ is the frequency parameter. In Eq. (23) the time variable has been separated and the time response of the system is

$$f\ (t)=\sin\left(\frac{\mu\, t}{\sqrt{I_0}}+\psi\right) \tag{24}$$

To obtain the mode shapes of the plate, r and θ must be separated from each other in Eq. (23). To this end, a new parameter η is used. So, Eq. (23) can be rearranged in the following separated form

$$KA_{44}\left(1+k_2\right)\left(\frac{p_{,\theta\theta}}{p}\right)=-KA_{55}\left(1+k_1\right)\left(r^2\frac{g_{,rr}}{g}+r\frac{g_{,r}}{g}\right)-r^2\mu^2=-\eta^2 \tag{25}$$

Mode shapes for circular laminated plate are obtained by solving Eq. (25). So, we have

$$p(\theta)=A_n\cdot J_\varrho\left(\frac{\mu\, r}{\sqrt{KA_{55}(1+k_1}}\right)+B_n\cdot Y_\varrho\left(\frac{\mu\, r}{\sqrt{KA_{55}(1+k_1}}\right) \tag{26}$$

where the order of Bessel function $\varrho=\eta/\sqrt{KA_{55}\left(1+k_1\right)}$ shows that in spite of the rectangular plates, the material property affects the mode shapes of circular plates. Assuming that the origin point of coordinate system is located at the center of circular plate, the boundary conditions for simply supported case are

$$
\begin{aligned}
&w_0\left(0,\theta,t\right) << \infty \Rightarrow B_n = 0 \\
&w_0\left(r,\frac{\pi}{2},t\right) = 0 \Rightarrow D_m = 0
\end{aligned}
\tag{27}
$$

The required algebraic equations to determine unknown constants (i.e., η, μ, k_1 and k_2) include two equations for moments and two equations induced from boundary conditions which are in the following form, respectively

$$
\begin{aligned}
&D_{11}\left(g_{,rrr}+\frac{1}{r}g_{,rr}\right)p+\frac{D_{66}}{r^3}p_{,\theta\theta}g-\left(\frac{D_{22}}{r^2}+\frac{1+k_1}{k_1}A_{55}-\alpha\lambda^2\right)g_{,r}p=0 \\
&\frac{D_{22}}{r^2}p_{,\theta\theta\theta}g+\left\{\left(D_{12}+D_{66}\right)\frac{k_1}{k_2}+D_{66}\right\}p_{,\theta}g_{,rr}+\frac{1}{r}\left\{\left(D_{22}+D_{66}\right)\frac{k_1}{k_2}+D_{66}\right\}p_{,\theta}g_{,r} \\
&-\left(\frac{D_{66}}{r^2}+\frac{1+k_2}{k_2}A_{44}-\alpha\lambda^2\right)p_{,\theta}g=0 \\
&\frac{\lambda R}{\sqrt{KA_{55}\left(1+k_1\right)}}=e \\
&\frac{\eta}{\sqrt{KA_{44}\left(1+k_2\right)}}=m
\end{aligned}
\tag{28}
$$

In Eq. (28), e is the different roots for Bessel function of first kind with the order ϱ. Shown in Table 5 are natural frequencies of isotropic circular plates. In this table, the natural frequencies of CPT (Chakraverty 2009) and the dimensionless frequency are obtained using Eqs. (29)-(30), respectively

$$\omega_{CPT}=e^4\sqrt{\frac{D}{\rho h}} \tag{29}$$

$$\Omega_5=\omega R^2\sqrt{\frac{\rho h}{D}} \tag{30}$$

Table 5 shows the advantages of MFSDT very well. Because of simplifying assumptions of CPT, this theory yields to accurate solution only in the simplest problems (isotropic thin plates), while the MFSDT leads, the same accurate solution to be obtained in thick laminated composite plates. If the thickness of the plate is increased, the CPT results given in Table 5 do not change and the error is increased, while the results of MFSDT are affected by the ratio of h/R, since in this theory the shear strains are taken into account. On the other hand, although TSDT (Hosseini-Hashemi *et al* 2010) gives relatively accurate results, it needs much more efforts to solve the governing equations. In fact, MFSDT is much simpler than CPT whereas it is as accurate as FSDT.

An interesting point in Table 5 is that, although in the thin plates and fundamental mode transverse shear strains are negligible (Nyfeh and Frank 2004), results of CPT are not consistent with the results of FSDT and TSDT while, in this case the results of CPT must not have any significant difference with the results from shear deformation theories. This difference is

Table 5 Dimensionless frequency for isotropic circular plate with various thicknesses

				Mode	(n, m)			
h/R	Theory	(0, 0)	(1, 0)	(2, 0)	(3, 0)	(4, 0)	(5, 0)	(6, 0)
	HSDT[a]	4.9335	29.719	74.155	138.310	222.200	325.830	449.180
	FSDT[b]	4.9351	29.720	74.155	138.310	222.210	325.830	449.180
0.001	MFSDT	5.7832	30.471	74.886	139.035	222.922	326.544	449.895
	CPT[c]	5.7831	30.471	74.887	139.040	222.932	326.563	449.933
	HSDT[a]	4.9247	29.327	71.780	130.420	203.000	287.170	380.820
	FSDT[b]	4.9247	29.323	71.756	130.350	202.810	286.790	380.130
0.05	MFSDT	5.7677	30.053	72.452	131.043	203.544	287.628	381.133
	CPT[c]	5.7831	31.471	74.887	139.040	222.932	326.563	449.933
	HSDT[a]	4.8942	28.254	66.024	113.820	168.090	226.400	287.210
	FSDT[b]	4.8938	28.240	65.942	113.570	167.530	225.340	285.440
0.100	MFSDT	5.7227	28.915	66.559	114.205	168.256	226.238	286.559
	CPT[c]	5.7831	30.471	74.887	139.040	222.932	326.563	449.933
	HSDT[a]	4.8448	26.774	59.214	97.209	137.910	179.95	222.700
	FSDT[b]	4.8440	26.715	59.062	96.775	136.980	178.230	219.860
0.15	MFSDT	5.6501	27.319	59.593	97.333	137.646	179.073	220.902
	CPT[c]	5.7831	30.471	74.887	139.040	222.932	326.563	449.933

[a]Hosseini-Hashemi *et al.* (2010), [b]Liew KM *et al.* (1997), [c]Chakraverty (2009)

negligible for higher modes of thin plate. On the other hand, results of MFSDT and CPT are consistent with each other in all modes of thin plate. However in thicker plates and higher modes, the results of MFSDT are consistent with those of shear deformation theories. Thus, inconsistency of MFSDT results in fundamental mode compared with those of shear deformation theories can be attributed in circularity of the plate. Since this problem does not occur in rectangular plates and affects the results of CPT, it can be concluded that the cause of this phenomenon is circular shape of plate and not the MFSDT itself.

In Table 6, the effect of orthotropic ratio (E_1/E_2), the ratio of thickness to radius (h/R) on the dimensionless natural frequencies of the system and the ratio of rotation are given. The dimensionless natural frequency is obtained using Eq. (31).

$$\Omega_6 = \omega R^2 \sqrt{\frac{\rho h}{D_{22}}} \tag{31}$$

It is seen that for each specified orthotropic ratio, in the thicker plates and higher modes, the ratio of rotation approaches to zero. However, in thin plates and lower modes, this parameter tends to be -1. On the other hand, for a specified mode and thickness, it is clear that ratio of E_1/E_2 changes the value of rotation ratios. Based on this matter, for some cases of fundamental mode which E_1/E_2 and h/R lead to the rotation ratios approach to -1, the last two equations of Eq. (28) which include $\sqrt{KA_{55}\left(1+k_1\right)}$ and $\sqrt{KA_{44}\left(1+k_2\right)}$ become ill conditioned. So, the round off errors decreases the accuracy. As a result k_1 and k_2 become less than -1 in these cases and an

Table 6 The effect of orthotropic ratio and thickness on dimensionless frequency and rotation ratio for moderately thick orthotropic circular plate
(E_2=40, G_{12}=G_{13}=0.6E_2, G_{23}=0.5E_2, v_{12}=0.5)

$\frac{E_1}{E_2}$	h/R	Frequency & rotation	Mode (n, m)					
			(0, 0)	(1, 0)	(2, 0)	(3, 0)	(4, 0)	(5, 0)
		ω	11.059	28.188	51.148	81.450	119.81	166.56
0.2		k_1	-0.9996	-0.9995	-0.9994	-0.9991	-0.9989	-0.9985
		ω	7.4603	29.907	69.657	126.91	201.65	293.77
0.8	0.01	k_1	-0.9998	-0.9994	-0.9988	-0.9979	-0.9967	-0.9952
		ω	Error	33.077	96.511	186.96	304.72	449.44
2		k_1	-1.0002	-0.9992	-0.9974	-0.9948	-0.9915	-0.9874
		ω	Error	39.916	142.93	285.88	470.63	696.47
5		k_1	-1.0013	-0.9990	-0.9974	-0.9887	-0.9810	-0.9716
		ω	11.005	27.956	50.423	79.614	115.85	159.00
0.2		k_1	-0.9911	-0.9891	-0.9855	-0.9806	-0.9744	-0.9671
		ω	7.4407	29.632	68.224	122.339	190.578	271.371
0.8	0.05	k_1	-0.9957	-0.9870	-0.9720	-0.9516	-0.9268	-0.8987
		ω	Error	34.782	99.038	185.736	291.177	411.434
2		k_1	-1.0068	-0.9821	-0.9411	-0.8885	-0.8292	-0.7672
		ω	Error	39.332	133.97	251.62	386.22	531.44
5		k_1	-1.0351	-0.9757	-0.8854	-0.7824	-0.6802	-0.5867
		ω	10.843	27.277	43.393	74.766	106.081	141.647
0.2		k_1	-0.9654	-0.9585	-0.9469	-0.9317	-0.9143	-0.8957
		ω	7.3805	28.830	64.379	111.235	166.547	228.005
0.8	0.1	k_1	-0.9830	-0.9510	-0.9005	-0.8401	-0.7765	-0.7140
		ω	Error	31.600	84.653	148.631	218.783	292.087
2		k_1	-1.0279	-0.9333	-0.8054	-0.6770	-0.5635	-0.4690
		ω	Error	37.687	114.55	194.24	274.35	353.635
5		k_1	-1.1562	-0.9109	-0.6650	-0.4813	-0.3546	-0.2680

imaginary value is obtained for the corresponding frequencies. The word "Error" in this table is used for these frequencies. Moreover, it is observed from Table 6 that increasing the orthotropic ratio (E_1/E_2) similar to thickening the plate causes the ratio of rotation to approach to zero.

5. Conclusions

In this research, the FSDT and CPT have been compared with each other. It is shown that there is a kinematic relation among rotation functions and slope curves of the plate. By considering the rotations as a linear ratio of the slope curves of the plate, we proposed the displacement field for modified shear deformation theories. The behavior of transverse shear strains was assumed to be similar to the slope of the plate. Exact mathematical solution for governing equations of thick laminated plates was accessible from the method of separation of variables, for the first time. It was shown that the new method is very simple and has good accuracy. The results of newly developed theory MFSDT were compared with those of CPT, FSDT and TSDT from both qualitative and quantitative point of views. In qualitative review, it was shown that using MFSDT leads to mathematical solution for a wider range of problems. Whereas MFSDT governing equations are a decoupled second order PDE and two algebraic equations, CPT leads to a fourth order PDE which usually cannot be decoupled (in-plane deformations and their related governing equations were neglected). On the other hand, FSDT and TSDT have three coupled PDEs. So, qualitative review showed that governing equations of MFSDT are much simpler in comparison with those of CPT, FSDT and TSDT. As a consequence, it can cover a more extensive spectrum of the problems by mathematical solution. In quantitative review, natural frequencies of CPT were only applicable for thin plates and lower modes, while the results of MFSDT were consistent with those of FSDT and TSDT in thick plates and higher modes.

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