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Vibration analysis of free-fixed hyperbolic cooling tower shells

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Abstract. A three-dimensional (3-D) method of analysis is presented for determining the free vibration frequencies of hyperboloidal shells free at the top edge and clamped at the bottom edge like a hyperboloidal cooling tower by the Ritz method based upon the circular cylindrical coordinate system instead of related 3-D shell coordinates which are normal and tangent to the shell midsurface. The Legendre polynomials are used as admissible displacements. Convergence to four-digit exactitude is demonstrated. Natural frequencies from the present 3-D analysis are also compared with those of straight beams with circular cross section, complete (not truncated) conical shells, and circular cylindrical shells as special cases of hyperboloidal shells from the classical beam theory, 2-D thin shell theory, and other 3-D methods.

Keywords: cooling tower; hyperbolic shell; free vibration; Legendre polynomials

1. Introduction

Hyperboloidal shell types of structures have been used widespread both in industrial and public buildings; for example, cooling towers, water towers, TV towers, supports of electric power transmission lines, reinforced concrete water vessels, high factory chimneys, and so forth, since they give rise to optimum conditions for good aerodynamics, strength, and stability.

A vast published literature exists for free vibrations of shells. The monograph of Leissa (1993) summarized approximately 1000 relevant publications world-wide through the 1960’s. Almost all of these dealt with shells of revolution (e.g., circular cylindrical, conical, spherical). Among them were three references (Anon 1965, Neal 1967, Carter *et al*. 1968) considering hyperboloidal shells. A review article by Krivoshapko (2002) describes some additional research on free vibrations of hyperboloidal shells of revolution for the period 1975-2000. Recently, hyperbolic shells are investigated by Viladkar *et al*. (2006), Ghoneim (2008), Díaz and Sanchez-Palencia (2009), Zhang *et al*. (2011), Ghoneim and Noor (2013), and Jia (2013).

However, these above analyses were all based upon shell theory, which is mathematically two-dimensional (2-D). That is, for thin shells one assumes the Kirchhoff hypothesis that normals to the shell middle surface remain normal to it during deformations (vibratory, in this case), and unstretched in length. This yields an eighth order set of partial differential equations of motion. For hyperboloidal shells they involve variable coefficients, making them quite difficult to solve. Even so, conventional shell theory is only applicable to *thin* shells. A higher order shell theory (Artioli *et*

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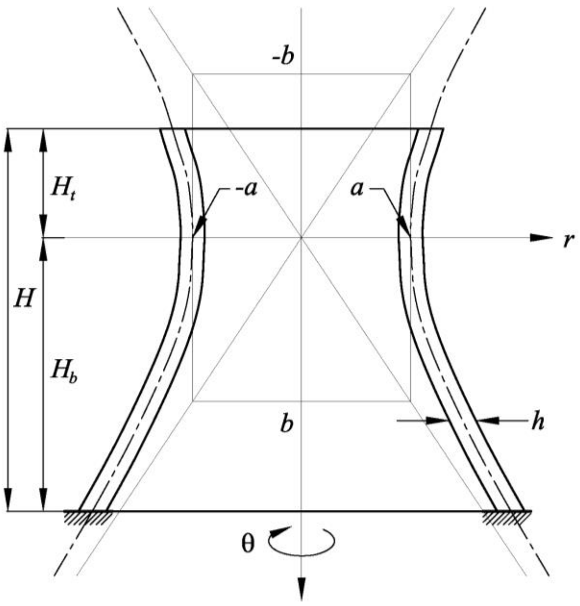

Fig. 1 A representative cross-section of a hyperboloidal shell of revolution free at top edge and clamped at the bottom edge (F-C) with the cylindrical coordinate system (r, θ, z)

al. 2005) could be derived which considers the effects of shear deformation and rotary inertia, and would be useful for the low frequency modes of moderately thick shells. Such a theory would also be 2-D. But for hyperboloidal shells the resulting equations would be very complicated. The Generalized coordinate methods is very efficient for finding the natural frequencies of similar structures and for solving boundary value problem in general (Viola *et al.* 2013).

Three-dimensional (3-D) analysis of structural elements has long been a goal of those who work in the field. With the current availability of computers of increased speed and capacity, it is now possible to perform 3-D structural analyses of bodies to obtain accurate values of static displacements, free vibration frequencies and mode shapes, and buckling loads and mode shapes. Kang and Leissa (2005) investigated vibrations of hyperboloidal shells based on 3-D analysis, but they presented results only for completely free boundary conditions.

In the present work hyperboloidal shells are analyzed by a 3-D approach. Instead of attempting to solve equations of motion, an energy approach is followed which, as sufficient freedom is given to the three displacement components, yields frequency values as close to the exact ones as desired. The Ritz method is applied using the Legendre polynomials as an admissible displacement function. To evaluate the energy integrations over the shell volume, displacements and strains are expressed in terms of the circular cylindrical coordinates, instead of related 3-D shell coordinates which are normal and tangent to the shell midsurface. Results are obtained for hyperboloidal shells free at the top edge and clamped at the bottom edge (F-C) like a hyperbolic cooling tower. These data serve as benchmarks against which other approximate methods (e.g., finite elements, finite differences) and other, improved 2-D shell theories may be tested. Natural frequencies from the present 3-D analysis are also compared with those of straight beams with circular cross section, complete (not truncated) conical shells, and circular cylindrical shells as special cases of

Table 1 Convergence of frequencies $\omega a\sqrt{\rho/G}$ of a hyperboloidal shell free at the top edge and clamped at the bottom edge (F-C) and for the five lowest bending modes (n=2) with b/a=3, h/a=0.4, H_b/a=4, and H_t/H_b=1 (ν=0.3) using the Legendre polynomials

TR	*TZ*	*DET*	1	2	3	4	5
4	2	24	0.3988	0.6385	1.525	2.024	2.489
4	4	48	0.2562	0.4798	0.6731	1.094	1.439
4	6	72	0.2509	0.4608	0.5423	0.8284	1.317
4	8	96	0.2499	0.4575	0.5236	0.7368	0.9958
4	10	120	0.2497	0.4571	0.5199	0.7254	0.9706
4	12	144	0.2497	0.4569	0.5194	0.7231	0.9592
5	2	30	0.3985	0.6378	1.525	2.024	2.489
5	4	60	0.2558	0.4765	0.6677	1.077	1.438
5	6	90	0.2506	0.4599	0.5385	0.8144	1.305
5	8	120	0.2498	0.4571	0.5222	0.7316	0.9823
5	10	150	0.2497	0.4568	0.5195	0.7243	0.9652
5	11	165	0.2496	0.4567	0.5193	0.7232	0.9610
5	12	180	0.2496	0.4567	0.5192	0.7227	0.9582
6	2	36	0.3984	0.6373	1.525	2.024	2.489
6	4	72	0.2558	0.4760	0.6671	1.076	1.437
6	6	108	0.2506	0.4598	0.5379	0.8133	1.303
6	8	144	0.2498	0.4570	0.5220	0.7306	0.9775
6	10	180	0.2496	0.4568	0.5194	0.7239	0.9630
6	11	198	0.2496	0.4567	0.5192	0.7229	0.9597
6	12	216	0.2496	0.4567	0.5192	0.7227	0.9577

TR=Total numbers of the Legendre polynomial terms used in the r (or ψ) direction
TZ=Total numbers of the Legendre polynomial terms used in the z (or ζ) direction
DET=Frequency determinant order

hyperboloidal shells of revolution from the classical beam theory, 2-D thin shell theory, and other 3-D methods

2. Method of analysis

A representative cross-section of hyperboloidal shells of constant thickness (h) in the radial direction (r), and height $H(=H_t+H_b)$ of the shell in the axial direction (z), where H_t and H_b are the lengths from the r-axis to the top and bottom ends of the shell, respectively, is shown in Fig. 1. The lengths of major and minor axes of the mid-surface of the hyperboloidal shell are $2a$ and $2b$, respectively, and so slopes of asymptotes are $\pm b/a$. The cylindrical coordinate system (r, θ, z), also shown in the figure, is used in the analysis, where θ is the circumferential angle. The equation of the hyperboloidal mid-surface is $(r/a)^2-(z/b)^2=1$. Thus the domain (Ω) of the shell is given by

$$\frac{a}{b}\sqrt{z^2+b^2}-\frac{h}{2}\le r\le\frac{a}{b}\sqrt{z^2+b^2}+\frac{h}{2},\quad 0\le\theta\le 2\pi,\quad -H_t\le z\le H_b\,. \tag{1}$$

Utilizing tensor analysis, the three equations of motion in the cylindrical coordinate system (r, θ, z) are found to be (Sokolnikoff 1956)

Table 2 Convergence of frequencies $\omega a\sqrt{\rho/G}$ of a hyperboloidal shell free at the top edge and clamped at the bottom edge (F-C) and for the five lowest bending modes (n=2) with b/a=3, h/a=0.4, H_b/a=4, and H_t/H_b=1 (ν=0.3) using the simple polynomials

TR	*TZ*	*DET*	1	2	3	4	5
4	2	24	0.4221	0.6422	1.425	2.221	2.377
4	4	48	0.2562	0.4798	0.6731	1.094	1.439
4	6	72	0.2509	0.4608	0.5423	0.8284	1.317
4	8	96	0.2499	0.4575	0.5236	0.7368	0.9958
4	10	120	0.2497	0.4571	0.5199	0.7254	0.9706
4	12	144	0.2498	0.4573*	0.5198	0.7231	0.9590
5	2	30	0.4139	0.6398	1.321	2.014	2.321
5	4	60	0.2558	0.4765	0.6677	1.077	1.438
5	6	90	0.2506	0.4599	0.5385	0.8144	1.305
5	8	120	0.2498	0.4571	0.5222	0.7316	0.9823
5	10	150	0.2497	0.4568	0.5195	0.7243	0.9652
5	11	165	0.2496	0.4567	0.5193	0.7232	0.9610
5	12	180	0.2497*	0.4569*	0.5194*	0.7243*	0.9682*
6	2	36	0.3991	0.6312	1.310	2.008	2.288
6	4	72	0.2558	0.4760	0.6671	1.076	1.437
6	6	108	0.2506	0.4598	0.5379	0.8133	1.303
6	8	144	0.2498	0.4570	0.5220	0.7306	0.9775
6	10	180	0.2496	0.4568	0.5194	0.7239	0.9630
6	11	198	0.2497*	0.4568*	0.5193	0.7229	0.9597
6	12	216	0.2489	0.4569*	0.5194*	0.7220	0.9689*

TR=Total numbers of the algebraic simple polynomial terms used in the r (or ψ) direction
TZ=Total numbers of the algebraic simple polynomial terms used in the z (or ζ) direction
DET=Frequency determinant order

$$\sigma_{rr,r}+\sigma_{rz,z}+\frac{1}{r}(\sigma_{rr}-\sigma_{\theta\theta}+\sigma_{r\theta,\theta})=\rho\ddot{u}_r\,,$$
$$\sigma_{r\theta,r}+\sigma_{\theta z,z}+\frac{1}{r}(2\sigma_{r\theta}+\sigma_{\theta\theta,\theta})=\rho\ddot{u}_\theta\,,$$
$$\sigma_{rz,r}+\sigma_{zz,z}+\frac{1}{r}(\sigma_{rz}+\sigma_{\theta z,\theta})=\rho\ddot{u}_z \tag{2}$$

where the σ_{ij} are the normal (i=j) and shear (i≠j) stress components; u_r, u_θ, and u_z are the displacement components in the r, θ, and z directions, respectively; ρ is mass density per unit volume; the commas indicate spatial derivatives; and the dots denote time derivatives. The well-known relationships between the tensorial stresses (σ_{ij}) and strains (ε_{ij}) of isotropic, linear elasticity are

$$\sigma_{ij}=\lambda\varepsilon\delta_{ij}+2G\varepsilon_{ij} \tag{3}$$

where λ and G are the Lamé parameters, expressed in terms of Young's modulus (E) and Poisson's ratio (ν) for an isotropic solid as

$$\lambda=\frac{E\nu}{(1+\nu)(1-2\nu)}\,,\quad G=\frac{E}{2(1+\nu)} \tag{4}$$

$\varepsilon \equiv \varepsilon_{rr}+\varepsilon_{\theta\theta}+\varepsilon_{zz}$ is the trace of the strain tensor, and δ_{ij} is Kronecker's delta. The 3-D tensorial strains (ε_{ij}) are found to be related to the three displacements u_r, u_θ, and u_z, by (Sokolnikoff 1956).

$$\varepsilon_{rr} = u_{r,r}\,, \quad \varepsilon_{\theta\theta} = \frac{u_{\theta,\theta} + u_r}{r}\,, \quad \varepsilon_{zz} = u_{z,z}\,,$$
$$\varepsilon_{r\theta} = \frac{1}{2}\left[u_{\theta,r} + \frac{u_{r,\theta} - u_\theta}{r}\right], \quad \varepsilon_{rz} = \frac{1}{2}(u_{r,z} + u_{z,r})\,, \quad \varepsilon_{\theta z} = \frac{1}{2}\left[u_{\theta,z} + \frac{u_{z,\theta}}{r}\right]. \tag{5}$$

Substituting Eqs. (3) and (5) into (2), one obtains a set of second-order partial differential equation in u_r, u_θ, and u_z governing free vibrations. However, in the case of hyperboloidal shells, exact solutions are intractable because of the variable coefficients that appear in many terms. Alternatively, one may approach the problem from an energy perspective.

During vibratory deformation of the body, its strain (potential) energy (V) is the integral over the domain (Ω)

$$V = \frac{1}{2}\int_\Omega (\sigma_{rr}\varepsilon_{rr} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} + \sigma_{zz}\varepsilon_{zz} + 2\sigma_{r\theta}\varepsilon_{r\theta} + 2\sigma_{rz}\varepsilon_{rz} + 2\sigma_{\theta z}\varepsilon_{\theta z})\, r\, dr\, d\theta\, dz\,. \tag{6}$$

Substituting Eqs. (3) and (5) into (6) results in the strain energy in terms of the three displacements

$$V = \frac{1}{2}\int_\Omega [\lambda(\varepsilon_{rr} + \varepsilon_{\theta\theta} + \varepsilon_{zz})^2 + 2G\{\varepsilon_{rr}{}^2 + \varepsilon_{\theta\theta}{}^2 + \varepsilon_{zz}{}^2 + 2(\varepsilon_{r\theta}{}^2 + \varepsilon_{rz}{}^2 + \varepsilon_{\theta z}{}^2)\}]\, r\, dr\, d\theta\, dz \tag{7}$$

where the tensorial strains ε_{ij} are expressed in terms of the three displacements by equations (5). The kinetic energy (T) is

$$T = \frac{1}{2}\int_\Omega \rho(\dot{u}_r{}^2 + \dot{u}_\theta{}^2 + \dot{u}_z{}^2)\, r\, dr\, d\theta\, dz\,. \tag{8}$$

For mathematical convenience, the radial r and axial z coordinates are made dimensionless as $\psi \equiv r/h$ and $\zeta \equiv z/H$. Thus the ranges of the nondimensional cylindrical coordinates (ψ, θ, ζ) are given by

$$\psi_1(\zeta) \le \psi \le \psi_2(\zeta)\,, \quad 0 \le \theta \le 2\pi\,, \quad -\frac{H_t}{H} \le \zeta \le \frac{H_b}{H} \tag{9}$$

where

$$\psi_1(\zeta) \equiv \frac{\sqrt{(H^*\zeta)^2 + k^2}}{kh^*} - \frac{1}{2}\,, \qquad \psi_2(\zeta) \equiv \frac{\sqrt{(H^*\zeta)^2 + k^2}}{kh^*} + \frac{1}{2} \tag{10}$$

and H^*($\equiv H/a$) and h^*($\equiv h/a$) are the nondimensional height and thickness of the shell, respectively, and k($\equiv b/a$) is the axis ratio (and asymptote slope). For the free, undamped vibration, the time (t) response of the three displacements is sinusoidal and, moreover, the circular symmetry of the body allows the displacements to be expressed by

$$u_r(\psi,\theta,\zeta,t) = U_r(\psi,\zeta)\cos n\theta \sin(\omega t + \alpha)\,,$$

Table 3 Comparisons of non-dimensional frequencies $\omega l^2\sqrt{\rho A/EI}$ of a beam with circular cross-section with l/r=500 from the classical beam theory and nearly straight beam with circular cross-section from the present 3-D method with b/a=1000, h/a=1.98, H_b/a=500, and H_t/H_b=1 for n=1 (ν =0.3)

Mode	Method	Boundary Conditions		
		F-F	C-F	C-C
1	Present	21.89	3.522	22.53
	Young and Felgar	22.37	3.516	22.37
2	Present	61.51	22.37	61.88
	Young and Felgar	61.67	22.03	61.67
3	Present	121.0	61.73	121.3
	Young and Felgar	120.9	61.70	120.9

Note: F=Free, C=Clamped

$$u_\theta(\psi,\theta,\zeta,t) = U_\theta(\psi,\zeta)\sin n\theta \sin(\omega t+\alpha)\,,$$
$$u_z(\psi,\theta,\zeta,t) = U_z(\psi,\zeta)\cos n\theta \sin(\omega t+\alpha) \tag{11}$$

where U_r, U_θ, and U_z are displacement functions of ψ and ζ, ω is a natural frequency, and α is an arbitrary phase angle determined by the initial conditions. The circumferential wave number is taken to be an integer (n=0, 1, 2, 3, …, ∞), to ensure periodicity in θ. Then Eq. (11) account for all free vibration modes except for the torsional ones. These modes arise from an alternative set of solutions which are the same as Eq. (11), except that $\cos n\theta$ and $\sin n\theta$ are interchanged. For n>0, this set duplicates the solutions of Eq. (11), with the symmetry axes of the mode shapes being rotated. But for n=0 the alternative set reduces to u_r=u_z=0, $u_\theta = U_\theta^*(r,z)\sin(\omega t+\alpha)$, which corresponds to the torsional modes. The displacements uncouple by circumferential wave number (n), leaving only coupling in r and z.

The Ritz method uses the maximum potential (strain) energy ($V_{\max}$) and the maximum kinetic energy ($T_{\max}$) functionals in a cycle of vibratory motion. The functionals are obtained by setting $\sin^2(\omega t+\alpha)$ and $\cos^2(\omega t+\alpha)$ equal to unity in Eqs. (7) and (8) after the displacements (11) are substituted, and by using the nondimensional coordinates ψ and ζ as follows

$$V_{\max} = \frac{GH}{2}\int_{-H_t/H}^{H_b/H}\int_{\psi_1}^{\psi_2}\left[\left\{\frac{\lambda}{G}(\kappa_1+\kappa_2+\kappa_3)^2 + 2(\kappa_1{}^2+\kappa_2{}^2+\kappa_3{}^2)+\kappa_4{}^2\right\}\Gamma_1 + (\kappa_5{}^2+\kappa_6{}^2)\Gamma_2\right]\psi\, d\psi\, d\zeta \tag{12}$$

$$T_{\max} = \frac{\rho H h^2\omega^2}{2}\int_{-H_t/H}^{H_b/H}\int_{\psi_1}^{\psi_2}[(U_r{}^2+U_z{}^2)\Gamma_1 + U_\theta{}^2\Gamma_2]\,\psi\, d\psi\, d\zeta\,, \tag{13}$$

where

$$\kappa_1 \equiv \frac{U_r+nU_\theta}{\psi},\quad \kappa_2 \equiv \frac{h^*}{H^*}U_{z,\zeta},\quad \kappa_3 \equiv U_{r,\psi}\,,$$

$$\kappa_4 \equiv \frac{h^*}{H^*}U_{r,\zeta}+U_{z,\psi},\quad \kappa_5 \equiv \frac{U_\theta+nU_r}{\psi}-U_{\theta,\psi},\quad \kappa_6 \equiv \frac{nU_z}{\psi}-\frac{h^*}{H^*}U_{\theta,\zeta} \tag{14}$$

and Γ_1 and Γ_2 are constants, defined by

$$\Gamma_1 \equiv \int_0^{2\pi} \cos^2 n\theta = \begin{cases} 2\pi & \text{if } n=0 \\ \pi & \text{if } n \ge 1 \end{cases}, \qquad \Gamma_2 \equiv \int_0^{2\pi} \sin^2 n\theta = \begin{cases} 0 & \text{if } n=0 \\ \pi & \text{if } n \ge 1 \end{cases}. \tag{15}$$

From Eq. (4) it is seen that the nondimensional constant λ/G in (12) involves only ν; i.e.

$$\frac{\lambda}{G} = \frac{2\nu}{1-2\nu}. \tag{16}$$

According to Mikhlin (1964) and Mikhlin and Smolitskiy (1967), a set of algebraic polynomial functions is not minimal, and numerical instability of the Ritz system may occur even within a relatively small number of terms of polynomial functions. They suggested the use of strongly minimal or orthonormal set of functions in D_A or other space similar to D_A if those functions are available, where D_A is a Hillbert space or energy space defined by the differential operators in the governing equations of motion. Using the orthogonal polynomials instead of ordinary ones as admissible functions permits one to use higher degrees before encountering ill-conditioning, thereby obtaining more accurate frequencies. In practice, the orthonormalized sets of admissible functions in the energy space or its similar space may be pursued by the use of either the classified orthogonal functions such as Bessel, Legendre, Hermite, Laguerre, Chebyshev, Jacobi, and so forth, or the Gram-Schmidt orthogonalization. In particular, the Gram-Schmidt procedure by means of recurrence formula (Beckmann 1973) may provide an efficient tool to produce orthonormal admissible functions numerically.

The Legendre polynomials $P_n(x)$ are defined by Rodrigues' formula (Lebedev 1972)

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n, \quad (n=0,1,2,...) \tag{17}$$

for arbitrary real or complex values of the variable x. The general expression for the nth Legendre polynomial is obtained from Eq. (17) by using the familiar binomial expansion

$$(x^2-1)^n = \sum_{k=0}^{n} \frac{(-1)^k n!}{k!(n-k)!} x^{2n-2k} \tag{18}$$

which implies

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k (2n-2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k} \tag{19}$$

where the symbol $[\mu]$ denotes the largest integer $\le \mu$. Alternatively, we can produce the Legendre polynomials from the recursion formula given by Courant and Hillbert (1953)

$$P_{n+1}(x) = \frac{1}{n+1}\left[(2n+1)xP_n(n) - nP_{n-1}(x)\right] (n = 0,1,2,\ldots). \tag{20}$$

Thus from Eqs. (17, (19), or (20) the first few Legendre polynomials are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2-1), \quad P_3(x) = \frac{1}{2}(5x^3-3x), \ldots. \tag{21}$$

Table 4 Comparisons of non-dimensional frequencies $\overline{\Omega}^2=\omega^2 R^2 \rho / E\cos^2\alpha$ for axisymmetric mode (n=0^A) of a complete conical shell with α=45° and h/R=0.01478 (K=100,000) from 2-D thin shell theory and nearly complete conical shell with b/a=1, h/a=1.98, H_b/a=134.0, and H_t/H_b =0 from the present 3-D method (ν=0.3)

Mode	Method	Boundary Conditions	
		Completely Free	Clamped
1	Present	1.155	1.741
	Dreher and Leissa	1.251	1.802
2	Present	1.686	2.302
	Dreher and Leissa	1.977	2.431
3	Present	2.622	3.121
	Dreher and Leissa	2.973	3.449

The orthogonality condition is given by

$$\int_{-1}^{1} P_m(x)P_n(x)dx=\begin{cases}0 \quad \text{for } m\neq n \\ 2/(2m+1) \quad \text{for } m=n\end{cases} \tag{22}$$

The displacement functions U_r, U_θ, and U_z in Eq. (11) are further assumed as the Legendre polynomials

$$U_r(\psi,\zeta)=\eta_r(\psi,\zeta)\sum_{i=0}^{I}\sum_{j=0}^{J}A_{ij}P_i(\psi)P_j(\zeta),$$

$$U_z(\psi,\zeta)=\eta_z(\psi,\zeta)\sum_{k=0}^{K}\sum_{l=0}^{L}B_{kl}P_k(\psi)P_l(\zeta),$$

$$U_\theta(\psi,\zeta)=\eta_\theta(\psi,\zeta)\sum_{m=0}^{M}\sum_{n=0}^{N}C_{mn}P_m(\psi)P_n(\zeta) \tag{23}$$

and similarly for U_θ^*, where i, j, k, l, m, and n are integers; I, J, K, L, M, and N are the highest degrees taken in the polynomial terms; A_{ij}, B_{kl} and C_{mn} are arbitrary coefficients to be determined, and the η are functions depending upon the geometric boundary conditions to be enforced. For example:

1. $\eta_r=\eta_z=\eta_\theta=\zeta-H_b/H$ (free at the top edge and clamped at the bottom edge, F-C)
2. $\eta_r=\eta_z=\eta_\theta=\zeta+H_t/H$ (clamped at the top edge and free at the bottom edge, C-F)
3. $\eta_r=\eta_z=\eta_\theta=(\zeta+H_t/H)(\zeta-H_b/H)$ (clamped at both edges, C-C)

The functions of η shown above, impose only the necessary geometric constraints. Together with the Legendre polynomials in Eq. (23), they form function sets which are mathematically complete (Kantorovich and Krylov 1958). Thus, the function sets are capable of representing any 3-D motion of the body with increasing accuracy as the indices I, J,…, N are increased. In the limit, as sufficient terms are taken, all internal kinematic constraints vanish, and the functions (23) will approach the exact solution as closely as desired.

The eigenvalue problem is formulated by minimizing the free vibration frequencies with respect to the arbitrary coefficients A_{ij}, B_{kl} and C_{mn}, thereby minimizing the effects of the internal

Table 5 Comparisons of non-dimensional frequencies $\omega a\sqrt{(1-\nu^2)\rho/E}$ of clamped-clamped (C-C) circular cylindrical shells with h/a=1 and ν=0.3

h/H	Method	n		
		0^A	1	2
0.1	Present	0.2076	0.1272	0.5888
	Loy and Lam	0.2076	0.1273	0.5888
0.4	Present	0.8301	0.7306	0.9183
	Loy and Lam	0.8301	0.7308	0.9184
0.8	Present	1.660	1.554	1.6428
	Loy and Lam	1.660	1.555	1.6428

n=circumferential mode number
A=Axisymmetric mode

constraints present, when the function sets are finite. This corresponds to the equations (Leissa 2005)

$$\frac{\partial}{\partial A_{ij}}(V_{\max}-\omega^2 T^*_{\max})=0,\quad (i=0,1,2,\ldots,I;\ j=0,1,2,\ldots,J)\,,$$
$$\frac{\partial}{\partial B_{kl}}(V_{\max}-\omega^2 T^*_{\max})=0,\quad (k=0,1,2,\ldots,K;\ l=0,1,2,\ldots,L)\,,$$
$$\frac{\partial}{\partial C_{mn}}(V_{\max}-\omega^2 T^*_{\max})=0,\quad (m=0,1,2,\ldots,M;\ n=0,1,2,\ldots,N) \tag{24}$$

where $T_{\max}=\omega^2 T^*_{\max}$. Eq. (24) yield a set of (I+1)(J+1)+(K+1)(L+1)+(M+1)(N+1) linear, homogeneous, algebraic equations (or Ritz system) in the unknowns A_{ij}, B_{kl} and C_{mn}. In the present problem, the Ritz system has the following form

$$\begin{bmatrix} \mathrm{K}_{ij\hat{i}\hat{j}} & \mathrm{K}_{ij\hat{k}\hat{l}} & \mathrm{K}_{ij\hat{m}\hat{n}} \\ \mathrm{K}_{kl\hat{i}\hat{j}} & \mathrm{K}_{kl\hat{k}\hat{l}} & \mathrm{K}_{kl\hat{m}\hat{n}} \\ \mathrm{K}_{mn\hat{i}\hat{j}} & \mathrm{K}_{mn\hat{k}\hat{l}} & \mathrm{K}_{mn\hat{m}\hat{n}} \end{bmatrix}\begin{bmatrix} A_{\hat{i}\hat{j}} \\ B_{\hat{k}\hat{l}} \\ C_{\hat{m}\hat{n}} \end{bmatrix}=\Lambda\begin{bmatrix} \mathrm{M}_{ij\hat{i}\hat{j}} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathrm{M}_{kl\hat{k}\hat{l}} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathrm{M}_{mn\hat{m}\hat{n}} \end{bmatrix}\begin{bmatrix} A_{\hat{i}\hat{j}} \\ B_{\hat{k}\hat{l}} \\ C_{\hat{m}\hat{n}} \end{bmatrix}, \tag{25}$$

where

$$\mathrm{K}_{ij\hat{i}\hat{j}}=\Gamma_1\left[\left(\frac{\lambda}{G}+2\right)\left(\frac{h}{H}\right)^2\left\langle P_{ij,\zeta},P_{\hat{i}\hat{j},\zeta}\right\rangle+\left\langle P_{ij,\psi},P_{\hat{i}\hat{j},\psi}\right\rangle\right]+n^2\Gamma_2\left\langle\frac{P_{ij}}{\psi},\frac{P_{\hat{i}\hat{j}}}{\psi}\right\rangle,$$

$$\mathrm{K}_{kl\hat{k}\hat{l}}=\Gamma_1\left[\left(\frac{\lambda}{G}+2\right)\left\{\left\langle\frac{P_{kl}}{\psi},\frac{P_{\hat{k}\hat{l}}}{\psi}\right\rangle+\left\langle P_{kl,\psi},P_{\hat{k}\hat{l},\psi}\right\rangle\right\}+\left(\frac{h}{H}\right)^2\left\langle P_{kl,\zeta},P_{\hat{k}\hat{l},\zeta}\right\rangle\right.$$
$$\left.+\left(\frac{\lambda}{G}\right)\left\{\left\langle\frac{P_{kl,\psi}}{\psi},P_{\hat{k}\hat{l}}\right\rangle+\left\langle\frac{P_{kl}}{\psi},P_{\hat{k}\hat{l},\psi}\right\rangle\right\}\right]+n^2\Gamma_2\left\langle\frac{P_{kl}}{\psi},\frac{P_{\hat{k}\hat{l}}}{\psi}\right\rangle,$$

$$\mathrm{K}_{mn\hat{m}\hat{n}} = n^2\Gamma_1\left(\frac{\lambda}{G}+2\right)\left\langle\frac{P_{mn}}{\psi},\frac{P_{\hat{m}\hat{n}}}{\psi}\right\rangle + \Gamma_2\left[\left(\frac{h}{H}\right)^2\left\langle P_{mn,\zeta},P_{\hat{m}\hat{n},\zeta}\right\rangle + \left\langle\frac{P_{mn}}{\psi},\frac{P_{\hat{m}\hat{n}}}{\psi}\right\rangle + \left\langle P_{mn,\psi},P_{\hat{m}\hat{n},\psi}\right\rangle\right.$$

$$\left. - \left\langle\frac{P_{mn}}{\psi},P_{\hat{m}\hat{n},\psi}\right\rangle - \left\langle\frac{P_{mn,\psi}}{\psi},P_{\hat{m}\hat{n}}\right\rangle\right],$$

$$\mathrm{K}_{ij\hat{k}\hat{l}} = \Gamma_1\left(\frac{h}{H}\right)\left[\left(\frac{\lambda}{G}\right)\left\{\left\langle\frac{P_{ij,\zeta}}{\psi},P_{\hat{k}\hat{l}}\right\rangle + \left\langle P_{ij,\zeta},P_{\hat{k}\hat{l},\psi}\right\rangle\right\} + \left\langle P_{ij,\psi},P_{\hat{k}\hat{l},\zeta}\right\rangle\right],$$

$$\mathrm{K}_{ij\hat{m}\hat{n}} = n\left(\frac{h}{H}\right)\left[\Gamma_1\left(\frac{\lambda}{G}\right)\left\langle\frac{P_{ij,\zeta}}{\psi},P_{\hat{m}\hat{n}}\right\rangle - \Gamma_2\left\langle\frac{P_{ij}}{\psi},P_{\hat{m}\hat{n},\zeta}\right\rangle\right],$$

$$\mathrm{K}_{kl\hat{m}\hat{n}} = n\Gamma_1\left[\left(\frac{\lambda}{G}+2\right)\left\langle\frac{P_{kl}}{\psi},\frac{P_{\hat{m}\hat{n}}}{\psi}\right\rangle + \frac{\lambda}{G}\left\langle\frac{P_{kl,\psi}}{\psi},P_{\hat{m}\hat{n}}\right\rangle\right] + n\Gamma_2\left[\left\langle\frac{P_{kl}}{\psi},\frac{P_{\hat{m}\hat{n}}}{\psi}\right\rangle - \left\langle\frac{P_{kl}}{\psi},P_{\hat{m}\hat{n},\zeta}\right\rangle\right],$$

$$\mathrm{M}_{ij\hat{i}\hat{j}} = \Gamma_1\left\langle P_{ij},P_{\hat{i}\hat{j}}\right\rangle, \qquad \mathrm{M}_{kl\hat{k}\hat{l}} = \Gamma_1\left\langle P_{kl},P_{\hat{k}\hat{l}}\right\rangle, \qquad \mathrm{M}_{mn\hat{m}\hat{n}} = \Gamma_2\left\langle P_{mn},P_{\hat{m}\hat{n}}\right\rangle \tag{26}$$

where $P_{\alpha\beta}$ is defined by $P_{\alpha\beta}\equiv P_\alpha(\psi)P_\beta(\zeta)$ ($\alpha=i,k,m$, $\beta=j,l,n$) and $K_{\alpha\beta\hat{\alpha}\hat{\beta}}$ and $M_{\alpha\beta\hat{\alpha}\hat{\beta}}$ ($\alpha=i,k,m$, $\beta=j,l,n$; $\hat{\alpha}=\hat{i},\hat{k},\hat{m},\hat{\beta}=\hat{j},\hat{l},\hat{n}$) denote the submatrices of the stiffness and mass matrices, respectively. The notation of < , > denotes an inner product defined by

$$\langle f,g\rangle \equiv \eta(\psi,\zeta)\int_{-H_t/H}^{H_b/H}\int_{\psi_1}^{\psi_2} f(\psi,\zeta)\, g(\psi,\zeta)\,\psi d\zeta d\psi\,. \tag{27}$$

For a nontrivial solution, the determinant of the coefficient matrix is set equal to zero, which yields the frequencies (eigenvalues); that is to say $|\mathbf{K}-\Lambda\mathbf{M}|=0$, where Λ is a square of non-dimensional frequency as $\omega^2a^2\rho/G$. These frequencies are upper bounds on the exact values.

3. Convergence study

To guarantee the accuracy of frequencies obtained by the procedure described above, it is necessary to conduct some convergence studies to determine the number of terms required in the Legendre polynomials of Eq. (23). A convergence study is based upon the fact that, if the displacements are expressed as power series, all the frequencies obtained by the Ritz method should converge to their exact values in an upper bound manner. If the results do not converge properly, or converge too slowly, it would be likely that the assumed displacement functions chosen are poor ones, or be missing some functions from a minimal complete set of polynomials.

Table 1 is such a study shows the convergence of frequencies $\omega a\sqrt{\rho/G}$ of a hyperboloidal shell free at the top edge and clamped at the bottom edge (F-C) for the five lowest bending modes (n=2) with b/a=3, h/a=0.4, H_b/a=4, and H_t/H_b=1 for ν=0.3. To make the study of convergence less complicated, equal numbers of the Legendre polynomial terms were taken in both the r (or ψ) coordinate (i.e., I=K=M) and z (or ζ) coordinate (i.e., J=L=N), although some computational optimization could be obtained for some configurations and some mode shapes by using unequal

Table 6 Non-dimensional frequencies $\omega a\sqrt{\rho/G}$ of hyperboloidal shells free at the top edge free and clamped at the bottom (F-C) with h/a=0.4 and H_a/a=4 (ν=0.3)

n	***s***	*b/a*=1			*b/a*=3		
		H_t/H_b=0	H_t/H_b=1/4	H_t/H_b=1	H_t/H_b=0	H_t/H_b=1/4	H_t/H_b=1
	1	0.6074	0.4968	**0.07394(3)**	**0.5203(3)**	**0.4091(2)**	**0.1691(2)**
	2	1.082	0.7718	0.6435	1.201	0.9544	0.6227
$\mathbf{0^T}$	**3**	1.631	1.292	0.8640	1.960	1.567	0.9874
	4	2.223	1.815	1.169	2.727	2.184	1.368
	5	2.830	2.324	1.445	3.497	2.803	1.752
	1	**0.4414(5)**	**0.3943(5)**	0.1726	0.6927	0.5541	**0.3085(5)**
	2	0.5151	0.4884	0.3382	1.272	1.271	0.8951
$\mathbf{0^A}$	**3**	0.7410	0.6929	0.4606	1.438	1.370	1.028
	4	0.9739	0.9565	0.4938	1.605	1.543	1.236
	5	1.241	1.138	0.6213	1.741	1.600	1.275
	1	**0.3308(3)**	**0.3065(2)**	**0.04641(1)**	**0.2866(1)**	**0.2012(1)**	**0.06952(1)**
	2	0.4651	**0.3268(4)**	0.1885	**0.6377(4)**	**0.4925(4)**	**0.2614(4)**
1	**3**	0.5577	0.4904	0.3396	1.080	0.9063	0.5349
	4	0.7960	0.6738	0.3796	1.300	1.155	0.7831
	5	0.9846	0.8992	0.5221	1.630	1.344	0.9175
	1	**0.2577(1)**	**0.2652(1)**	**0.05645(2)**	**0.4672(2)**	**0.4710(3)**	**0.2496(3)**
	2	0.4493	0.4615	0.2276	**0.6448(5)**	**0.5488(5)**	0.4567
2	**3**	0.6465	0.5291	0.2805	1.082	0.8437	0.5191
	4	0.9079	0.6742	0.4647	1.605	1.247	0.7229
	5	1.156	0.9421	0.5007	1.878	1.689	0.9597
	1	**0.3234(2)**	**0.3260(3)**	**0.1022(4)**	0.9636	0.9658	0.5908
	2	0.5980	0.6078	0.3203	1.226	1.214	0.9666
3	**3**	0.8932	0.8914	0.3276	1.459	1.311	0.9885
	4	1.188	0.9532	0.5999	1.905	1.611	1.249
	5	1.439	1.236	0.6158	2.443	1.998	1.342
	1	**0.4290(4)**	0.4299	**0.1687(5)**	1.496	1.496	1.018
	2	0.7581	0.7607	0.4318	1.922	1.927	1.497
4	**3**	1.116	1.122	0.4380	2.163	2.082	1.552
	4	1.493	1.412	0.7597	2.451	2.263	1.933
	5	1.752	1.516	0.7642	2.910	2.560	1.983
	1	0.5589	0.5600	0.2524	2.067	2.067	1.504
	2	0.9337	0.9356	0.5625	2.585	2.586	2.069
5	**3**	1.333	1.336	0.5769	2.974	2.978	2.147
	4	1.753	1.763	0.9366	3.183	3.014	2.595
	5	2.037	1.956	0.9399	3.524	3.295	2.655

Notes: *T*=Torsional mode; *A*=Axisymmetric mode.
Numbers in parentheses identify frequency sequence.

number of the polynomial terms.

The symbols ***TR*** and ***TZ*** in the table indicate the total numbers of the Legendre polynomial terms used in the r (or ψ) and z (or ζ) directions, respectively. Note that the frequency determinant order ***DET*** is related to ***TR*** and ***TZ*** as follows

$$DET = \begin{Bmatrix} TR \times TZ & \text{for torsional modes } (n = 0^{\mathrm{T}}) \\ 2 \times TR \times TZ & \text{for axisymmetric modes } (n = 0^{\mathrm{A}}) \\ 3 \times TR \times TZ & \text{for general modes } (n \geq 1) \end{Bmatrix}. \quad (28)$$

Table 1 shows the monotonic convergence of all five frequencies as ***TR*** (=I+1, K+1 and M+1 in Eq. (23)) are increased, as well as ***TZ*** (=J+1, L+1 and N+1 in Eq. (23)). One sees, for example, that the fundamental (i.e., lowest) nondimensional frequency $\omega a\sqrt{\rho/G}$ converges to four digits (0.2496) when as few as (5×11) terms are used, which results in ***DET***=165. Moreover, this accuracy requires using at least five terms through the radial (***TR***=5) and 11 terms through the axial direction (***TZ***=11). Underlined, bold-faced values in Table 1 represent the converged results (up to four significant figures) achieved with the smallest determinant size. Table 2 shows the convergence study using the algebraic simple polynomials to compare with Table 1 using the Legendre polynomials. The frequencies with * stand for those encountering the numerical instability. When using the Legendre polynomials as admissible functions, the frequencies are converged with a smaller size of determinant compared with simple polynomials.

4. Comparisons

Nearly straight beams with solid circular cross-section with $b/a\to\infty$ and $a\to h/2$ and nearly complete conical shells with H_t=0 and $a\to h/2$ can be special cases of hyperboloidal shells of revolution. Table 3 shows comparisons of the first three non-dimensional frequencies $\omega l^2\sqrt{\rho A/EI}$ of a straight beam (l/r=500) for bending modes (n=1) from the present analysis and the classical beam theory (Young and Felgar 1949), where EI is the flexural rigidity of a beam, I is the area moment of inertia, l is beam length, A is area of cross-section, and r is radius of circular cross-section. Table 3 shows good agreement in frequencies from the two different methods on the whole. But it is strange that most of the results from the present 3-D analysis are a little bit higher than those from the study by Young and Felgar (1949), since an accurate 3-D analysis should typically yield lower frequencies than the classical beam theory, mainly because shear deformation and rotary inertia effects are accounted for in a 3-D analysis, but not in the classical beam theory.

Dreher and Leissa (1968) used the exact solution procedure involving expansion of thedisplacements in terms of power series to study the axisymmetric (n=0^A) free vibrations of complete (not truncated) conical shells, where the Donnell-Mushtari shell theory was used. They used K=$12(1-\nu^2)R^2\cos^2\alpha/h^2\sin^4\alpha$ as a shell stiffness parameter, where α is vertex half angle, h is shell thickness, and R is the radius of mid-surface of a cone at the bottom face. Table 4 shows comparisons of the first three non-dimensional frequencies $\overline{\Omega}^2=\omega^2R^2\rho/E\cos^2\alpha$ for axisymmetric mode (n=0^A) of complete conical shell with α=45° and h/R=0.01478 (K=100,000) from 2D thin shell theory (Dreher and Leissa 1968) and nearly complete conical shell with b/a=1, h/a=1.98, H_b/a=134.0, and H_t/H_b=0 from the present 3-D Ritz method for ν=0.3. All the frequencies from the present 3-D Ritz method are smaller than those from 2-D thin shell theory (Dreher and Leissa 1968) as expected.

Loy and Lam (1999) presented an approximate analysis using a layerwise approach to study the vibration of thick circular cylindrical shells on the basis of 3-D theory of elasticity. Table 5 shows

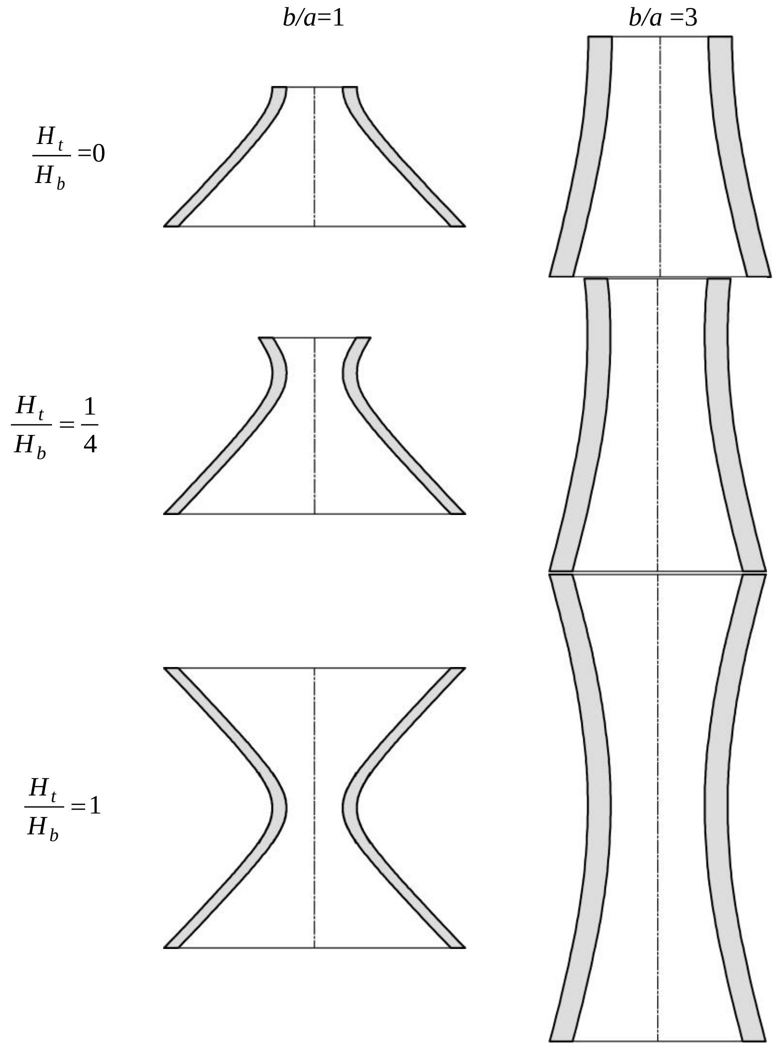

Fig. 2 Hyperboloidal shells h/a=0.4 and H_b/a=4

comparisons of the first non-dimensional frequencies $\omega a\sqrt{(1-\nu^2)\rho/E}$ for n=0^A, 1, and 2 of clamped-clamped (C-C) hollow circular cylindrical shells as a special case of the hyperboloidal shells (b/a=1000) with h/a=1 and ν=0.3 from the 3-D theory by Loy and Lam (1999) and from the present 3-D Ritz method. All the frequencies shows good agreement.

5. Numerical results

Table 6 presents the non-dimensional frequencies $\omega a\sqrt{\rho/G}$ of hyperboloidal shells free at the top edge and fixed at the bottom edge (F-C) and with h/a=0.4 and H_b/a=4 for ν=0.3. The shell configurations for Table 6 are shown in Fig. 2. Thirty five frequencies are given for each shell configuration, which arise from seven circumferential wave numbers (n=0^T, 0^A, 1, 2, 3, 4, 5) and the first five modes (s=1, 2, 3, 4, 5) for each value of n, where the superscripts T and A indicate torsional and axisymmetric modes, respectively. The numbers in parentheses identify the first five frequencies for each shell configuration. The zero frequencies of rigid body modes are omitted from the tables. It is seen that most of the frequencies become larger as H_t/H_b decreases. It is also observed that the torsional (n=0^T) modes are more important as the values of H_t/H_b and a/b become larger. That is, they are among the lowest frequencies of the shells.

6. Conclusions

Accurate frequency data determined by the 3-D Ritz analysis using the Legendre as admissible functions have been presented for hyperboloidal shells free at the top edge and clamped at the bottom edge (F-C) like a cooling tower. The analysis uses the 3-D equations of the theory of elasticity in their general forms for isotropic materials. They are only limited to small strains. No other constraints are placed upon the displacements. This is in stark contrast with the classical 2-D thin shell theories, which make very limiting assumptions about the displacement variation through the shell thickness.

The method is capable of determining frequencies as close to the exact ones as desired. Therefore, the data in Table 6 may be regarded as benchmark results against which 3-D results obtained by other methods, such as finite elements and finite differences, may be compared to determine the accuracy of the latter. Moreover, the frequency determinants required by the present method are at least an order of magnitude smaller than those needed by finite element analyses of comparable accuracy. This was demonstrated extensively in a paper by McGee and Leissa (1991). The Ritz method guarantees upper bound convergence of the frequencies in terms of functions sets that are mathematically complete, such as algebraic polynomials. Some finite element methods can also accomplish this, but at much greater costs, and others cannot.

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