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A new nonlocal hyperbolic shear deformation theory for nanobeams embedded in an elastic medium

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Abstract. This work presents a new nonlocal hyperbolic shear deformation beam theory for the static, buckling and vibration of nanoscale-beams embedded in an elastic medium. The present model is able to capture both the nonlocal parameter and the shear deformation effect without employing shear correction factor. The nonlocal parameter accounts for the small size effects when dealing with nanosize structures such as nanobeams. Based on the nonlocal differential constitutive relations of Eringen, the equations of motion of the nanoscale-beam are obtained using Hamilton's principle. The effect of the surrounding elastic medium on the deflections, critical buckling loads and frequencies of the nanobeam is investigated. Both Winkler-type and Pasternak-type foundation models are used to simulate the interaction of the nanobeam with the surrounding elastic medium. Analytical solutions are presented for a simply supported nanoscale-beam, and the obtained results compare well with those predicted by the other nonlocal theories available in literature.

Keywords: nonlocal theory; nanobeam; elastic medium

1. Introduction

The great advancement in the application of nanostructures in nano-engineering industries, mainly MEMS/NEMS devices, because of their higher physical properties, rendered a sudden momentum in modeling the structures of nano length scale. It has been remarked that there is a significant difference in the structural response of material at nano-scale when compared to their bulk counterpart. The problem in employing the classical theory is that this one is not able to capture the size influences. The classical theory overpredicts the behavior of nanoscale structures. In addition, in the classical model, the particles influence one another by contact forces. Thus, at nanoscale the size influences cannot be neglected. Therefore size-dependent continuum based

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theories (Zhou and Li 2001, Fleck and Hutchinson 1997, Yang *et al*. 2002) are becoming popular in modeling small sized structures as it provides much faster solutions than molecular dynamics simulations for various engineering problems. Furthermore, the nonlocal size-dependent continuum mechanics models are used because at small length scales, the material microstructures (such as lattice spacing between individual atoms) become increasingly important and its effect can no longer be neglected.

The most widely used theory for investigating small scale structures is the nonlocal elasticity theory initiated by Eringen (1972, 1983). In nonlocal elasticity model the small-scale influences are introduced by considering that the stress at a point as a function not only of the strain at that point but also a function of the strains at all other points of the continuum domain. Such constitutive theories account for the forces between atoms and the internal length scale. Based on the nonlocal constitutive relation of Eringen, a number of papers have been reported attempting to develop nonlocal beam theories and use them to investigate the bending (Wang and Liew 2007, Pijaudier-Cabot and Bazant 1987, Reddy and Pang 2008, Larbi Chaht *et al*. 2015), buckling (Zhang *et al*. 2004, Zhang *et al*. 2006, Wang *et al*. 2006, Murmu and Pradhan 2009a, Amara *et al*. 2010, Narendar and Gopalakrishnan 2011a, Tounsi *et al*. 2013a,b, Zidour *et al*. 2014, Benguediab *et al*. 2014a, Berrabah *et al*. 2013, Eltaher *et al*. 2014, Adda Bedia *et al*. 2015), vibration (Zhang *et al*. 2005, Benzair *et al*. 2008, Murmu and Pradhan 2009b, 2010, Eltaher *et al*. 2012, 2013, Belkorissat *et al*. 2015, Besseghier *et al*. 2015, Zemri *et al*. 2015), wave propagation (Lu *et al*. 2007, Tounsi *et al*. 2008, Heireche *et al*. 2008a, b, Song *et al*. 2010, Narendar and Gopalakrishnan 2011b), and thermo-mechanical (Mustapha and Zhong 2010, Zidour *et al*. 2012) responses of nanobeams.

Mechanical response of the nanobeams combined with the effect of the surrounding elastic medium is of practical importance. Recently, considerable attention has been turned to the mechanical behavior of carbon nanotubes embedded in polymer or metal matrix (Ru 2001, Kuzumaki *et al*. 1998, Schadler *et al*. 1998, Wagner *et al*. 1998, Bower *et al*. 1999, Qian *et al*. 2000, Besseghier *et al*. 2011). Besseghier *et al*. (2011) employed Winkler-type model for wave propagation analysis of double-walled carbon nanotubes embedded in an elastic medium. Further, Pradhan and Murmu (2009a) investigated vibration analysis of beam embedded by elastic medium by employing nonlocal beam theory and Winkler foundation model. It should be noted that these works were based on nonlocal Euler–Bernoulli beam model. However, the Winkler-type model is regarded as a crude approximation of the real mechanical response of the elastic material, and this is due to the inability of the model to consider the continuity or cohesion of the medium. A more realistic representation of the elastic medium can be reached by considering a two-parameter foundation model. One such mechanical foundation model is the Pasternak-type foundation model (Pasternak 1954) which is often called as two-parameter foundation model. The first parameter of this model considers the normal pressure, while the second parameter introduces the transverse shear stress due to interaction of shear deformation of the surrounding elastic medium (Bouderba *et al*. 2013). Winkler and Pasternak foundation models were reported by Pradhan and Murmu (2009b) for the vibration investigation of single-walled carbon nanotubes embedded in polymer matrix. In this work, authors used only the nonlocal Timoshenko beam model. It is clear from the literature investigation that the employ of two-parameter foundation model for modelling the surrounding elastic medium are limited in literature.

In the present work, a new nonlocal hyperbolic shear deformation beam theory is developed for the static, buckling and vibration of nanoscale-beams embedded in an elastic medium. The most interesting feature of this theory is that it accounts for a hyperbolic variation of the transverse shear strains across the thickness and satisfies the zero traction boundary conditions on the top and bottom surfaces of the nanobeam without using shear correction factors. Contrary to the hyperbolic shear deformation theory proposed by Benguediab *et al*. (2014b), it is noted that the present theory uses a new and different hyperbolic function to that presented by Benguediab *et al*. (2014b). In addition, the present nonlocal hyperbolic shear deformation theory is based on the partition of the transverse displacement into the bending and shear parts. Both Winkler-type and Pasternak-type models are used to simulate the interaction of the nanobeams with a surrounding elastic medium. Based on the nonlocal constitutive relations of Eringen, equations of motion of nanoscale beams are obtained by employing Hamilton's principle. Effect of the nonlocal parameter, Winkler modulus parameter, Pasternak shear modulus parameter, and aspect ratio of the nanoscale beam on deflection, critical buckling load and frequency of the nanobeam are investigated and discussed. It is hoped that the present analysis will be useful to researchers and engineers working on nanostructures.

2. Mathematical formulations

2.1 Kinematics

Based on work presented by Berrabah *et al.* (2013), Ould Larbi *et al*. (2013) and Al-Basyouni *et al*. (2015), the displacement field of the present theory can be obtained as

$$u(x,z,t) = -z\frac{\partial w_b}{\partial x} - f(z)\frac{\partial w_s}{\partial x} \tag{1a}$$

$$w(x,z,t) = w_b(x,t) + w_s(x,t) \tag{1b}$$

where w_b and w_s are the bending and shear components of transverse displacement along the mid-plane of the beam. In this work, a novel shape function $f(z)$ is proposed based on a hyperbolic function as

$$f(z) = \frac{h\sinh\left(\frac{10z}{h}\right)}{10\cosh(5)} - \frac{h}{100} \tag{2}$$

The non-zero strains are given by

$$\varepsilon_x = z\,k_x^b + f(z)\,k_x^s \tag{3a}$$

$$\gamma_{xz} = g(z)\,\gamma_{xz}^s \tag{3b}$$

where

$$k_x^b = -\frac{\partial^2 w_b}{\partial x^2},\quad k_x^s = -\frac{\partial^2 w_s}{\partial x^2},\quad \gamma_{xz}^s = \frac{\partial w_s}{\partial x} \tag{3c}$$

$$g(z) = 1 - f'(z) \text{ and } f'(z) = \frac{df(z)}{dz} \tag{3d}$$

2.2 Constitutive relations

Behavior of materials at the nanoscale is different from those of their bulk counterparts. In the model of nonlocal elasticity (Eringen 1972, 1983), the stress at a reference point ***x*** is assumed to be a functional of the strain field at every point in the body. For example, in the non-local elasticity, the uniaxial constitutive law is expressed as (Eringen 1972, 1983, Heireche *et al*. 2008)

$$\sigma_x - \mu \frac{d^2\sigma_x}{dx^2} = E\varepsilon_x \tag{4a}$$

$$\tau_{xz} - \mu \frac{d^2\tau_{xz}}{dx^2} = G\gamma_{xz} \tag{4b}$$

where E and G are the elastic modulus and shear modulus of the nanobeam, respectively; $\mu=(e_0 a)^2$ is the nonlocal parameter, e_0 is a constant appropriate to each material and a is an internal characteristic length.

2.3 Equations of motion

In this section, Hamilton's principle is employed to derive the equations of motion (Ait Yahia *et al*. 2015, Bourada *et al*. 2015, Mahi *et al*. 2015, Nedri *et al*. 2014, Hebali *et al*. 2014, Belabed *et al*. 2014, Draiche *et al*. 2014, Benachour *et al*. 2011, Bessaim *et al*. 2013, Bourada *et al*. 2012)

$$\int_0^T (\delta U + \delta V - \delta K) dt = 0 \tag{5}$$

where δU is the virtual variation of the strain energy; δV is the virtual variation of the potential energy; and δK is the virtual variation of the kinetic energy. The variation of the strain energy of the beam can be stated as

$$\begin{aligned} \delta U &= \int_0^L \int_A (\sigma_x \delta \varepsilon_x + \tau_{xz} \delta \gamma_{xz}) dA dx \\ &= \int_0^L \left(-M_b \frac{d^2 \delta w_b}{dx^2} - M_s \frac{d^2 \delta w_s}{dx^2} + Q \frac{d\delta w_s}{dx} \right) dx \end{aligned} \tag{6}$$

where M_b, M_s and Q are the stress resultants defined as

$$(M_b, M_s) = \int_A (z, f)\, \sigma_x dA \text{ and } Q = \int_A g \tau_{xz} dA \tag{7}$$

The variation of the potential energy by the applied loads can be written as

$$\delta V = -\int_0^L (q + f_e) \delta(w_b + w_s) dx - \int_0^L N_0 \frac{d(w_b + w_s)}{dx} \frac{d\delta(w_b + w_s)}{dx} dx \tag{8}$$

where q and N_0 are the transverse and axial loads, respectively. The Winkler-type and Pasternak-type models are utilized to simulate the interaction of the nanobeams with a surrounding elastic medium as follows (Ait Atmane *et al*. 2010, Bouderba *et al*. 2013, Zidi *et al*. 2014, Khalfi *et al*. 2014, Ait Amar Meziane *et al*. 2014)

$$f_e = k_w w - k_s \frac{\partial^2 w}{\partial x^2} \tag{9}$$

with k_w and k_g are the Winkler and the Pasternak moduli of the surrounding elastic medium, respectively.

The variation of the kinetic energy can be expressed as

$$\begin{aligned}\delta K &= \int_0^L \int_A \rho [\dot{u}\delta \dot{u} + \dot{w}\delta \dot{w}] dA dx \\ &= \int_0^L \{ I_0 (\dot{w}_b + \dot{w}_s)(\delta \dot{w}_b + \delta \dot{w}_s) + I_2 \left(\frac{d\dot{w}_b}{dx} \frac{d\delta \dot{w}_b}{dx} \right) + K_2 \left(\frac{d\dot{w}_s}{dx} \frac{d\delta \dot{w}_s}{dx} \right) \\ &\quad + J_2 \left(\frac{d\dot{w}_b}{dx} \frac{d\delta \dot{w}_s}{dx} + \frac{d\dot{w}_s}{dx} \frac{d\delta \dot{w}_b}{dx} \right) \} dx\end{aligned} \tag{10}$$

where dot-superscript convention indicates the differentiation with respect to the time variable t; ρ is the mass density; and (I_0, I_2, J_2, K_2) are the mass inertias expressed as

$$(I_0, I_2, J_2, K_2) = \int_A (1, z^2, z\, f, f^2) \rho \, dA \tag{11}$$

By introducing Eqs. (6), (8), and (10) into Eq. (5) and integrating by parts versus both space and time variables, and collecting the coefficients of δu_0, δw_b, and δw_s, the following equations of motion of the nanobeam are obtained

$$\delta w_b : \frac{d^2 M_b}{dx^2} + q - f_e - N_0 \frac{d^2 (w_b + w_s)}{dx^2} = I_0 (\ddot{w}_b + \ddot{w}_s) - I_2 \frac{d^2 \ddot{w}_b}{dx^2} - J_2 \frac{d^2 \ddot{w}_s}{dx^2} \tag{12a}$$

$$\delta w_s : \frac{d^2 M_s}{dx^2} + \frac{dQ}{dx} + q - f_e - N_0 \frac{d^2 (w_b + w_s)}{dx^2} = I_0 (\ddot{w}_b + \ddot{w}_s) - J_2 \frac{d^2 \ddot{w}_b}{dx^2} - K_2 \frac{d^2 \ddot{w}_s}{dx^2} \tag{12b}$$

and the boundary conditions of the present model are

$$\text{specify } w_b \text{ or } V_b \equiv \frac{dM_b}{dx} - (k_s + N_0) \frac{d(w_b + w_s)}{dx} + I_2 \frac{d\ddot{w}_b}{dx} + J_2 \frac{d\ddot{w}_s}{dx} \tag{12c}$$

$$\text{specify } w_s \text{ or } V_s \equiv \frac{dM_s}{dx} + Q - (k_s + N_0) \frac{d(w_b + w_s)}{dx} + J_2 \frac{d\ddot{w}_b}{dx} + K_2 \frac{d\ddot{w}_s}{dx} \tag{12d}$$

$$\text{specify } \frac{dw_b}{dx} \text{ or } M_b \tag{12e}$$

$$\text{specify } \frac{dw_s}{dx} \text{ or } M_s \tag{12f}$$

when the shear deformation effect is omitted (w_s=0), the equilibrium equations in Eq. (12) become those derived from the Euler–Bernoulli beam theory.

By substituting the stress-strain relations expressed by Eq. (4) into the definitions of force and moment resultants given in Eq. (7) the following constitutive equations are obtained

$$M_b - \mu \frac{d^2 M_b}{dx^2} = -D \frac{d^2 w_b}{dx^2} - D_s \frac{d^2 w_s}{dx^2} \tag{13a}$$

$$M_s - \mu \frac{d^2 M_s}{dx^2} = -D_s \frac{d^2 w_b}{dx^2} - H_s \frac{d^2 w_s}{dx^2} \tag{13b}$$

$$Q - \mu \frac{d^2 Q}{dx^2} = A_s \frac{dw_s}{dx} \tag{13c}$$

where

$$(D, D_s, H_s) = \int_A (z^2, z\, f,\, f^2) E dA, \quad A_s = \int_A g^2 G dA \tag{13d}$$

By substituting Eq. (13) into Eq. (12), the nonlocal equations of motion can be written in terms of displacements (w_b, w_s) as

$$\begin{aligned} &-D \frac{d^4 w_b}{dx^4} - D_s \frac{d^4 w_s}{dx^4} + q - \mu \frac{d^2 q}{dx^2} - f_e + \mu \frac{d^2 f_e}{dx^2} - N_0 \left(\frac{d^2 (w_b + w_s)}{dx^2} - \mu \frac{d^4 (w_b + w_s)}{dx^4} \right) \\ &= I_0 \left((\ddot{w}_b + \ddot{w}_s) - \mu \frac{d^2 (\ddot{w}_b + \ddot{w}_s)}{dx^2} \right) - I_2 \left(\frac{d^2 \ddot{w}_b}{dx^2} - \mu \frac{d^4 \ddot{w}_b}{dx^4} \right) - J_2 \left(\frac{d^2 \ddot{w}_s}{dx^2} - \mu \frac{d^4 \ddot{w}_s}{dx^4} \right) \end{aligned} \tag{14a}$$

$$\begin{aligned} &-D_s \frac{d^4 w_b}{dx^4} - H_s \frac{d^4 w_s}{dx^4} + A_s \frac{d^2 w_s}{dx^2} + q - \mu \frac{d^2 q}{dx^2} - f_e + \mu \frac{d^2 f_e}{dx^2} - N_0 \left(\frac{d^2 (w_b + w_s)}{dx^2} - \mu \frac{d^4 (w_b + w_s)}{dx^4} \right) \\ &= I_0 \left((\ddot{w}_b + \ddot{w}_s) - \mu \frac{d^2 (\ddot{w}_b + \ddot{w}_s)}{dx^2} \right) - J_2 \left(\frac{d^2 \ddot{w}_b}{dx^2} - \mu \frac{d^4 \ddot{w}_b}{dx^4} \right) - K_2 \left(\frac{d^2 \ddot{w}_s}{dx^2} - \mu \frac{d^4 \ddot{w}_s}{dx^4} \right) \end{aligned} \tag{14b}$$

The equations of motion of local beam theory can be determined from Eq. (14) by taking the scale parameter μ equal to zero.

3. Analytical solutions

In this section, explicit solutions of deflection, critical buckling and frequency as a function of geometrical parameters, material constants, Winkler modulus parameter, Pasternak shear modulus parameter, and nonlocal scale parameter are presented. The assumed displacement field in the case of simply-supported beams can be described by the following harmonic functions that satisfy the boundary conditions

$$\begin{Bmatrix} w_b \\ w_s \end{Bmatrix} = \sum_{n=1}^{\infty} \begin{Bmatrix} W_{bn} \sin(\alpha\, x)\, e^{i\omega t} \\ W_{sn} \sin(\alpha\, x)\, e^{i\omega t} \end{Bmatrix} \tag{15a}$$

where W_{bn}, and W_{sn} are arbitrary parameters to be determined, ω is the eigenfrequency associated with nth eigenmode, and $\alpha = n\pi/L$. It is known that the boundary bending moment and/or shear force conditions for nonlocal beam models are different from those for their classical counterpart beam theories as discussed by Lu (2007). The simply supported boundary conditions are specified by w=0 and M_b=0, where the nonlocal M_b is the bending moment given from in Eqs. (12a), (13a) as follows

$$\begin{aligned} M_b = -D\frac{d^2 w_b}{dx^2} - D_s\frac{d^2 w_s}{dx^2} + \mu\Bigg[-q + f_e + N_0\frac{d^2(w_b + w_s)}{dx^2} \\ + I_0(\ddot{w}_b + \ddot{w}_s) - I_2\frac{d^2\ddot{w}_b}{dx^2} - J_2\frac{d^2\ddot{w}_s}{dx^2}\Bigg] = 0 \quad \text{at } x = 0,\ L \end{aligned} \tag{15b}$$

The transverse load q is also expanded in the Fourier sine series as

$$q(x) = \sum_{n=1}^{\infty} Q_n \sin\alpha\, x, \quad Q_n = \frac{2}{L}\int_0^L q(x)\sin(\alpha\, x)\, dx \tag{16}$$

The Fourier coefficients Q_n associated with some typical loads are given

$$Q_n = q_0, \quad n = 1 \quad \text{for sinusoidal load,} \tag{17a}$$

$$Q_n = \frac{4q_0}{n\pi}, \quad n = 1,3,5..... \text{ for uniform load,} \tag{17b}$$

$$Q_n = \frac{2q_0}{L}\sin\frac{n\pi}{2}, \ n = 1,2,3.... \text{ for point load } Q_0 \text{ at the midspan,} \tag{17c}$$

Substituting the expansions of w_b, w_s, and q from Eqs. (15) and (16) into Eq. (14), the closed-form solutions can be obtained from the following equations

$$\left(\begin{bmatrix} S_{11} & S_{12} \\ S_{12} & S_{22} \end{bmatrix} + \lambda\left(k_w + k_s\alpha^2\right)\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \lambda N_0\alpha^2 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} - \lambda\omega^2 \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix} \right) \begin{Bmatrix} W_{bn} \\ W_{sn} \end{Bmatrix} = \begin{Bmatrix} \lambda Q_n \\ \lambda Q_n \end{Bmatrix} \tag{18}$$

where

$$\begin{aligned} S_{11} = D\alpha^4, \ S_{12} = D_s\alpha^4, \ S_{22} = H_s\alpha^4 + A_s\alpha^2, \ \lambda = 1 + \mu\alpha^2 \\ m_{11} = I_0 + I_2\alpha^2, \ m_{12} = I_0 + J_2\alpha^2, \ m_{22} = I_0 + K_2\alpha^2 \end{aligned} \tag{19}$$

3.1 Bending

The static deflection is obtained from Eq. (18) by setting N_0 and all time derivatives to zero

$$w(x)=\sum_{n=1}^{\infty}\frac{(S_{11}+S_{22}-2S_{12})}{(S_{11}S_{22}-S_{12}^2)}\lambda Q_n \sin\alpha x \tag{20}$$

3.2 Buckling

The buckling load is obtained from Eq. (18) by setting q and all time derivatives to zero

$$N_0=\frac{S_{11}S_{22}-S_{12}^2}{\lambda\alpha^2\left(S_{11}-2S_{12}+S_{22}\right)} \tag{21}$$

3.3 Vibration

By setting q and N_0 in Eq. (18) equal to zero, the natural frequency can be obtained from the following equation

$$\left(m_{11}m_{22}-m_{12}^2\right)\lambda^2\omega^4+\left(2S_{12}m_{12}-S_{11}m_{22}-S_{22}m_{11}\right)\lambda\omega^2+\left(S_{11}S_{22}-S_{12}^2\right)=0 \tag{22}$$

4. Numerical results

Numerical results illustrated in this section demonstrate the influence of shear deformation functions, nonlocal parameter, the foundation parameters, and slenderness ratio on deflections, critical buckling loads and frequencies of nanobeams. For all calculations, the Poisson's ratio is taken as 0.3. However, for calculation carried out using Timoshenko beam theory (TBT), the shear correction factor is taken as 5/6. It can be noted that, since the TBT violates the conditions of zero transverse shear stresses on the top and bottom surfaces of the beam, a shear correction factor which depends on many parameters is required to compensate for the error due to a constant shear strain assumption through the thickness. The higher-order shear deformation theories (HSDTs) such as the present nonlocal hyperbolic shear deformation theory account for the shear deformation effects, and satisfy the zero transverse shear stresses on the top and bottom surfaces of the beam, thus, a shear correction factor is not required. The side of nanobeam L is assumed to be 10 nm. For convenience, the following nondimensional quantities are employed in the current study:

- $\bar{w}=100w\dfrac{EI}{q_0L^4}$ for uniform load;
- $\bar{N}=N_{cr}\dfrac{L^2}{EI}$ critical buckling load parameter;
- $\bar{\omega}=\omega L\sqrt{\dfrac{I_0}{EI}}$ frequency parameter;
- $K_w=\dfrac{k_wL^4}{EI}$ Winkler parameter;

- $K_s = \dfrac{k_s L^4}{EI}$ Pasternak parameter.

4.1 Validation of the results

To show the validity and the accuracy of the present formulations, results for static, buckling and free vibration using the present new theory, have been performed out with the results of the available published works.

The results for deflections of nanobeam under uniform load are documented in Table 1. Results are computed by utilizing 100 terms in the series Eq. (20). The obtained results are compared with those of Euler-Bernoulli beam theory (EBT), Timoshenko beam theory (TBT) and Reddy beam theory (RBT). It can be observed that the present results are in good agreement with those predicted by TBT and RBT for all values of slenderness L/h and nonlocal scale parameter μ. However, the small differences between the results obtained by the present theory and RBT are due to the used function. This may be explained by the fact that the hyperbolic function is very much richer than the cubic function used in RBT. This point is also discussed by Idlbi *et al*. (1997) for the sinusoidal function. Our results are also in good agreement with those obtained by Berrabah *et al*. (2013). For all theories, it is seen that the deflection increases as the nonlocal scale parameter increases at a specified slenderness ratio. Moreover, for high slenderness (L/h=100) ratio, all theories are approximately identical in predicting the deflection, which confirms the accuracy of the Euler-Bernoulli theory in the case of thin nanobeams. However, it is noted that for very thin beams the shear locking problem is often found when the numerical approaches are used.

Table 1 Comparison of dimensionless maximum center deflection under uniform load for simply supported nanobeams (K_w=K_s=0)

μ	L/h	EBT	TBT	RBT	Present
0	5	1.302083332	1.432083343	1.43195457	1.428485570
	10	1.302083332	1.334583333	1.33457528	1.333693217
	20	1.302083332	1.310208335	1.31020782	1.309986384
	100	1.302083332	1.310208332	1.30240832	1.302399463
1	5	1.427083338	1.567516123	1.56735522	1.563603269
	10	1.427083338	1.462191536	1.46217859	1.461223604
	20	1.427083338	1.435860392	1.43585833	1.435619542
	100	1.427083338	1.427434425	1.42743441	1.427424835
2	5	1.552083348	1.702948909	1.70275587	1.698720972
	10	1.552083348	1.589799737	1.58977651	1.588753993
	20	1.552083348	1.561512444	1.56150884	1.561252708
	100	1.552083348	1.552460510	1.55246049	1.552450205
3	5	1.677083354	1.838381690	1.83815652	1.833838667
	10	1.677083354	1.717407940	1.71737713	1.716284376
	20	1.677083354	1.687164510	1.68715934	1.686885860
	100	1.677083354	1.677486600	1.67748656	1.677475585
4	5	1.802083363	1.973814474	1.97355717	1.968956364
	10	1.802083363	1.845016141	1.84497774	1.843814764
	20	1.802083363	1.821816558	1.81280985	1.812519017
	100	1.802083363	1.802512690	1.80251265	1.802500960

Table 2 Comparison of dimensionless critical buckling load for simply supported nanobeams (K_w=K_s=0)

μ	L/h	EBT	TBT	RBT	Present
0	5	9.869604404	8.950853970	8.951871004	8.974047798
	10	9.869604404	9.622677158	9.622750681	9.629280748
	20	9.869604404	9.806692087	9.806696870	9.808400323
	100	9.869604404	9.867072414	9.867072422	9.867141506
1	5	8.983016238	8.146797307	8.147722982	8.167907631
	10	8.983016238	8.758270509	8.758337429	8.764280898
	20	8.983016238	8.925755354	8.925759696	8.927310132
	100	8.983016238	8.980711698	8.980711706	8.980774586
2	5	8.242583614	7.475290722	7.476140097	7.494661010
	10	8.242583614	8.036362732	8.036424136	8.041877711
	20	8.242583614	8.190042498	8.190046485	8.191469124
	100	8.242583614	8.240469024	8.240469030	8.240526726
3	5	7.614917659	6.906053489	6.906838185	6.923948745
	10	7.614917659	7.424400328	7.424457054	7.429495345
	20	7.614917659	7.566377508	7.566381185	7.567695495
	100	7.614917659	7.612964091	7.612964099	7.613017402
4	5	7.076079994	6.417375620	6.418104790	6.434004594
	10	7.076079994	6.899043823	6.899096537	6.903778313
	20	7.076079994	7.030974581	7.030978001	7.032199309
	100	7.076079994	7.074264665	7.074264671	7.074314201

Table 3 Comparison of dimensionless frequency for simply supported nanobeams (K_w=K_s=0)

μ	L/h	EBT	TBT	RBT	Present
0	5	9.711154959	9.274039718	9.274524576	9.285413206
	10	9.829265945	9.707477241	9.707513458	9.710755916
	20	9.859473249	9.828127157	9.828129833	9.828979632
	100	9.869198561	9.867932739	9.867932748	9.867967287
1	5	9.264715845	8.847695570	8.848158141	8.858546197
	10	9.377397054	9.261207197	9.261241753	9.264335148
	20	9.406215678	9.376310614	9.376312881	9.377123898
	100	9.415493897	9.414286272	9.414286280	9.414319232
2	5	8.874679788	8.475215682	8.475658783	8.485609512
	10	8.982617220	8.871318852	8.871351954	8.874315118
	20	9.010222607	8.981576525	8.981578701	8.982355570
	100	9.019110221	9.017953440	9.017953449	9.017985006
3	5	8.530090071	8.146136526	8.146562422	8.156126778
	10	8.633836471	8.526859633	8.526891455	8.529739564
	20	8.660369980	9.632836179	8.632838264	8.633584980
	100	8.668912503	8.667800637	8.667800645	8.667830981
4	5	8.222755525	7.852635618	7.853046169	7.862265926
	10	8.322763999	8.219641480	8.219672150	8.222417645
	20	8.348341519	8.321799749	8.321801759	8.322521570
	100	8.356576261	8.355504449	8.355504457	8.355533705

This problem is solved by several researchers such as Reddy (1997).

Tables 2 and 3 present respectively the variation of critical buckling load and frequency with

Table 4 Comparison of the first three nondimensional frequencies $\bar{\omega}$ of simply supported nanobeam (L/h=5, K_w=K_w=0)

Modes (n)	μ	EBT	TBT	RBT	Present
1	0	9.711154958	9.274039718	9.274524576	9.285413206
	1	9.264715840	8.847695570	8.848158141	8.858546197
	2	8.874679787	8.475215682	8.475658783	8.485609512
	3	8.530090070	8.146136526	8.146562422	8.156126778
	4	8.222755528	7.852635618	7.853046169	7.862265926
2	0	37.11199316	32.16650095	32.18471414	32.28600171
	1	31.42394991	27.23643837	27.25186007	27.33762361
	2	27.74215119	24.04527099	24.05888581	24.13460082
	3	25.11035858	21.76418739	21.77651062	21.84504284
	4	23.10878430	20.02934007	20.04068100	20.10375044
3	0	78.02342092	61.45806331	61.57462785	61.84593057
	1	56.77976434	44.72470331	44.80953050	45.00696486
	2	46.82458071	36.88313089	36.95308533	37.11590358
	3	40.75681631	32.10362950	32.16451890	32.30623834
	4	36.56566000	28.80230858	28.85693653	28.98408249

respect to nonlocal parameter and slenderness ratios. The obtained results are also compared with those of EBT, TBT and RBT. It can be noticed from this investigation that the present results are in good agreement with other theories. For all theories, it is seen that at a specified slenderness ratio both the buckling load and frequency decrease as the nonlocal parameter increases. However, for high slenderness (L/h=100) ratio, all theories are approximately identical in predicting both the buckling load and frequency, which confirms the accuracy of the Euler-Bernoulli theory in the case of thin nanobeams. Again, our results are in good agreement with those obtained by Berrabah *et al.* (2013).

Table 4 illustrates the variation of the three first frequencies with respect to nonlocal scale parameter and proposed theories. For all theories, it is noted that the frequency decreases as the nonlocal parameter increases at a specified mode of vibration. However, the effect of nonlocal parameter is more significant at higher modes.

4.2 Results for effect of Winkler modulus parameter

To study the effects of the surrounding elastic medium on the bending, buckling and vibration responses of nanobeam, variations of deflection, critical buckling load and frequency ratios with foundation parameters are plotted. The quantity ratio serves as an index to assess quantitatively the nonlocal scale parameter effect on bending, buckling and vibration solutions of nanobeam. The deflection, buckling load and frequency ratios are defined as

$$\text{Deflection ratio} = \frac{w_{NL}}{w_L}, \quad \text{Buckling load ratio} = \frac{N_{NL}}{N_L}, \quad \text{Frequency ratio} = \frac{\omega_{NL}}{\omega_L} \tag{23}$$

where the subscripts NL and L refer to the quantities computed using the nonlocal model and the local model, respectively.

The elastic medium is modeled as both (i) Winkler-type foundation and (ii) Pasternak-type foundation. The Winkler-type and Pasternak foundations are characterized by foundation stiffness,

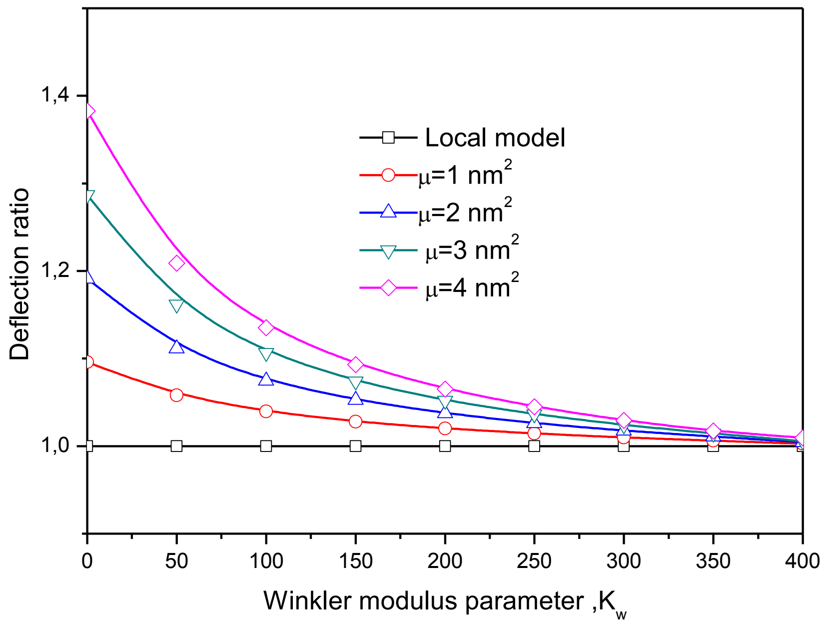

Fig. 1 Effect of Winkler modulus parameter on the deflection ratio of nanobeam for various nonlocal parameters (K_s=0, L/h=10)

K_w and K_s, respectively.

For the present work, the nonlocal coefficient values of nanobeam are considered as μ=0, 1, 2, 3, and 4 nm^2. The Winkler modulus parameter (K_w) values were taken in the range of 0-400.

Fig. 1 demonstrates the effect of nonlocal scale parameter on the bending response of embedded nanobeam with elastic medium modeled as Winkler-type foundation. It can be seen from this figure that there is significant effect of small size on the bending response of embedded nanobeam. The deflection considering nonlocal model is always higher than the classical model (μ=0). This means that the use of the local hyperbolic shear deformation beam theory for nanobeam analysis would lead to an under-prediction of the deflection if the small length scale effects between the individual atoms of nanobeam are ignored. Further, with increase in μ values, the deflections computed by nonlocal hyperbolic shear deformation theory become higher compared to classical model (μ=0). Furthermore, it is observed that as the Winkler modulus parameter increase the deflection ratio decreases. This increasing trend indicates that the nanobeam becomes stiffer with including the elastic medium. With higher values of Winkler modulus the rate of decrease of deflection ratio reduces. This implies that the nonlocal scale parameter effect in bending response of nanobeam looses its significance as the Winkler modulus values increase. Thus the small-scale effect tends to become more considerable without the presence of elastic medium.

Figs. 2 and 3 show the trend of variation of buckling and frequency ratios of embedded nanobeam with Winkler-type foundation versus the nonlocal scale parameter. The results show that the critical buckling load and frequency considering nonlocal model are always smaller than the classical model (μ=0) which highlights the significance of small size on the buckling and vibration responses of embedded nanobeam. This means that the use of the local hyperbolic shear

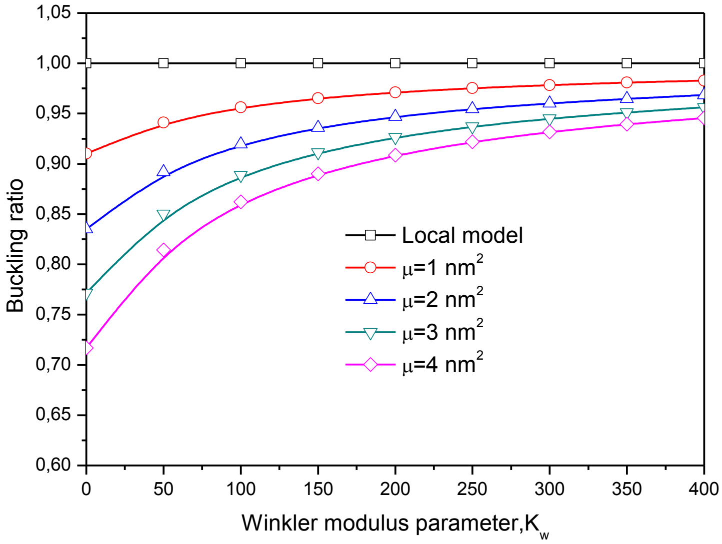

Fig. 2 Effect of Winkler modulus parameter on the buckling load ratio of nanobeam for various nonlocal parameters (K_s=0, L/h=10)

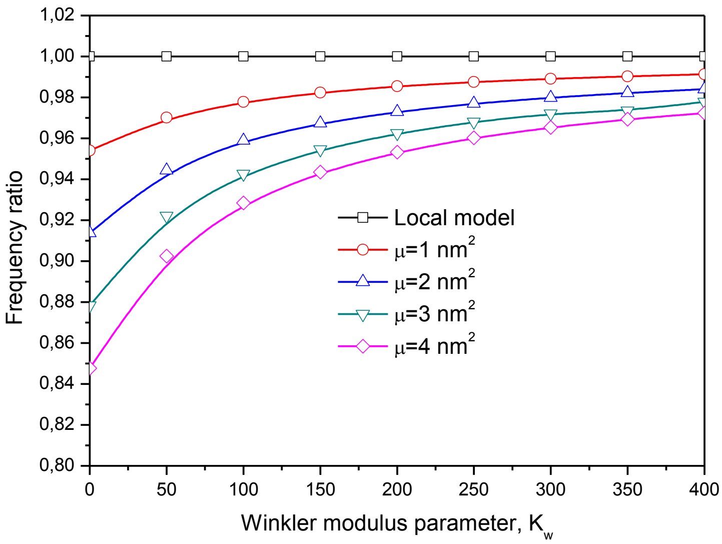

Fig. 3 Effect of Winkler modulus parameter on the frequency ratio of nanobeam for various nonlocal parameters (K_s=0, L/h=10)

deformation beam theory for nanobeam investigation would lead to an over-prediction of the both critical buckling load and frequency if the nonlocal scale parameter effects between the individual atoms of nanobeam are not considered. Increasing the nonlocal scale parameter from 0 to 4 results in significant decrease in buckling load and frequency. So, it can be concluded that the buckling load and frequency are highly decreased with higher values of the nonlocal scale parameter. In addition, it is found that the small-scale effect tends to become more considerable without the

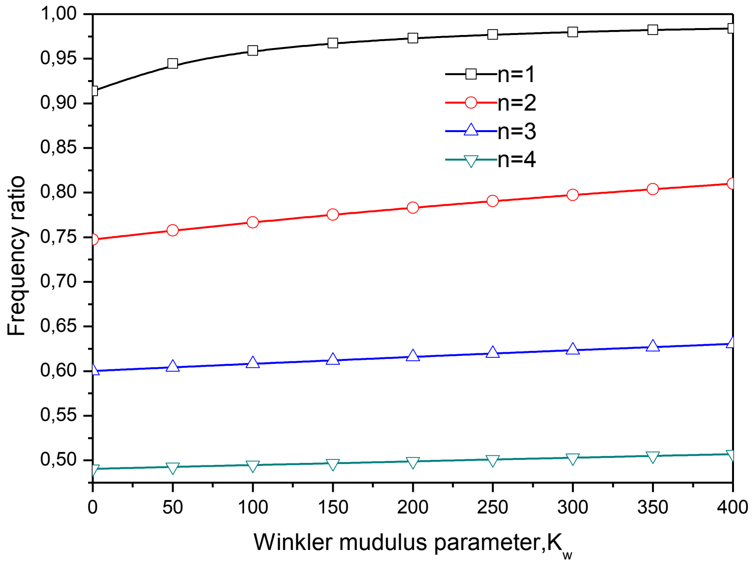

Fig. 4 Effect of Winkler modulus parameter on the frequency ratio of nanobeam for various mode numbers *n* (K_s=1, L/h=10, μ=2).

presence of elastic medium. Indeed, with higher values of Winkler modulus the rate of increase of buckling load and frequency ratios reduces. This means that the nonlocal scale parameter effect in buckling and vibration response of nanobeam looses its importance as the Winkler modulus values increase. Thus the small-scale effect tends to become more considerable without the presence of elastic medium.

Fig. 4 demonstrates the variation of frequency ratio with Winkler modulus parameter. Different modes of vibration are considered in this example. It is seen that the small-scale effects on vibration behavior are more significant for higher modes of vibration. Furthermore, as the spring constant factor increases, the frequency ratios increase marginally for higher modes except for first mode of vibration. This means that the there is comparatively less influence of elastic medium on higher mode frequency of nanobeam.

4.3 Results for effect of Pasternak modulus parameter

For the present study, the nonlocal coefficient values of nanobeam are considered as μ=0, 1, 2, 3, and 4 nm^2. The Pasternak shear modulus parameter (K_s) values were taken in the range of 0–10. The Winkler modulus parameter is assumed as K_w=100.

Fig. 5 illustrates the influence of nonlocal scale parameter on the bending response of nanobeam with elastic medium modeled as Pasternak-type foundation. As the shear modulus parameter increases, the deflection ratio increases. However, the deflection considering the nonlocal model is always higher than the classical model (μ=0). With higher μ values the deflections are comparatively high. Contrary to the variation of deflection ratio with Winkler modulus, which is nonlinear, the variation of deflection ratio considering Pasternak-type foundation is almost linear in nature.

Figs. 6 and 7 illustrate the trend of variation of buckling and frequency ratios of nanobeam with

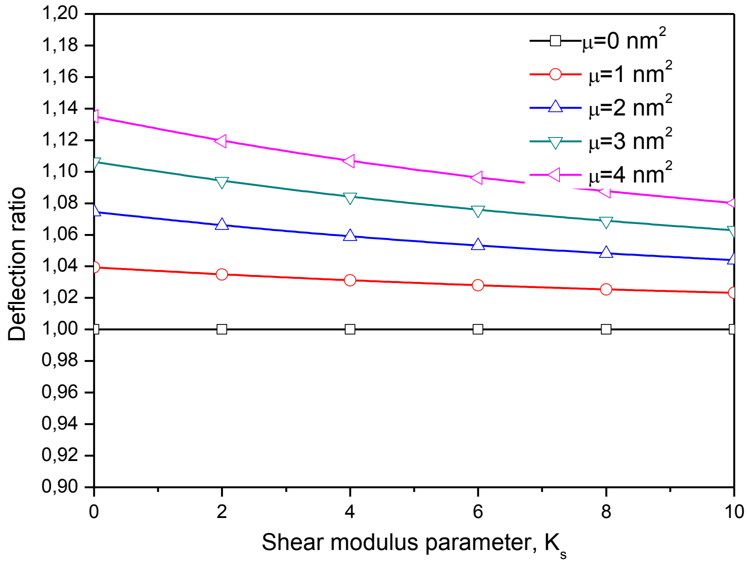

Fig. 5 Effect of Pasternak shear modulus parameter on the deflection ratio of nanobeam for various nonlocal parameters (K_w=100, L/h=10)

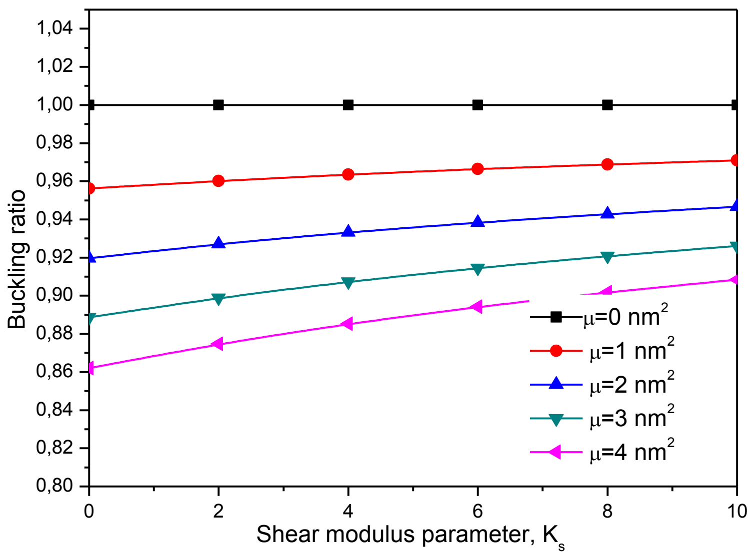

Fig. 6 Effect of Pasternak shear modulus parameter on the buckling load ratio of nanobeam for various nonlocal parameters (K_w=100, L/h=10)

Pasternak-type foundation versus the nonlocal scale parameter. Increasing the Pasternak shearing layer modulus parameter leads to an increase of the buckling load and frequency ratios. The buckling load and frequency considering nonlocal model are always smaller than the classical model (μ=0). With higher μ values the buckling loads and frequencies are comparatively less. Contrary to the variation of buckling load and frequency ratios with Winkler modulus, which is

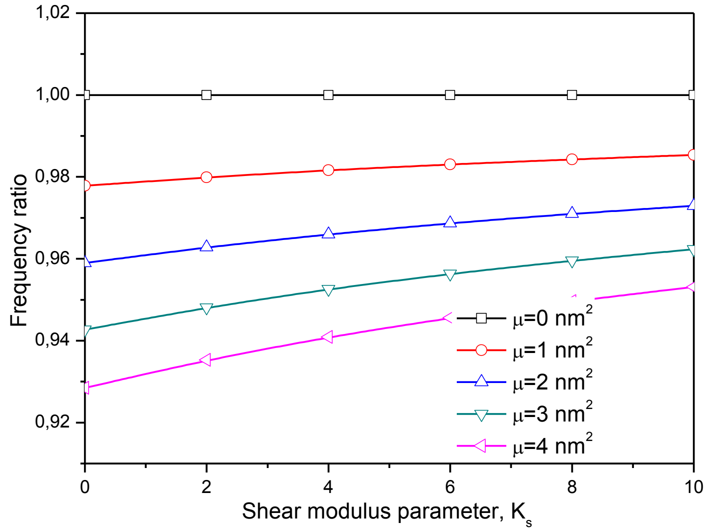

Fig. 7 Effect of Pasternak shear modulus parameter on the frequency ratio of nanobeam for various nonlocal parameters (K_w=100, L/h=10)

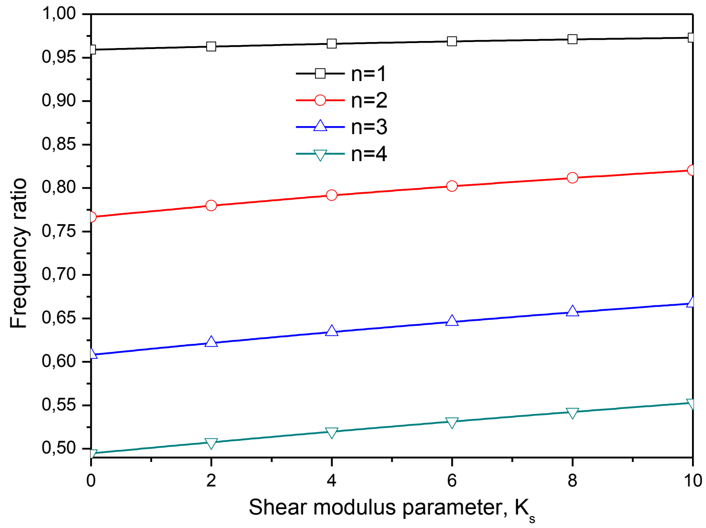

Fig. 8 Effect of Pasternak shear modulus parameter on the frequency ratio of nanobeam for various mode numbers n (K_w=100, L/h=10, μ=2)

nonlinear, it is observed that these quantities increase almost linearly as the Pasternak shearing layer modulus parameter increases.

Fig. 8 displays the variation of frequency ratio with respect to Pasternak shear modulus parameter for different modes of vibration. It is seen from figure that the nonlocality effects on vibration response are more considerable for higher modes of vibration. Furthermore, the increase in the Pasternak shearing layer modulus parameter yields in higher frequency ratios. The trend of increase in frequency ratio is linear in nature for all the modes of vibration considered.

5. Conclusions

In this study, the static, buckling and vibration behaviors of nanoscale-beams embedded in an elastic medium is investigated by proposing a new nonlocal hyperbolic shear deformation beam theory. The nonlocal Eringen's elasticity model is considered to account for small-scale effects. Both Winkler-type and Pasternak-type models are utilized to simulate the interaction of the nanobeam with a surrounding elastic medium. The results demonstrate that the nonlocality parameter has a notable effect on the bending, buckling and vibration response of nanobeam. Deflections, buckling loads and frequencies of the nanobeams vary nonlinearly when the elastic medium is modeled as Winkler-type foundation. While, these quantities of the nanobeams vary linearly with elastic medium modeled as Pasternak-type foundation. The formulation lends itself particularly well to nanostructures studied with advanced shear deformation theories (El Meiche *et al*. 2011, Houari *et al.* 2013, Saidi *et al*. 2013, Tounsi *et al*. 2013c, Bousahla *et al*. 2014, Fekrar *et al*. 2014, Hamidi *et al*. 2015, Chattibi *et al*. 2015, Bouchafa *et al*. 2015, Bennai *et al*. 2015), which will be considered in the near future.

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