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Buckling analysis in hybrid cross-ply composite laminates on elastic foundation using the two variable refined plate theory

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Abstract. This paper presents the effect of hybridization material on variation of critical buckling load with different cross-ply laminates plate resting on elastic foundations of Winkler and Pasternak types subjected to combine uniaxial and biaxial loading by using two variable refined plate theories. Governing equations are derived from the principle of virtual displacement; the formulation is based on a new trigonometric shape function of displacement taking into account transverse shear deformation effects vary parabolically across the thickness satisfying shear stress free surface conditions. These equations are solved analytically using the Navier solution of a simply supported. The influence of the various parameters geometric and material, the thickness ratio, and the number of layers symmetric and antisymmetric hybrid laminates material has been investigated to find the critical buckling loads. The numerical results obtained through the present study with several examples are presented to verify and compared with other models with the ones available in the literature.

Keywords: buckling; hybrid, cross-ply laminates; winkler and pasternak; elastic foundation; two variables plate theory

1. Introduction

With advances in science and technology, there is increasing interest in composite materials, both in scientific research and for engineering applications like aircraft runway, automobile ships, and trains. In particular, hybrid materials are increasingly in demand for structural applications in the aerospace, automotive ,marine industries, civil construction and other systems due to their advantageous excellent specific mechanical performance (mechanical properties /density ratio) and design flexibility compared with conventional materials. The realization of hybrid laminates also allows a reduction of the mass of structures and, hence, often a reduction of economic cost.

However, these materials are prone to a wide range of defects and damage that can cause significant reduction in their properties and their lifetime Sadowski (2009, 2012).In practice; these composite plates are subject to main loading conditions and mechanical stresses. In order to obtain

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an effective structural design, which can often be idealized as equivalent laminated plates, they need a study of their stability. In response to this need; the buckling performance of composite plates has been analyzed by many investigations. Creating a more efficient method to predict the buckling resistance of composite laminates is of great interest to many researchers. Some exact results solutions for the buckling load of orthotropic plates can be found as early as Thielemann (1950).

Based on different assumptions for displacement fields considering the transverse shear deformation effect, a number of theories for buckling analysis have been devised. The classical laminated plate theory (CLPT); which neglects the transverse shear deformation effect, provides reasonable results for thin plates. This theory was employed for buckling analysis of orthotropic plate by Das (1963), Harik and Ekambaram (1988), Bao *et al*. (1997), Hwang and Lee (2006), among others. As a consequence, the evaluation of mechanical behavior of hybrid laminates needs more advanced theories, as the higher-order shear deformation theories (HSDT) use by Phan and Reddy (1985), this theory is capable of considering a quadratic variation for transverse shear strains along the thickness and thus a shear correction factor is necessary. Reddy and Khdeir (1989) have conducted an analytical solution by ESL theories (CLPT-HSDT) to study the buckling behavior of cross ply laminated composite plates. Another approach is presented by Akavci (2007) using a hyperbolic displacement model with five unknowns which accounts the transverse shear strains and their parabolic variation through the thickness with 3D Elasticity by Noor (1975) have presented a solution for stability of multi layered composite plates based on three dimensional elasticity theory by solving equations with the finite difference method. A three dimensional elasticity employing layer-wise theory has been refined to take into account the variation of the variables through the thickness by Setoodeh (2004). In addition of these methods, Rajasekaran and Wilson (2013) are used finite difference technique for determination of buckling loads, Singhatanadgid and Sukajit (2011) are used a vibration correlation technique (VCT) to identify the buckling load of a rectangular thin plate. Felix *et al*. (2011) are proposed an analytical solution using the Ritz method to calculate the critical buckling load of clamped, orthotropic, rectangular thin plates subjected to different linear distributed in-plane forces.

Even as plates on elastic foundations are often encountered in many practical applications, the analysis of a general problem of thick reinforced laminated plate resting on elastic foundation are the focus of attention for mechanical and structural engineers and has been studied by a considered number of investigations. Winkler's elastic foundation model which consists of infinitely many closed-spaced linear springs is a one parameter model that is extensively used in practice. Some authors have dealt buckling plates on Winkler's foundation Gupta (2006), Saha (1997), Lee (1998), Utku (2000), El-Zafrany (1995). The limitation of this method is that it assumes no interaction between the springs. The Pasternak model takes into account the effect of shear interaction among the point in the foundation has presented by Malekzadeh and Karami (2004), Omurtag and Kadioglu (1998), Xiang *et al*. (1996), Hui-Shen *et al*. (2003).

The buckling behavior of composite hybrid plates can be critical and therefore must be properly represented by the various models of structures usually employed either to predict their behavior to identify experimentally their properties.

In the current work, plates composed of graphite, glass fibers and epoxy resin subjected to in- plane loads will be investigated by a new shear deformation theory. The accuracy of this theory has been demonstrated for static bending and free vibration behaviors of plates by Shimpi and Patel (2006). This theory was successfully expanded to the buckling behaviors of laminated composite plates subjected to in plane loading by Kim *et al*. (2009), El Meiche et *al*. (2011)

provides analyze of new hyperbolic shear deformation theory taking into account transverse shear deformation effects is presented for the buckling and free vibration analysis of thick functionally graded sandwich plates, Mechab *et al.* (2012) are also performed this function for analyze of thick orthotropic laminated composite under thermo- mechanical loading.

Recently, El meiche developed a new function of the displacement of shear deformation with only four unknown functions. However, various higher-order shear deformation theories are developed using five unknown functions.

A present method with this new function has been employed to find analytical solutions for buckling load and mode shape. The buckling loads and modes with respect to plate aspect ratios for different hybrid cross-ply composite laminates will be obtained, to illustrating the full mechanical behavior, this approach can coupled between compression, bending and shear deformation modes which considerably, reduce the number of equations, and the complexity in the formulation and resolution of different higher order theories. The results obtained by the present theory are compared with other theories existing in the literature.

2. Modeling

In this section the analytical model and the applied theories in this study are briefly outlined. The plate theories which are applied in the analytical formulation are the classical laminate plate theory (CLPT), the first and the higher order shear deformation theories

$$\begin{aligned}\bar{u}(x,y,z) &= u(x,y) - z\frac{\partial w}{\partial x}\\ \bar{v}(x,y,z) &= v(x,y) - z\frac{\partial w}{\partial y}\\ \bar{w}(x,y,z) &= w(x,y)\end{aligned} \tag{1}$$

And the FSDT assumptions are

$$\begin{aligned}\bar{u}(x,y,z) &= u(x,y) + z\phi_x(x,y)\\ \bar{v}(x,y,z) &= v(x,y) + z\phi_y(x,y)\\ \bar{w}(x,y,z) &= w(x,y)\end{aligned} \tag{2}$$

And the Reddy's third order shear deformation theory is written as

$$\begin{aligned}\bar{u}(x,y,z) &= u(x,y) + z\phi_x(x,y) - z^3 \mathrm{C}_1(\phi_x + \frac{\partial w}{\partial x})\\ \bar{v}(x,y,z) &= v(x,y) + z\phi_y(x,y) - z^3 \mathrm{C}_1(\phi_y + \frac{\partial w}{\partial y})\\ \bar{w}(x,y,z) &= w(x,y)\end{aligned} \tag{3}$$

Where $\bar{u}$,$\bar{v}$ and $\bar{w}$ are components of displacements at a general point. u, $v$and w are similar components at the middle plane, ϕ_x and ϕ_y are the rotations of the mid plane normal about y and x axis respectively.

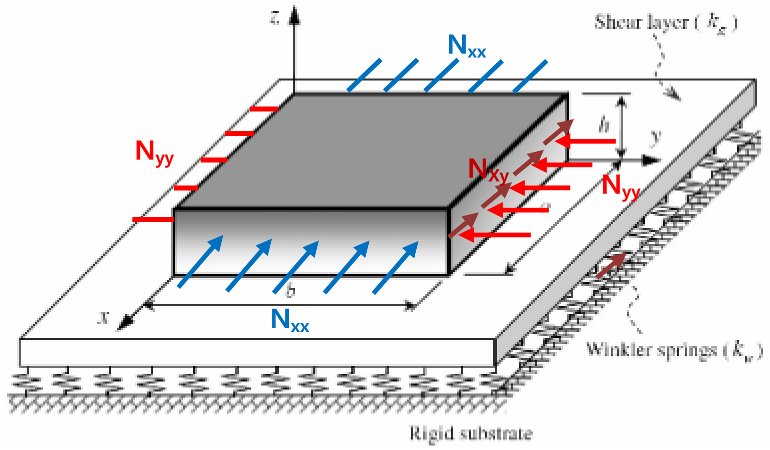

Fig. 1 Geometry and loading conditions of a laminated plate on elastic foundation
(N_{xx}=$\gamma_1 \overline{N}_{xx}$, N_{yy}=$\gamma_2 \overline{N}_{yy}$, N_{xy}=0)

Setoodeh (2004) presents the displacements with interpolating functions as

$$
\begin{aligned}
u_1(x,y,z) &= \sum_{j=1}^{n}\sum_{i=1}^{p} u_{ij}(x_{ij}, y_{ij})\psi_i(x,y)\phi_j(z) \\
u_2(x,y,z) &= \sum_{j=1}^{n}\sum_{i=1}^{p} v_{ij}(x_{ij}, y_{ij})\psi_i(x,y)\phi_j(z) \\
u_3(x,y,z) &= \sum_{j=1}^{n}\sum_{i=1}^{p} w_{ij}(x_{ij}, y_{ij})\psi_i(x,y)\phi_j(z)
\end{aligned}
\tag{4}
$$

2.1 Formulation of the theory

As shown in Fig. 1 Consider a flat laminated plate resting on an elastic foundation with fiber orientation angle of θ with respect to x-axis.the plate has a length a, width b, constant thickness h. It is assumed subject to uniformly compressive distributed in-plane forces of $\overline{N}_{xx}$ and $\overline{N}_{yy}$.

2.1.1 Assumptions of the theory

• The displacements are small in comparison with the plate thickness h and, there for, strains involved are infinitesimal.

• The transverse displacement w includes two components of bending w_b and shear w_s. Both these components are functions of coordinates x, y

$$w(x,y,z) = w_b(x,y) + w_s(x,y) \tag{5}$$

• The transverse normal stress σ_z is negligible in comparison with in-plane stresses σ_x, σ_y.
• The displacements u in x direction and v in y direction consist of extension, bending and shear components

$$U = u_0 + u_b + u_s, \quad V = v_0 + v_b + v_s \tag{6}$$

The expression for bending components u_b and v_b can be given as

$$u_b = -z\frac{\partial w_b}{\partial x} \quad , \quad v_b = -z\frac{\partial w_b}{\partial y} \tag{7}$$

The shear components u_s and v_s give rise in conjunction with w_s to the parabolic variations of shear strains γ_{xy}, γ_{xz} and γ_{yz} hence to shear stresses τ_{xz}, τ_{yz}. Through the thickness of the plate, h, in such a way that shear stresses τ_{xz}, τ_{yz} are zero at the top and bottom faces of the plate. The expression for u_s and v_s can be given as

$$u_s = -f(z)\frac{\partial w_s}{\partial x}, \quad v_s = -f(z)\frac{\partial w_s}{\partial y} \tag{8}$$

The subscripts s and b indicate the shear and bending quantities, respectively.

2.1.2 The sisplacement field model

Based on the assumptions made in the preceding sections, the displacement fields defined by new hyperbolic shape function developed by Noureddine El meiche and I. Mechab in unified form as follows:

$$\begin{aligned} u(x,y,z) &= u_0(x,y) - z\frac{\partial w_b}{\partial x} - f(z)\frac{\partial w_s}{\partial x} \\ v(x,y,z) &= v_0(x,y) - z\frac{\partial w_b}{\partial y} - f(z)\frac{\partial w_s}{\partial y} \\ w(x,y,z) &= w_b(x,y) + w_s(x,y) \end{aligned} \tag{9}$$

$$f(z) = \frac{z}{2+\pi}\left(\pi + 2\cos(z\pi)\right) \tag{10}$$

Where u, v and w are components of displacements at a general point x, y, z direction, whilst u_0, v_0 are similar components at the middle surface (z=0). With the function $f(z)$ the displacement field accounts for zero transverse shear stresses on boundary conditions on the top and bottom faces of the plate and the quadratic variation of transverse shear strains (and hence stress) through the thickness. Thus there is no need to use correction factors.

Using Eq. (9) in the strain-displacement equation of the elasticity; gives the following expressions for normal and shear strains

$$\varepsilon_{xx} = \frac{\partial u_0}{\partial x} - z\frac{\partial^2 w_b}{\partial x^2} - f(z)\frac{\partial^2 w_s}{\partial x^2}$$

$$\varepsilon_{yy} = \frac{\partial v_0}{\partial y} - z\frac{\partial^2 w_b}{\partial y^2} - f(z)\frac{\partial^2 w_s}{\partial y^2}$$

$$\gamma_{xy}=\frac{\partial v_0}{\partial x}+\frac{\partial u_0}{\partial y}-2z\frac{\partial^2 w_b}{\partial x\partial y}-2f(z)\frac{\partial^2 w_s}{\partial x\partial y}$$
$$\gamma_{xz}=(1-f'(z))\frac{\partial w_s}{\partial x} \tag{11}$$
$$\gamma_{yz}=(1-f'(z))\frac{\partial w_s}{\partial y}$$

$$f'(z)=\frac{df(z)}{dz} \tag{12}$$

Where constituve equation are

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{yz}\\ \tau_{zx}\\ \tau_{xy}\end{Bmatrix}=\begin{bmatrix}Q_{11} & Q_{12} & 0 & 0 & 0\\ Q_{12} & Q_{22} & 0 & 0 & 0\\ 0 & 0 & Q_{44} & 0 & 0\\ 0 & 0 & 0 & Q_{55} & 0\\ 0 & 0 & 0 & 0 & Q_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{yz}\\ \gamma_{zx}\\ \gamma_{xy}\end{Bmatrix}$$

$$Q_{11}=\frac{E_1}{1-\upsilon_{12}\upsilon_{21}},Q_{22}=\frac{E_2}{1-\upsilon_{12}\upsilon_{21}},Q_{12}=\frac{\upsilon_{12}E_2}{1-\upsilon_{12}\upsilon_{21}}$$
$$Q_{66}=G_{12}\qquad ,Q_{44}=G_{23}\qquad ,Q_{55}=G_{13} \tag{13}$$

2.2 Equilibrum equations

The governing equation for the buckling analysis of structural system can be derived using a principle of the virtual work. This gives by Reddy (1981).

$$\delta\int(U+U_f-W)=0 \tag{14}$$

Where U is the strain energy, U_f is the strain energy of foundation and W is the work of external forces. Employing the minimum of the total energy principle and integrating by parts

$$\int_{-h/2}^{h/2}\int_{\Omega}[\sigma_x\delta\varepsilon_x+\sigma_y\delta\varepsilon_y+\tau_{xy}\delta\gamma_{xy}+\tau_{xz}\delta\gamma_{xz}+\tau_{yz}\delta\gamma_{yz}]d\Omega dz$$
$$+\int_{\Omega}[F_e\delta w+\overline{N}_x w_{,xx}\delta w+\overline{N}_y w_{,yy}\delta w]d\Omega=0 \tag{15}$$

Where $\overline{N}_x$ and $\overline{N}_y$ are inplane compressive loading on the side of plate.
F_e is the density of reaction force of foundation; follows for two parameters Pasternak model as

$$F_e=K_0 w-K_1\nabla^2 w \tag{16}$$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is Laplace's operator, K_0 and K_1 are the normal and shear stiffness of the foundation, respectively. This model is simply known as Winkler type when K_1=0, Aiello (1999).

Using Eqs. (11)-(15) takes the following form

$$\int_\Omega [\delta u \frac{\partial N_{xx}}{\partial x} + \delta u \frac{\partial N_{xy}}{\partial y} + \delta v \frac{\partial N_{xy}}{\partial x} + \delta v \frac{\partial N_{yy}}{\partial y} + \delta w_b \frac{\partial^2 M_{xx}^b}{\partial x^2} + 2\delta w_b \frac{\partial^2 M_{xy}^b}{\partial x \partial y} + \delta w_b \frac{\partial^2 M_{yy}^b}{\partial y^2}$$

$$+ \delta w_s \frac{\partial^2 M_{xx}^s}{\partial x^2} + 2\delta w_s \frac{\partial^2 M_{xy}^s}{\partial x \partial y} + \delta w_s \frac{\partial^2 M_{yy}^s}{\partial y^2} + \delta w_s \frac{\partial R_x}{\partial x} + \delta w_s \frac{\partial R_y}{\partial y}] d\Omega$$

$$- \int_\Omega [F_e(\delta w_s + \delta w_b) + \delta w_b \overline{N}_x (\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_s}{\partial x^2}) + \delta w_b \overline{N}_y (\frac{\partial^2 w_b}{\partial y^2} + \frac{\partial^2 w_s}{\partial y^2}) +$$

$$\delta w_s \overline{N}_x (\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_s}{\partial x^2}) + \delta w_s \overline{N}_y (\frac{\partial^2 w_b}{\partial y^2} + \frac{\partial^2 w_s}{\partial y^2})] d\Omega = 0 \tag{17}$$

Where stress and moment resultants are defined as

$$(N_i, M_i^b, M_i^s) = \int_{-h/2}^{h/2} (1, z, f(z)) \sigma_i dz (i = xx, yy, xy),$$

$$R_i = \int_{-h/2}^{h/2} (1 - f'(z)) \sigma_i dz (i = xx, yy) \quad (18)\#$$

The governing equations of equilibrium can be derived from Eq. (17) by integrating the displacement gradients by parts and setting the coefficients u_0, v_0, w_b and w_s. The equilibrium equations associated with the present theory are

$$\delta u_0 : \frac{\partial N_{xx}}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\delta v_0 : \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_{yy}}{\partial y} = 0$$

$$\delta w_b : \frac{\partial^2 M_{xx}^b}{\partial x^2} + 2\frac{\partial^2 M_{xy}^b}{\partial x \partial y} + \frac{\partial^2 M_{yy}^b}{\partial y^2} - Fe - \overline{N}_x (\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_s}{\partial x^2}) - \overline{N}_y (\frac{\partial^2 w_b}{\partial y^2} + \frac{\partial^2 w_s}{\partial y^2}) = 0$$

$$\delta w_S : \frac{\partial^2 M_{xx}^S}{\partial x^2} + 2\frac{\partial^2 M_{xy}^S}{\partial x \partial y} + \frac{\partial^2 M_{yy}^S}{\partial y^2} - Fe - \overline{N}_x (\frac{\partial^2 w_b}{\partial x^2} + \frac{\partial^2 w_S}{\partial x^2}) - \overline{N}_y (\frac{\partial^2 w_b}{\partial y^2} + \frac{\partial^2 w_S}{\partial y^2}) + \frac{\partial R_x}{\partial x} + \frac{\partial R_y}{\partial y} = 0 \tag{19}$$

In the above equations, the stress resultants are defined as

$$\begin{Bmatrix} N \\ M^b \\ M^s \end{Bmatrix} = \begin{bmatrix} A_{ij} & B_{ij} & B_{ij}^s \\ A_{ij} & D_{ij} & D_{ij}^s \\ B_{ij}^s & D_{ij}^s & C_{ij}^s \end{bmatrix} \begin{Bmatrix} \varepsilon \\ \varepsilon^b \\ \varepsilon^s \end{Bmatrix} \quad (i, j = 1,2,6), \quad \{R\} = [F_{ij}]\{\gamma\} \quad (i, j = 4,5) \tag{20}$$

Where

$$\varepsilon = \begin{Bmatrix} \frac{\partial u_0}{\partial x} \\ \frac{\partial v_0}{\partial y} \\ \frac{\partial v_0}{\partial x} + \frac{\partial u_0}{\partial y} \end{Bmatrix} \quad \varepsilon^b = -\begin{Bmatrix} \frac{\partial^2 w_b}{\partial x?} \\ \frac{\partial^2 w_b}{\partial y?} \\ 2\frac{\partial^2 w_b}{\partial x \partial y} \end{Bmatrix} \quad ,\varepsilon^s = \begin{Bmatrix} \frac{\partial^2 w_s}{\partial x?} \\ \frac{\partial^2 w_s}{\partial y?} \\ 2\frac{\partial^2 w_s}{\partial x \partial y} \end{Bmatrix}, \quad \gamma = \begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \tag{21}$$

And stiffness components are given as

$$\begin{gathered} \{A_{ij}, B_{ij}, B_{ij}^s, D_{ij}, D_{ij}^s, C_{ij}\} = \int_{-h:2}^{h:2} \{1, z, f(z), z?\, zf(z), [f(z)]^2\} \overline{Q_{ij}}\, dz \quad i, j = 1,2,6 \\ \{F_{ij}\} = \int_{-h:2}^{h:2} [f'(z)]^2\, \overline{Q_{ij}}\, dz \quad i, j = 4,5 \end{gathered} \tag{22}$$

2.3 Analytical solution

Assuming the four edges of the plate is simply supported; the geometric boundary conditions are given as follows

$$v_0(0, y) = w_b(0, y) = w_s(0, y) = \frac{\partial w_b}{\partial y}(0, y) = \frac{\partial w_s}{\partial y}(0, y) = 0$$

$$v_0(a, y) = w_b(a, y) = w_s(a, y) = \frac{\partial w_b}{\partial y}(a, y) = \frac{\partial w_s}{\partial y}(a, y) = 0$$

$$N_x(0, y) = M_x^b(0\; y) = M_x^s(0, y) = N_x(a, y) = M_x^b(a, y) = M_x^s(a, y) = 0$$

$$u_0(x, 0) = w_b(x,0) = w_s(x,0) = \frac{\partial w_b}{\partial x}(x,0) = \frac{\partial w_s}{\partial x}(x,0) = 0$$

$$u_0(x,b) = w_b(x,b) = w_s(x,b) = \frac{\partial w_b}{\partial x}(x,b) = \frac{\partial w_s}{\partial x}(x,b) = 0$$

$$N_y(x,0) = M_y^b(x,0) = M_y^s(x,0) = N_y(x,b) = M_y^b(x,b) = M_y^s(x,b) = 0 \tag{23}$$

Navier method is used for the analytical solutions of Eq. (19). In order to satisfy the geometric boundary conditions the displacement functions may be approximated by a truncated FOURRIER series as

$$u_0(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn}cos(\lambda\, x)sin(\mu\, y)$$

$$v_0(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn}sin(\lambda\, x)cos(\mu\, y)$$

$$w_b(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{bmn}sin(\lambda\, x)sin(\mu\, y)$$

$$w_s(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{smn}sin(\lambda\, x)sin(\mu\, y) \tag{24}$$

Where U_{mn}, V_{mn}, W_{bmn}, W_{smn} are unknown constants that will determinate the buckling mode shape and *m, n* are integers that will determine the number of terms of the truncated series.

$$u = m\pi / a \quad , \quad \lambda = n\pi / b$$

3. Buckling of a simply supported plate under compressive loads

Substituting Eqs. (22)-(24) into equilibrium Eq. (19), the buckling problem is

$$([K]-[N])\{\Delta\}=\{0\} \tag{25}$$

Where

$$[N]=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha^2\overline{N}_x+\beta^2\overline{N}_y & \alpha^2\overline{N}_x+\beta^2\overline{N}_y \\ 0 & 0 & \alpha^2\overline{N}_x+\beta^2\overline{N}_y & \alpha^2\overline{N}_x+\beta^2\overline{N}_y \end{bmatrix} \qquad \{\Delta\}=\begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{bmn} \\ W_{smn} \end{Bmatrix}$$

$$[\mathrm{K}]=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix} \tag{26}$$

As the constants of the matrix of rigidity [*K*] are written as

$$a_{11}=-\left(A_{11}\lambda^2+A_{66}\mu^2\right)$$

$$a_{12}=-\lambda\,\mu\left(A_{12}+A_{66}\right)$$

$$a_{13}=\lambda\,[B_{11}\lambda^2+(B_{12}+2B_{66})\,\mu^2\,]$$

$$a_{14}=-\lambda\,[B^s_{11}\lambda^2+(B^s_{12}+2B^s_{66})\,\mu^2\,]$$

$$a_{22} = -\left(A_{66}\lambda^2 + A_{22}\mu^2\right)$$

$$a_{23} = \mu\,[(B_{12} + 2B_{66})\,\lambda^2 + B_{22}\mu^2\,]$$

$$a_{24} = -\mu\,[(B^s_{12} + 2B^s_{66})\,\lambda^2 + B^s_{22}\mu^2\,]$$

$$a_{33} = k1(-\lambda^2 - \mu^2) - k0 - \left(D_{11}\lambda^4 + 2(D_{12} + 2D_{66})\lambda^2\mu^2 + D_{22}\mu^4\right)$$

$$a_{34} = k1(\lambda^2 - \mu^2) - k0 + \left(D^s_{11}\lambda^4 + 2(D^s_{12} + 2D^s_{66})\lambda^2\mu^2 + D^s_{22}\mu^4\right)$$

$$a_{44} = k1(\lambda^2 - \mu^2) - k0 - \left(H^s_{11}\lambda^4 + 2(H^s_{11} + 2H^s_{66})\lambda^2\mu^2 + H^s_{22}\mu^4 + A^s_{55}\lambda^2 + A^s_{44}\mu^2\right) \tag{27}$$

For nontrivial solutions of Eq. (25) the following determinations should be zero

$$\left|[K] - [N]\right| = 0 \tag{28}$$

The solutions of Eq. (28) give the buckling loads of the plate with different mode shape.

4. Numerical results and discussion

In order to validate the accuracy and efficiency of the method, some numerical calculations are carried out. In the examples considered the buckling analysis of symmetric and antisymmetric cross- ply thick laminates on elastic foundation by a new shear deformation model are suggested for investigation and the comparisons are made with available solutions in literature.

4.1 Validation study

A square limited plate is composed of two layers oriented at (0°/90°) to the x axis. This plate is simply supported on the four sides. The sum of the thickness of the 90° layer is the same as the

Table 1 Uniaxial non-dimensional buckling load factors of two cross play square (0°/90°) laminate for different orthotropic ratios (a/h=10)

		load direction			E_1/E_2		
k_0		($\gamma 1$, $\gamma 2$)	Method		20	30	40
0	0	(-1,0)	Noor	(1975)	7.8196	9.3746	10.8170
			HSDT[a]	(1989)	8.1151	9.8695	11.5630
			FSDT[b]	(1989)	8.0423	9.7347	11.3530
			Setoodeh	(2004)	8.0455	9.6995	11.2382
			Akavci	(2007)	8.1223	9.8826	11.5828
			Present		8.1173	9.8549	11.5258
100	0	(-1,0)	Setoodeh	(2004)	15.3245	17.5249	19.3401
			Akavci	(2007)	15.5347	18.2844	20.7765
			Present		15.5484	18.2501	20.6802
100	10	(-1,0)	Setoodeh	(2004)	27.5347	29.7616	31.5981
			Akavci	(2007)	28.0347	30.7844	33.2765
			Present		28.0484	30.7501	33.1802

Table 2 Comparison of non-dimensional critical buckling load of square plates subjected to biaxial compressive load

a/h	load direction	Theories	Orthotropic		
	($\gamma 1$, $\gamma 2$)		E_1/E_2=10	E_1/E_2=25	E_1/E_2=40
5		Present	2.9442[a]	3.4433[a]	3.7976[a]
		RPT Kim *et al*. (2009)	2.8549[a]	3.3309[a]	3.4800[a]
		FSDT (*k*=2/3)	2.5042[a]	2.7332[a]	2.8303[a]
		FSDT (*k*=5/6)	2.8319[a]	3.1422[a]	3.2822[a]
	(-1,-1)	FSDT (*k*=1)	3.1027[a]	3.4933[a]	3.6793[a]
10		Present	4.7392[a]	6.1766[a]	7.4115[a]
		RPT Kim *et al*. (2009)	4.6718[a]	6.0646[a]	7.2536[a]
		FSDT (*k*=2/3)	4.4259	5.4351[a]	6.0797[a]
	(-1,-1)	FSDT (*k*=5/6)	4.6367	5.8370[a]	6.6325[a]
		FSDT (*k*=1)	4.7708	6.1425[a]	7.0690[a]
20		Present	5.3433	7.7105[a]	9.7346[a]
		RPT Kim *et al*. (2009)	5.3267	7.6643[a]	9.6614[a]
		FSDT (*k*=2/3)	5.2463	7.3701[a]	8.9895[a]
	(-1,-1)	FSDT (*k*=5/6)	5.3100	7.5546[a]	9.3049[a]
		FSDT (*k*=1)	5.3533	7.6834[a]	9.5297[a]
50		Present	5.5418	8.2871[a]	10.6720[a]
		RPT Kim *et al*. (2009)	5.5390	8.2784[a]	10.6576[a]
		FSDT (*k*=2/3)	5.5249	8.2199[a]	10.5111[a]
	(-1,-1)	FSDT (*k*=5/6)	5.5361	8.2566[a]	10.5810[a]
		FSDT (*k*=1)	5.5436	8.2812[a]	10.6282[a]
100		Present	5.5714	8.3766[a]	10.8209[a]
		RPT Kim *et al*. (2009)	5.5707	8.3744[a]	10.8172[a]
		FSDT (*k*=2/3)	5.5672	8.3593[a]	10.7788[a]
	(-1,-1)	FSDT (*k*=5/6)	5.5700	8.3657[a]	10.7972[a]
		FSDT (*k*=1)	5.5719	8.3751[a]	10.8095[a]
		FSDT	5.5814	8.4069	10.8715[a]

[a] Mode for plate is (m,n=1,2)

sum of the thickness of 0° layer; each ply has identical material properties with respect to the material axes. The following properties are assumed.

$$\frac{E_1}{E_2}=40\,,\frac{G_{12}}{E_2}=\frac{G_{13}}{E_2}=0.6\ ,\frac{G_{23}}{E_2}=0.5\ ,\nu_{12}=\nu_{13}=0.25$$

The buckling load factor $\overline{N}_x$ is non-dimensionalized according to the following relations and denoted by λ**.** Also, elastic stiffness coefficients (K_0 and K_1) are non-dimensionalized and denoted by K_0 and K_1.

Table 3 Comparison of non-dimensional critical buckling load of square plates subjected to tension in the x-axis direction and compression in the y-axis direction

a/h	load direction ($\gamma 1$, $\gamma 2$)	Theories	Orthotropic E_1/E_2=10	Orthotropic E_1/E_2=25	Orthotropic E_1/E_2=40
5	(1,-1)	Present	4.1630[b]	4.2219[c]	4.2675[c]
		RPT Kim *et al*. (2009)	4.0258[b]	4.1044[c]	4.1525[c]
		FSDT (k=2/3)	3.2849[d]	3.3001[e]	3.3053[e]
		FSDT (k=5/6)	3.9241[c]	3.9794[c]	4.0075[d]
		FSDT (k=1)	4.4488[b]	4.5691[c]	4.6073[c]
10	(1,-1)	Present	7.8987[a]	8.7441[b]	9.3882[b]
		RPT Kim *et al*. (2009)	7.7863[a]	8.5471[b]	9.1638[b]
		FSDT (k=2/3)	7.2656[a]	7.7820[b]	8.1208[b]
		FSDT (k=5/6)	7.7748[a]	8.4774[b]	8.9039[b]
		FSDT (k=1)	8.0651[a]	9.0153[b]	9.5197[b]
20	(1,-1)	Present	9.3219[a]	11.7306[b]	12.9192[b]
		RPT Kim *et al*. (2009)	9.2811[a]	11.6347[b]	12.8031[b]
		FSDT (k=2/3)	9.1310[a]	11.2544[b]	12.1990[b]
		FSDT (k=5/6)	9.2782[a]	11.6015[b]	12.6339[b]
		FSDT (k=1)	9.3790[a]	11.8453[b]	12.9428[b]
50	(1,-1)	Present	9.8174[a]	12.9723[b]	14.4415[b]
		RPT Kim *et al*. (2009)	9.8101[a]	12.9531[b]	14.4177[b]
		FSDT (k=2/3)	9.7830[a]	12.8751[b]	14.2839[b]
		FSDT (k=5/6)	9.8097[a]	12.9463[b]	14.3789[b]
		FSDT (k=1)	9.8275[a]	12.9942[b]	14.4430[b]
100	(1,-1)	Present	9.8925[a]	13.1715[b]	14.6888[b]
		RPT Kim *et al*. (2009)	9.8907[a]	13.1666[b]	14.6827[b]
		FSDT (k=2/3)	9.8838[a]	13.1463[b]	14.6474[b]
		FSDT (k=5/6)	9.8906[a]	13.1648[b]	14.6724[b]
		FSDT (k=1)	9.8951[a]	13.1772[b]	14.6891[b]
		FSDT	9.9179[a]	13.2393[b]	14.7732[b]

[a] Mode for plate is (m,n=1,2)
[b] Mode for plate is (m,n=1,3)
[c] Mode for plate is (m,n=1,4)
[d] Mode for plate is (m,n=1,5)
[e] Mode for plate is (m,n=1,6)

$$\lambda = \overline{N}_x \frac{a^2}{E_2 h^3} \quad ; \quad k_0 = \frac{K_0 L^4}{E_2 h^3} \quad ; \quad k_1 = \frac{K_1 L^2}{E_2 h^3}$$

4.2 Buckling of hybrid materials

After verifying the merit and accuracy of the present solutions, in order to obtain the following new results it is assumed that the plates made of graphite/epoxy and glass/epoxy.

The numerical results obtained from the analytical solutions of the different hybrid plates under uniaxial and biaxial non-dimensional buckling load with a variation of aspect ratio and thickness ratio are studied and discussed and shown in Tables 5 and 6.

Table 4 Material properties for graphite/epoxy and glass/epoxy unidirectional ply

Material	Graphite/Epoxy Groves *et al*. (1987)	Glass/Epoxy Joffe *et al*. (2001)
E_L (GPa)	144.8	44.73
E_T (GPa)	9.58	12.76
G_{LT} (GPa)	4.79	5.8
$G_{TT'}$ (GPa)	4.2	4.49
ν_{LT}	0.31	0.297
ν_{TL}	0.4	0.42
Ply thickness (mm)	0.127	0.144

Table 5 Uniaxial non-dimensional buckling load factors of different hybrid cross-ply material on elastic foundation

Hybrid laminates	*a/b*	(γ1,γ2)	*k*0	*k*1	*a/h*=5	*a/h*=10	*a/h*=20	*a/h*=50	*a/h*=100
0°/90° graphite/epoxy (0°) glass/epoxy (90°)	1	(-1 ,0)	0	0	2,0077	2,2025	2,2573	2,2731	2,2754
			100	0	19,6778	21,6677	22,3026	22,4031	22,5208
			100	10	69,6778	71,6677	72,3026	72,4031	72,5208
	2		0	0	2,3864	2,8839	3,0426	3,0903	3,0972
			100	0	58,7233	70,1858	75,7470	77,7000	77,9969
			100	10	228,7233	240,1858	245,7470	247,7000	247,9969
0°/90°/0° graphite/epoxy (0°) glass/epoxy (90°)	1	(-1 ,0)	0	0	9,6745	14,3048	16,2518	16,8958	16,9920
			100	0	33,4300	43,9184	48,2094	49,6139	49,8231
			100	10	83,4300	93,9184	98,2094	99,6139	99,8231
	2		0	0	5,8245	8,4465	9,5193	9,8704	9 ,9227
			100	0	70,5391	97,2127	113,4791	119,859	120,8629
			100	10	270,5391	267,2127	283,4791	289,859	290,8629
90°/0°/90°/0° graphite/epoxy(90°) glass/epoxy (0°)	1	(-1 ,0)	0	0	7,5255	10,2004	11,1942	11,5080	11,5543
			100	0	47,2856	87,0933	115,0078	126,8415	128,7528
			100	10	97,2856	137,0933	165,0078	176,8415	178,7528
	2		0	0	9,2884	19,2403	26,2189	29,1773	29,6552
			100	0	81,3461	170,4160	316,4267	441,7482	469,5332
			100	10	251,3461	340,4160	486,4267	611,7482	639,5332
0°/90°/90°/0° graphite/epoxy (0°) glass/epoxy (90°)	1	(-1 ,0)	0	0	9,5408	13,8101	15,5516	16,1210	16,2058
			100	0	35,1030	46,9013	51,8344	53,4627	53,7058
			100	10	85,1030	96,9013	101,8344	103,4627	103,7058
	2		0	0	6,2427	9,1923	10,4256	10,8326	10,8934
			100	0	73,6489	106,9003	129,2676	138,5246	140,0074
			100	10	243,6489	276,9003	299,2676	308,5246	310,0074

Table 6 Biaxial non-dimensional buckling load factors of different hybrid cross-ply material on elastic foundation

Hybrid laminates	*a/b*	(γ1,γ2)	*k*0	*k*1	*a/h*=5	*a/h*=10	*a/h*=20	*a/h*=50	*a/h*=100
$0^{\circ}/90^{\circ}$ graphite/epoxy (0°) glass/epoxy (90°)	1	(-1,-1)	0	0	1,0038	1,1012	1,1286	1,1365	1,1377
			100	0	3,9355	4,3335	4,4605	4,4986	4,5041
			100	10	13,9355	14,3335	14,4605	14,4986	14,5041
	2		0	0	0,4772	0,5767	0,6085	0,6180	0,6194
			100	0	3,4543	4,1285	4,4557	4,5706	4,5880
			100	10	13,4543	14,1285	14,4557	14 ,5706	14,5880
$0^{\circ}/90^{\circ}/0^{\circ}$ graphite/epoxy (0°) glass/epoxy (90°)	1	(-1,-1)	0	0	4.8372	7.1524	8.1258	8.4479	8.4960
			100	0	6.6860	8.7836	9.6418	9.9227	9.9646
			100	10	16.6860	18.7836	19.6418	19.9227	19.9646
	2		0	0	1.1648	1.6893	1.9038	1.9740	1.9845
			100	0	4.1493	5.7183	6.6752	7.0505	7.1095
			100	10	14.1493	15.7183	16.6752	17.0505	17.1095
90°/0°/90°/0° graphite/epoxy (90°) glass/epoxy (0°)	1	(-1,-1)	0	0	3.7627	5.1002	5.5970	5 .7540	5.7771
			100	0	9.4571	17.4186	23.0015	25.3700	25.7505
			100	10	19.4571	27.4186	33.0015	35.3700	35.7505
	2		0	0	1.8576	3.8480	5.2437	5.8354	5.9310
			100	0	4.7850	10.0244	18.6133	25.9852	27.6196
			100	10	14.7850	20.0244	28.6133	35.9852	37.6196

In Table 1, non-dimensional critical uniaxial buckling load of antisymmetric two cross-ply square plates on elastic foundation for different orthotropy ratios are shown and compared with different shear deformation theories; it can be seen that the buckling load obtained by the present theory show also a satisfied agreement with other theories.

In Tables 2-3 the non-dimensional critical buckling load subjected to biaxial and the tension in the x-direction and compression in the *y*-direction respectively with the variation of aspect ratio and side-to-thickness ratio of simply supported in the orthotropic square plates.

These comparisons show that the results from the present method with new trigonometric shape function of displacement using two variable refined plate theories are in good agreement with the existing results.

To show the effect of hybridization material with the variation of different parameters, geometrical and material on behavior non-dimensional critical uniaxial and biaxial buckling loads; the analysis for these elements is carried out considering configurations described in Tables 5-6.

Figs. 2-3, show the effect of hybridization and behavior of different hybrid laminates; the critical buckling loads of hybrid materials are evaluated varying the aspect ratio and thickness ratio a/h. It is established that the elastic foundations significantly affect the mechanical behavior of hybrid composites plates. It is also seen from the figures that increasing the values of the foundation stiffness and orthotropy ratio increases the buckling load of the plate. The results are the maximum for the symmetric plate of material I and the minimum for the antisymmetric plate of material II.

It can be seen from these various curves that without elastic foundations, the results of non-

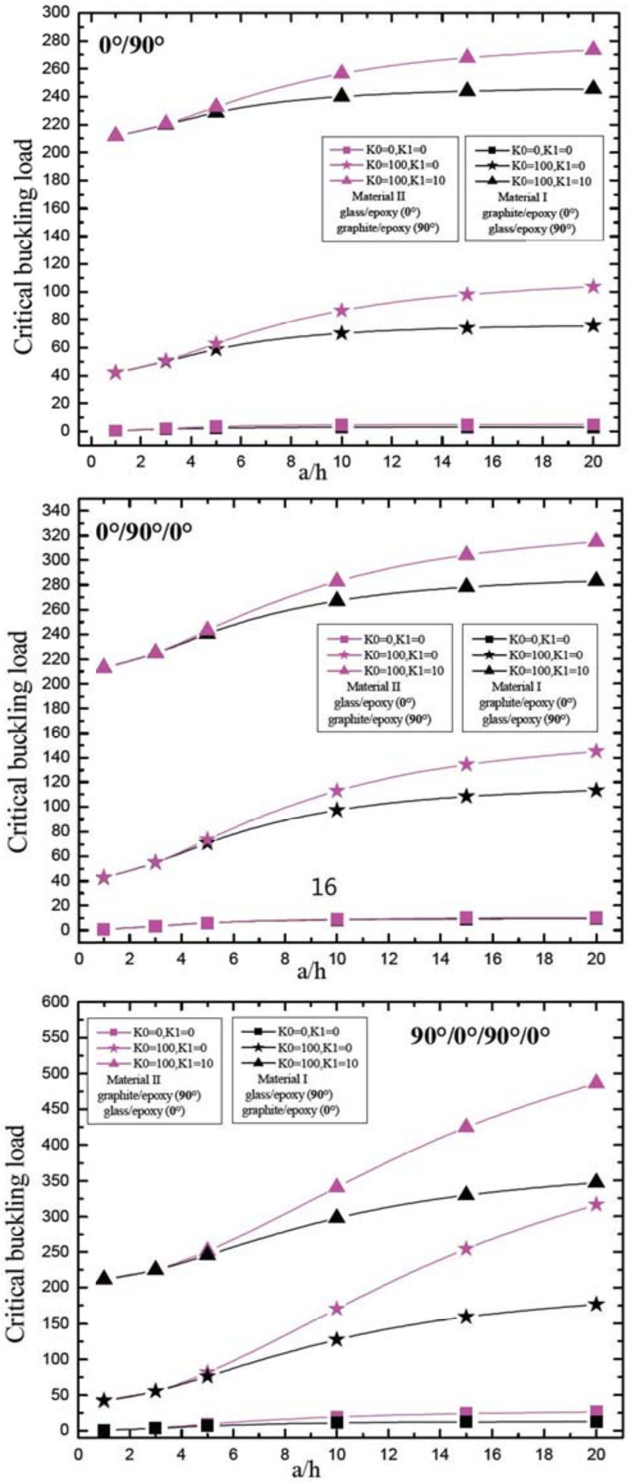

Fig. 2 Effect of hybridization material on variation of non-dimensional critical uniaxial buckling load for three different types of elastic foundation with (a/b=2)

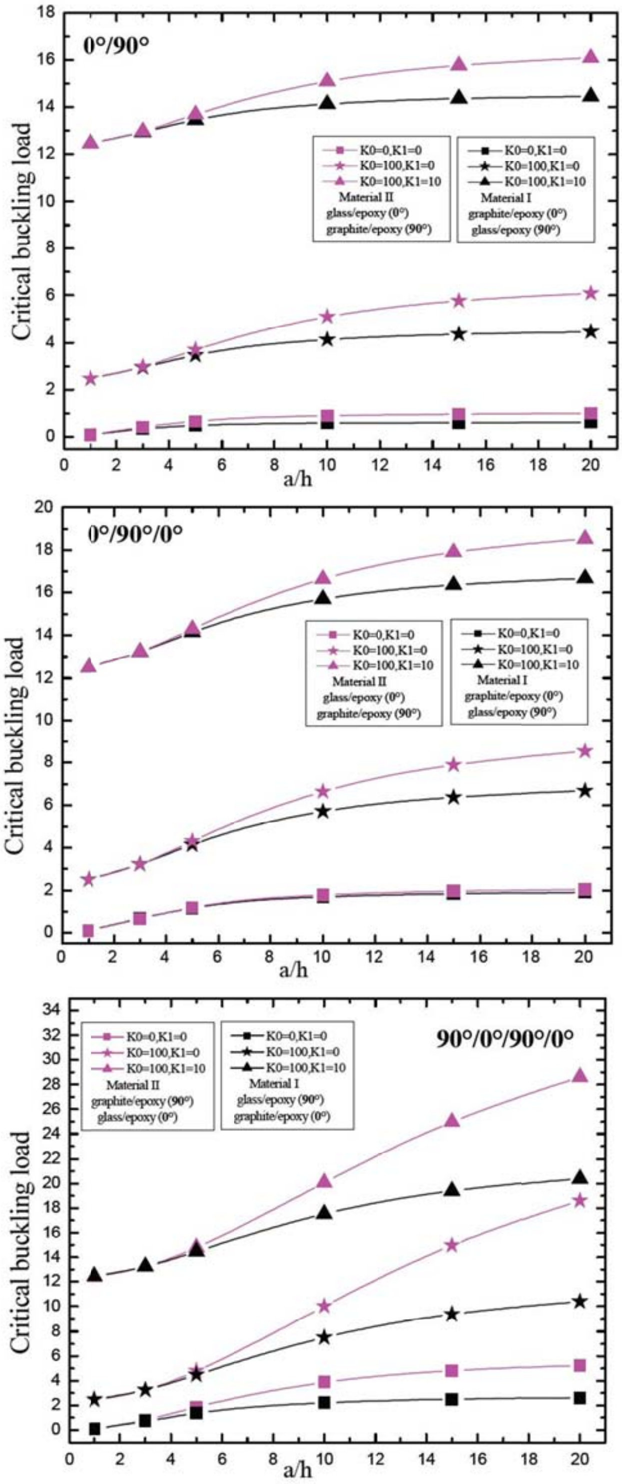

Fig. 3 Effect of hybridization material on variation of non-dimensional critical biaxial buckling load for three different types of elastic foundation with (a/b=2)

dimensional critical buckling load are the same and stabilized with long-thick for different hybrid laminates material symmetric and antisymmetric.

A numerical analysis enables us to find both the influence of elastic foundation, varying mechanical and geometrical parameters of the laminates, and advantages furnished from hybridization, by comparison with result relative to non hybrid laminates.

5. Conclusions

In this article, an approach to calculate the buckling load of plates made in different hybrid materials, resting on elastic foundation, under in plane loads was presented and investigated analytically. The closed-form solution of a simply supported rectangular plate subjected to in-plane loading has been obtained by using the Navier method with new trigonometric shape function based on high order theory and compared with, first-order shear deformable theory solutions (FSDT) ,refined plate theory (RPT) and FEM with 3D Elasticity.

A parametric study of the influence of different hybrid laminates material symmetric and antisymmetric, the effect of elastic foundations, slenderness ratio, on the critical buckling load. Based on the derived results, the following conclusions can be drawn:

• The present model, based on a new shear shape function without shear correction factor, gives results which are in good agreement with the solution of previous higher-order theories.

• The introduction of elastic foundations leads generally to increases the value of the critical buckling load with increasing a/b ratio and remains almost unaltered for *a/h* greater than 18.

• The effect of hybridization with symmetric and antisymmetric material without elastic foundations leads generally to appear to have no influence on the critical buckling load.

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