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Dynamic response of a Timoshenko beam to a continuous distributed moving load

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Abstract. In the paper we study dynamic response of a finite, simply supported Timoshenko beam subject to a moving continuously distributed forces. Three problems have been considered. The dynamic response of the Timoshenko beam under a uniform distributed load moving with a constant velocity v has been considered as the first problem. Obtained solutions allow to find the response of the beam under the interval of the finite length a uniformly distributed moving load. Part of the solutions are presented in a closed form instead of an infinite series. As the second problem the steady-state vibrations of the beam under uniformly distributed mass m_1 moving with the constant velocity has been considered. The vibrations of the beam caused by the interval of the finite length randomly distributed load moving with constant velocity is considered as the last problem. It is assumed that load process is space-time stationary stochastic process.

Keywords: Timoshenko beam; moving force; vibrations

1. Introduction

The problem of a dynamic response of a structure subjected to moving loads is interesting and important. This problem occurs in dynamics of bridges, roadways, railways and runways as well as missiles and aircrafts. Different types of structures and girders like beams, plates, shells, frames have been considered. Also different models of moving loads have been assumed (Kryloff 1905, Fryba 1999, Klasztorny and Langer 1990, Michaltsos 2002, Podwórna 2011). Deterministic and stochastic approaches have been presented (Tung 1969, Fryba 1976, Sieniawska and Śniady 1990). It would be interesting to study the problem of the dynamic response of Timoshenko (1921) beam to moving loads. This problem has been considered, among others, in the papers (Timoshenko 1922, Achenbach and Sun 1965, Florence 1965, Steel 1968, Tang 1966, Bogacz *et al*. 1986, Szcześniak, Katz *et al*. 1988, Zu and Han 1994, Lee 1995, Felszeghy 1996a, b, Wang 1997, Chen *et al*. 2011, Ariaei *et al*. 2011). Both the moving forces (Timoshenko 1922, Achenbach and Sun 1965, Florence 1965, Felszeghy 1996b, Chen *et al*. 2011) and the moving masses (Bogacz *et al*. 1986) have been assumed as the model of moving load. Also vibration

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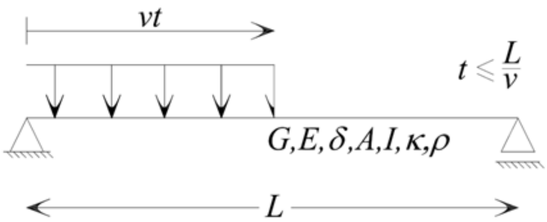


Fig. 1 Scheme of Timoshenko beam under moving load. Phase of entry of load

multi-span Timoshenko beam due to moving load have been considered (Wang 1997, Ariaei *et al*. 2011). The problem of vibration of the Euler-Bernoulli beam caused by uniform partially distributed moving mass has been presented in (Esmailzadeh and Ghorasi 1995). In the paper we study dynamic response of a finite, simply supported Timoshenko beam subject to a moving continuously distributed forces. Three problems have been considered. The dynamic response of the Timoshenko beam under a uniform distributed load moving with a constant velocity v has been considered as the first problem. Obtained solutions allow to find the response of the beam under the interval of the finite length a uniformly distributed moving load. Part of the solutions are presented in a closed form instead of an infinite series. The problems of finding the closed solutions in dynamics of string, beams and frames loaded by moving force is presented in the papers (Kączkowski 1963, Reipert 1969, 1970, Śniady 2008, Rusin *et al.* 2011). For a finite, simply supported Timoshenko beam closed forms of the solutions take different forms whether the velocity of the moving force is smaller or bigger than the velocities of shear and bending waves of the beam. This follows from the fact that for Timoshenko beam (contrary to Euler-Bernoulli beam) wave phenomena can occur. As the second problem the steady-state vibrations of the beam under uniformly distributed mass m_1 moving with the constant velocity has been considered. The vibrations of the beam caused by the interval of the finite length randomly distributed load moving with constant velocity is considered as the last problem. It is assumed that load process is space-time stationary stochastic process. The problems just mentioned may be applied to the vibration of highway and railway bridges.

2. Vibration of a Timoshenko beam under a moving force

We consider vibrations of a simply supported Timoshenko beam of finite length L subjected to a uniform partially distributed load moving with a constant velocity v (Fig. 1)

Vibrations of a beam are described by the equations

$$-\frac{GA}{\kappa}\frac{\partial^2 w(x,t)}{\partial x^2}+\frac{GA}{\kappa}\frac{\partial\phi(x,t)}{\partial x}+A\rho\frac{\partial^2 w(x,t)}{\partial t^2}=pH(vt-x) \tag{1}$$

$$EJ\frac{\partial^2\varphi(x,t)}{\partial x^2}+\frac{GA}{\kappa}\frac{\partial w(x,t)}{\partial x}-\frac{GA}{\kappa}\varphi(x,t)-J\rho\frac{\partial^2\varphi(x,t)}{\partial t^2}=0 \tag{2}$$

where A and J denote the cross-section area and inertia momentum, respectively, E and G are Young modulus and shear modulus, respectively, κ is the shear coefficient, ρ is the mass density, H( ) is Heaviside'a step function.

The functions $w(x,t)$ and $\varphi(x,t)$ describe the transverse displacement and the rotation of the cross-section of the beam. The bending moment $M(x,t)$ and the shear force $Q(x,t)$ are described by the relations

$$M(x,t) = -EJ\frac{\partial\varphi(x,t)}{\partial x}, \; Q(x,t) = \frac{GA}{\kappa}[\frac{\partial w(x,t)}{\partial x} - \varphi(x,t)]. \tag{3}$$

For a finite, simply supported beam the boundary conditions have forms

$$w(0,t) = w(L,t) = 0, \; \left.\frac{\partial\varphi(x,t)}{\partial x}\right|_{x=0} = \left.\frac{\partial\varphi(x,t)}{\partial x}\right|_{x=L} = 0, \tag{4}$$

where L is the span length.

After introducing the dimensionless variables

$$\xi = \frac{x}{L}, \; T = \frac{vt}{L}, \; \xi \in [0,1], \; T \in [0,1], \tag{5}$$

the Eqs. (1), (2) take the form

$$-w^{//}(\xi.T) + L\phi^{/}(\xi,T) + \eta^2 \ddot{w}(\xi,T) = p_0 H(T-\xi), \tag{6}$$

$$\frac{\lambda}{r} w(\xi,T) - \lambda^2\varphi(\xi,T) + \gamma^2\varphi''(\xi,T) - \eta^2\ddot{\varphi}(\xi,T) = 0 \tag{7}$$

where: $\eta = \dfrac{v}{v_s}$, $\gamma = \dfrac{v_g}{v_s}$, $\mathrm{r} = \sqrt{\dfrac{J}{A}}$, $p_0 = \dfrac{pL^2\kappa}{AG}$, $v_s = \sqrt{\dfrac{G}{\kappa\rho}}$, $v_g = \sqrt{\dfrac{E}{\rho}}$,

$\lambda = \dfrac{L}{r}$ is slenderness ratio of the beam.

Roman numerals denote differentiation with respect to the spatial coordinate ξ, and dots denote differentiation with respect to time T. For elastic materials the inequality $v_g \geq v_s (\gamma \geq 1)$ holds true. Respectively the quantities v_s and v_g represent the shear wave velocity and bending wave velocity.

The boundary conditions have forms

$$w(0,T) = w(1,T) = 0, \; \varphi^{/}(0,T) = \varphi^{/}(1,T) = 0. \tag{8}$$

Let the initial conditions have forms

$$w(\xi,0) = 0, \; \dot{\mathrm{w}}(\xi,0) = 0, \quad \varphi(\xi,0) = 0, \; \dot{\varphi}(\xi,0) = 0. \tag{9}$$

The response of the beam $w(\xi,T)$ and $\varphi(\xi,T)$ for boundary conditions (8) are assumed to be in the form of sine and cosine series

$$w(\xi,T) = \sum_{n=1}^{\infty} y_n(T) \sin n\pi\xi, \tag{10}$$

$$\varphi(\xi,T) = \sum_{n=1}^{\infty} \varphi_n(T) \cos n\pi\xi. \tag{11}$$

By substituting expressions (10), (11) into the system of the Eqs. (6), (7) the solutions are sums of the particular integrals w_A=(ξ,T), φ_A=(ξ,T) and general integrals w_S=(ξ,T) and φ_S=(ξ,T)

$$w(\xi,T) = w_A(\xi,T) + w_S(\xi,T), \tag{12}$$

$$\varphi(\xi,T) = \varphi_A(\xi,T) + \varphi_S(\xi,T), \tag{13}$$

where

$$\begin{aligned} w_A(\xi,T) = {} & 2p_0\frac{\lambda^2}{\gamma^2}\sum_{n=1}^{\infty}\frac{\sin n\pi\xi}{(n\pi)^5} + 2p_0\sum_{n=1}^{\infty}\frac{\sin n\pi\xi}{(n\pi)^3} - \frac{2p_0}{1-\eta^2}\sum_{n=1}^{\infty}\frac{\cos n\pi T\sin n\pi\xi}{(n\pi)^3} - \\ & -\frac{2p_0\lambda^2}{1-\eta^2}\sum_{n=1}^{\infty}\frac{\cos n\pi T\sin n\pi\xi}{(n\pi)^3[(n\pi)^2(1-\eta^2)(\gamma^2-\eta^2)-\eta^2\lambda^2]}, \end{aligned} \tag{14}$$

$$\varphi_A(\xi,T) = 2p_0\frac{\lambda^2}{L}\{\sum_{n=1}^{\infty}\frac{1}{(n\pi)^4\gamma^2} - \sum_{n=1}^{\infty}\frac{\cos n\pi T}{(n\pi)^2[(n\pi)^2(1-\eta^2)(\gamma^2-\eta^2)-\eta^2\lambda^2]}\}\cos n\pi\xi. \tag{15}$$

and

$$w_S(\xi,T) = \sum_{n=1}^{\infty}[A_n\cos r_nT + B_n\cos s_nT]\sin n\pi\xi, \tag{16}$$

$$\varphi_S(\xi,T) = \sum_{n=1}^{\infty}[C_n\cos r_nT + D_n\cos s_nT]\cos n\pi\xi, \tag{17}$$

where $r_n, s_n = \dfrac{\sqrt{2}}{2\eta}\sqrt{\lambda^2+(n\pi)^2(1+\gamma^2)\pm\sqrt{[\lambda^2+(n\pi)^2(1-\gamma)^2][\lambda^2+(n\pi)^2(1+\gamma)^2]}}$.

The constants A_n, B_n, C_n, D_n can be found from the initial conditions (9), and have the forms

$$A_n = \{r_n^2 + \frac{1}{\eta^2}[\gamma^2(n\pi)^2+\lambda^2]\}\frac{y_{sn}(0)}{s_n^2-r_n^2} + \frac{L(n\pi)}{\eta^2}\frac{\varphi_{sn}(0)}{s_n^2-r_n^2}, \tag{18}$$

$$B_n = -\{s_n^2 + \frac{1}{\eta^2}[\gamma^2(n\pi)^2+\lambda^2]\}\frac{y_{sn}(0)}{s_n^2-r_n^2} - \frac{L(n\pi)}{\eta^2}\frac{\varphi_{sn}(0)}{s_n^2-r_n^2}, \tag{19}$$

$$C_n = \frac{\lambda(n\pi)}{r\eta^2}\frac{y_{sn}(0)}{s_n^2-r_n^2} + [r_n^2 + (\frac{n\pi}{\eta})^2]\frac{\varphi_{sn}(0)}{s_n^2-r_n^2}, \tag{20}$$

$$D_n = -\frac{\lambda(n\pi)}{r\eta^2}\frac{y_{sn}(0)}{s_n^2-r_n^2} - [s_n^2 + (\frac{n\pi}{\eta})^2]\frac{\varphi_{sn}(0)}{s_n^2-r_n^2}, \tag{21}$$

where

$$y_{sn}(0) = \frac{2p_0}{(1-\eta^2)\gamma^2}\{\frac{[\lambda^2(1-\eta^2)-\eta^2\gamma^2(n\pi)^2][(n\pi)^2(1-\eta^2)(\gamma^2-\eta^2)-\eta^2\lambda^2]-\lambda^2\gamma^2(n\pi)^2}{(n\pi)^5[(n\pi)^2(1-\eta^2)\gamma^2-\eta^2\lambda^2]}\},$$

$$\varphi_{sn}(0) = 2p_0\frac{\lambda^2}{L}\{\frac{1}{\gamma^2(n\pi)^4} - \frac{1}{(n\pi)^2[(n\pi)^2(1-\eta^2)(\gamma^2-\eta^2)-\eta^2\lambda^2]}\}.$$

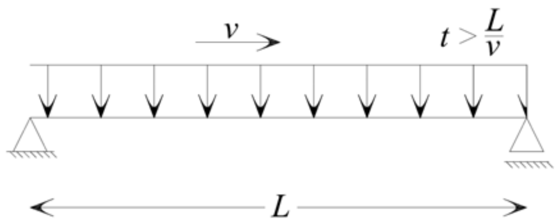


Fig. 2 Scheme of Timoshenko beam under moving load. Full load of the beams

Let consider the case when $T \geq 1$ (Fig. 2).
The solution have forms Eqs. (12) and (13), where aperiodic part are equal

$$w_A(\xi) = \frac{2p_0}{\gamma^2} \sum_{n=1}^{\infty} \frac{[1-(1)^n][\gamma^2 (n\pi)^2 + \lambda^2] \sin n\pi\xi}{(n\pi)^5}, \tag{22}$$

$$\varphi_A(\xi) = \frac{2p_0\lambda^2}{\gamma^2 L} \sum_{n=1}^{\infty} \frac{[1-(1)^n] \cos n\pi\xi}{(n\pi)^4}. \tag{23}$$

The general integrals can be presented in the forms,

$$w_S(\xi,T) = w_{S1}(\xi,T) + w_{S2}(\xi,T-1), \tag{24}$$

$$\varphi_S(\xi,T) = \varphi_{S1}(\xi,T) + \varphi_{S2}(\xi,T-1), \tag{25}$$

where functions $w_{S1}=(\xi,T)$ and $\varphi_{S1}=(\xi,T)$ are given by expressions (16)-(21), respectively, assuming that $T \geq 1$. The functions $w_{S2}=(\xi,T-1)$ and $\varphi_{S2}=(\xi,T-1)$ have forms

$$w_{S2}(\xi,T-1) = \sum_{n=1}^{\infty} [E_n \cos r_n (T-1) + F_n \cos s_n (T-1)] \sin n\pi\xi, \tag{26}$$

$$\varphi_{S2}(\xi,T-1) = \sum_{n=1}^{\infty} [G_n \cos r_n (T-1) + H_n \cos s_n (T-1)] \cos n\pi\xi, \tag{27}$$

where

$$E_n = \{r_n^2 + \frac{1}{\eta^2}[\gamma^2 (n\pi)^2 + \lambda^2]\} \frac{\tilde{y}_{sn}(0)}{s_n^2 - r_n^2} + \frac{L(n\pi)}{\eta^2} \frac{\tilde{\varphi}_{sn}(0)}{s_n^2 - r_n^2}, \tag{28}$$

$$F_n = -\{s_n^2 + \frac{1}{\eta^2}[\gamma^2 (n\pi)^2 + \lambda^2]\} \frac{\tilde{y}_{sn}(0)}{s_n^2 - r_n^2} - \frac{L(n\pi)}{\eta^2} \frac{\tilde{\varphi}_{sn}(0)}{s_n^2 - r_n^2}, \tag{29}$$

$$G_n = \frac{\lambda^2 (n\pi)}{L\eta^2} \frac{\tilde{y}_{sn}(0)}{s_n^2 - r_n^2} + [r_n^2 + (\frac{n\pi}{\eta})^2] \frac{\tilde{\varphi}_{sn}(0)}{s_n^2 - r_n^2}, \tag{30}$$

$$H_n = -\frac{\lambda^2 (n\pi)}{L\eta^2} \frac{\tilde{y}_{sn}(0)}{s_n^2 - r_n^2} - [s_n^2 + (\frac{n\pi}{\eta})^2] \frac{\tilde{\varphi}_{sn}(0)}{s_n^2 - r_n^2}, \tag{31}$$

where

$$\tilde{y}_{sn}(0) = -\frac{2p_0}{\gamma^2}\frac{[1-(-1)^n][(n\pi)^2\gamma^2+\lambda^2]}{(n\pi)^5},$$

$$\tilde{\varphi}_{sn}(0) = -\frac{2p_0\lambda^2}{\gamma^2 L}\frac{[1-(-1)^n]}{(n\pi)^4}.$$

3. Closed form solutions

The functions $w_A=(\xi,T)$ and $\varphi_A=(\xi,T)$ are aperiodic vibrations and satisfy the nonhomogeneous differential Eqs. (6)-(7). These functions do not satisfy the initial conditions of motion (9). The functions $w_S=(\xi,T)$ and $\varphi_S=(\xi,T)$ are free vibrations of the Timoshenko beam which satisfy the homogeneous differential Eqs. (6)-(7) (p_0=0) and together with the aperiodic functions the initial conditions of motion (9) are satisfied. Now we will present the aperiodic solution $w_A=(\xi,T)$ in closed forms.

Let notice that the first and second series in the expressions (14) have forms

$$w_{A1}(\xi) = 2p_0\frac{\lambda^2}{\gamma^2}\sum_{n=1}^{\infty}\frac{\sin n\pi\xi}{(n\pi)^5} = p_0\frac{L^2}{\gamma^2 r^2}[\frac{-\xi^5}{120}+\frac{\xi^4}{24}-\frac{\xi^3}{18}+\frac{\xi}{45}], \tag{32}$$

$$w_{A2}(\xi) = 2p_0\sum_{n=1}^{\infty}\frac{\sin n\pi\xi}{(n\pi)^3} = p_0(\frac{\xi^3}{6}-\frac{\xi^2}{2}+\frac{\xi}{3}). \tag{33}$$

The next two series in Eq. (14)

$$w_{A3}(\xi,T) = -\frac{2p_0}{1-\eta^2}\sum_{n=1}^{\infty}\frac{\cos n\pi T\sin n\pi\xi}{(n\pi)^3}, \tag{34}$$

and

$$w_{A4}(\xi,T) = -\frac{2p_0\lambda^2}{1-\eta^2}\sum_{n=1}^{\infty}\frac{\cos n\pi T\sin n\pi\xi}{(n\pi)^3[(n\pi)^2(1-\eta^2)(\gamma^2-\eta^2)-\eta^2\lambda^2]}, \tag{35}$$

are solutions of the ordinary equations

$$w_{A3}^{IV}(\xi,T) = \frac{p_0}{1-\eta^2}\delta'(\xi-T), \tag{36}$$

and

$$w_{A4}^{VI}(\xi,T) + \frac{\eta^2\lambda^2}{(1-\eta^2)(\gamma^2-\eta^2)}w_{A4}^{IV}(\xi,T) = \frac{-p_0\lambda^2}{(1-\eta^2)^2(\gamma^2-\eta^2)}\delta'(\xi-T), \tag{37}$$

for the boundary conditions

$$w_{Ai}(0,T) = w_{Ai}(1,T) = 0,\ \ w_{Ai}^{II}(0,T) = w_{Ai}^{II}(1,T) = 0,\ \ w_{A4}^{IV}(0,T) = w_{A4}^{IV}(1,T) = 0, \tag{38}$$

where i=3,4.

The variable T in Eqs. (36) and (37) is the only parameter which describes the location of the moving force on the beam. After solving the Eqs. (36) and (37) using, for example, the Laplace transform we can obtain the functions $w_{A3}=(\xi,T)$ and $w_{A4}=(\xi,T)$ in the closed form instead of a series. The function $w_{A3}=(\xi,T)$ has the form

$$w_{A3}(\xi,T)=\frac{p_0}{2(1-\eta^2)}[\frac{1}{3}\xi(1-\xi^2)-\xi(1-T)^2] \text{ for } \xi\le T,$$

$$w_{A3}(\xi,T)=\frac{p_0}{2(1-\eta^2)}[\frac{1}{3}\xi(1-\xi^2)-\xi(1-T)^2+(\xi-T)^2] \text{ for } \xi\ge T. \tag{39}$$

The closed form of the solutions $w_{A4}=(\xi,T)$ depends on the velocity of moving force. In the case if $\eta<1$ or $\eta>\gamma$ when the velocity of the force is smaller than the velocity of the shear wave ($\eta<1$) or larger than the velocity of the bending wave ($\eta>\gamma$) the solutions have forms

$$w_{A4}(\xi,T)=\frac{p_0}{(1-\eta^2)\eta^2}\{\frac{\sin a\xi\cos a(1-T)}{a^2\sin a}+\frac{1}{2}\xi(1-T)^2-\frac{1}{6}\xi(1-\xi^2)-\frac{1}{a^2}\xi\}, \text{ for } \xi\le T,$$

$$w_{A4}(\xi,T)=\frac{p_0}{\eta^2}[-\frac{\sin a(1-\xi)\cos aT}{a^2\sin a}+\frac{1}{2}(1-\xi)(\xi-T^2)-\frac{1}{6}\xi(1-\xi^2)+\frac{1}{a^2}(1-\xi)], \text{ for } \xi\ge T. \tag{40}$$

where

$$a^2=\frac{\eta^2\lambda^2}{(1-\eta^2)(\gamma^2-\eta^2)}. \text{ for } \eta<1, \quad \gamma<\eta,$$

and

$$w_{A4}(\xi,T)=\frac{p_0}{(\eta^2-1)\eta^2}[\frac{\sinh\bar{a}\xi\cosh\bar{a}(1-T)}{\bar{a}^2\sinh\bar{a}}-\frac{1}{2}(1-T)^2\xi-\xi\frac{1}{a^2}+\frac{1}{6}\xi(1-\xi^2)], \text{ for } \xi\le T,$$

$$w_{A4}(\xi,T)=\frac{p_0}{(\eta^2-1)\eta^2}[-\frac{\sinh a(1-\xi)\cosh aT}{a^2\sinh a}-\frac{1}{2}(1-\xi)(\xi-T^2)+ \frac{1}{6}\xi(1-\xi^2)+\frac{1}{a^2}(1-\xi)], \text{ for } \xi\ge T, \tag{41}$$

where

$$\bar{a}^2=\frac{\eta^2\lambda^2}{(\eta^2-1)(\gamma^2-\eta^2)}. \text{ for } \gamma>\eta>1.$$

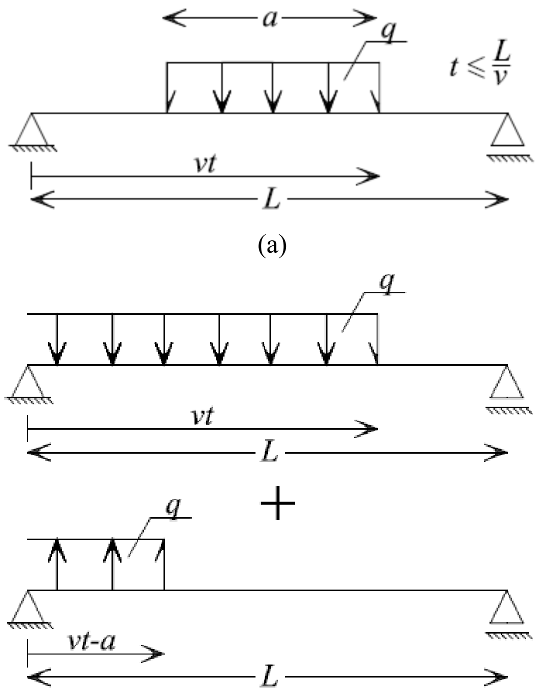
The part of the closed solution for angle is given at the section 5.

In the case if $T\ge1$ the closed solutions have forms

$$w_A(\xi)=p_0[(\frac{1}{2}+\frac{\lambda^2}{24\gamma^2})\xi-\frac{\xi^2}{2}-\frac{\lambda^2}{12\gamma^2}\xi^3+\frac{\lambda^2}{24\gamma^2}\xi^4], \tag{42}$$

$$\varphi_A(\xi)=\frac{p_0\lambda^2}{L\gamma^2}(\frac{1}{24}-\frac{\xi^2}{4}+\frac{\xi^3}{6}). \tag{43}$$

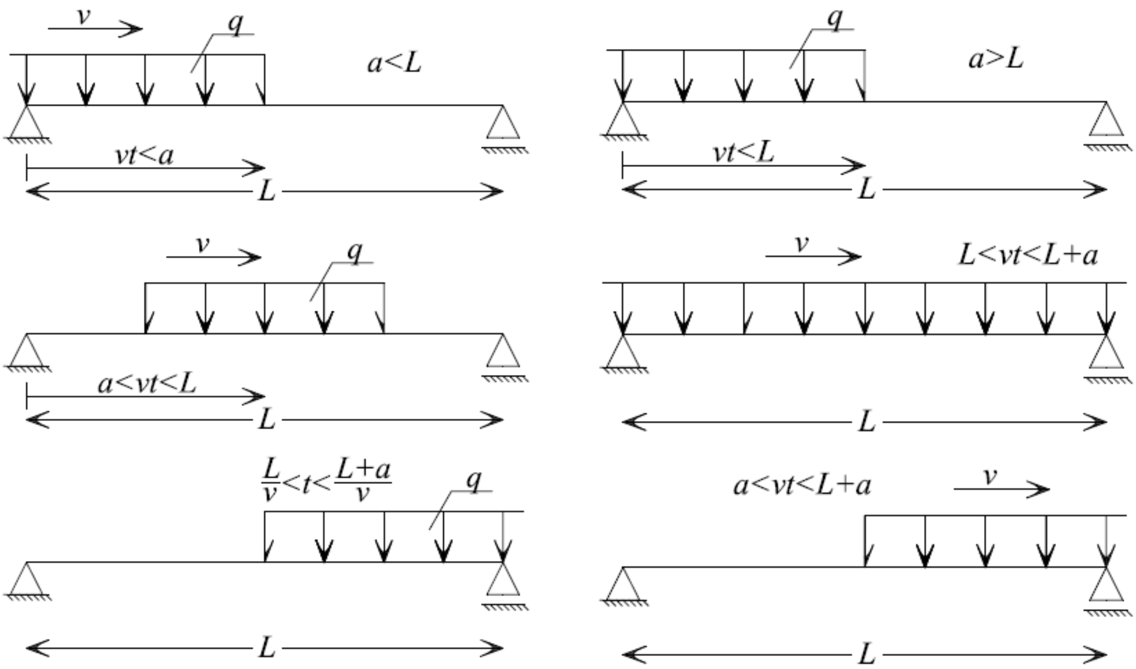
Let the vibrations of the beam be caused by the interval of the finite length *a* uniformly distributed moving load. (Fig. 3(a)). The solution of the problem can be obtained by superposition



(a)

(b)

Fig. 3 Interval uniformly distributed moving load



(a) (b)

Fig. 4 Phases of moving interval uniformly distributed load of the solutions which are given above (see Fig. 3(b)). We consider three phases, namely, the load drive on the beam, the full load process on the beam and drive off of the load (see Fig. 4)

For example for the case $a<L$ and $a\le vt\le L$, $\frac{a}{L}\le T\le 1$ (Fig. 4(a)) the solutions have forms

$$w(\xi,T)=w_A(\xi,T)+w_S(\xi,T)-w_A(\xi,T-T_0)-w_S(\xi,T-T_0), \tag{44}$$

$$\varphi(\xi,T)=\varphi_A(\xi,T)+\varphi_S(\xi,T)-\varphi_A(\xi,T-T_0)-\varphi_S(\xi,T-T_0), \tag{45}$$

where $T_0=\frac{a}{L}$ and the functions w_A=(ξ,T), w_S=(ξ,T), φ_A=(ξ,T) and φ_S=(ξ,T) are given in the sections 2 and 3 (see Eqs. (14)-(21) and also (32), (33), (39)-(41)).

4. The inertial moving load

Let us consider the vibrations of the beam under uniformly distributed moving mass m_1.

$$\begin{aligned}&-\frac{GA}{\kappa}\frac{\partial^2 w(x,t)}{\partial x^2}+\frac{GA}{\kappa}\frac{\partial\varphi(x,t)}{\partial x}+A\rho\frac{\partial^2 w(x,t)}{\partial t^2}=\\&[m_1 g-m_1(\frac{\partial^2 w(x,t)}{\partial t^2}+2v\frac{\partial^2 w(x,t)}{\partial x\partial t}+v^2\frac{\partial^2 w(x,t)}{\partial x^2})]H(vt-x),\end{aligned} \tag{46}$$

$$EJ\frac{\partial^2\varphi(x,t)}{\partial x^2}+\frac{GA}{\kappa}\frac{\partial w(x,t)}{\partial x}-\frac{GA}{\kappa}\varphi(x,t)-J\rho\frac{\partial^2\varphi(x,t)}{\partial t^2}=0 \tag{47}$$

where g is acceleration of gravity.

Let notice that the load process is the sum of the constant part and inertial part which changes in time. After using dimensionless variables (5) the Eqs. (46) and (47) have forms

$$\begin{aligned}&-w^{//}(\xi.T)+L\varphi^{/}(\xi,T)+\eta^2\ddot{w}(\xi,T)=\\&\{\frac{m_1 gL^2\kappa}{GA}-\frac{m_1 v^2\kappa}{GA}[\ddot{w}(\xi,T)+2\dot{w}^{I}(\xi,T)+w^{II}(\xi,T)]\}H(T-\xi),\end{aligned} \tag{48}$$

$$\frac{\lambda}{r}w(\xi,T)-\lambda^2\varphi(\xi,T)+\gamma^2\varphi''(\xi,T)-\eta^2\ddot{\varphi}(\xi,T)=0. \tag{49}$$

Let consider only steady-state solution for T>1 (Fig. 5). In this case w=(ξ,T)=w_{steady}=(ξ) and φ=(ξ,T)=φ_{steady}=(ξ) since the Eqs. (48) and (49) obtain the system of ordinary equations

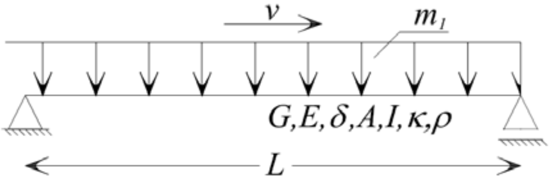


Fig. 5 Scheme of beam under moving uniformly distributed mass.

$$-(1-\frac{m_1 v^2 \kappa}{GA}) w^{II}_{steady}(\xi) + L\varphi^{I}_{steady}(\xi) = \frac{m_1 g L^2 \kappa}{GA}, \tag{50}$$

$$\frac{\lambda^2}{L} w^{I}_{steady}(\xi) - \lambda^2 \varphi_{steady}(\xi) + \gamma^2 \varphi^{II}_{steady}(\xi) = 0. \tag{51}$$

The response of the beam w_{steady}=(ξ) and φ_{steady}=(ξ) for boundary conditions (8) are assumed to be in the form of sine and cosine series

$$w_{steady}(\xi) = \sum_{n=1}^{\infty} y_n \sin n\pi\xi, \tag{52}$$

$$\varphi_{steady}(\xi) = \sum_{n=1}^{\infty} \varphi_n \cos n\pi\xi. \tag{53}$$

By substituting expressions (52), (53) into the system of the Eqs. (50), (51) the solutions have forms

$$w_{steady}(\xi) = 2\frac{gL^2\mu\eta^2}{v^2}\sum_{n=1}^{\infty}\frac{[1-(-1)^n][\lambda^2+\gamma^2(n\pi)^2]\sin n\pi\xi}{(n\pi)^3[(1-\mu\eta^2)\gamma^2(n\pi)^2-\mu\eta^2\lambda^2]}, \tag{54}$$

$$\varphi_{steady}(\xi) = 2\frac{gL\lambda^2\mu\eta^2}{v^2}\sum_{n=1}^{\infty}\frac{[1-(-1)^n]\cos n\pi\xi}{(n\pi)^2[(1-\mu\eta^2)\gamma^2(n\pi)^2-\mu\eta^2\lambda^2]}, \tag{55}$$

where $\mu = \frac{m_1}{\rho A}$ is ratio of moving mass to the beam mass.

The critical velocity v_{cr} of moving mass is equal to

$$v_{cr} = v_g \frac{\pi}{\sqrt{\mu(\gamma^2\pi^2+\lambda^2)}} = v_g \frac{1}{\sqrt{\mu[\gamma^2+(\frac{\lambda}{\pi})^2]}}. \tag{56}$$

The closed solutions have forms if $\mu\eta^2$<1 $(v < v_s\sqrt{\frac{1}{\mu}})$, than

$$w_{steady}(\xi) = \frac{gL^2}{v^2}\{\frac{\gamma^2}{\mu\eta^2\lambda^2}[\frac{\sin b\xi - \sin b + \sin b(1-\xi)}{\sin b}] - \frac{1}{2}\xi(1-\xi)\}, \tag{57}$$

$$\varphi_{steady}(\xi) = \frac{gL}{v^2}[\frac{\cos b\xi - \cos b(1-\xi)}{b\sin b} - \frac{1}{2} + \xi], \tag{58}$$

where $b = \frac{v\lambda}{v_g}\sqrt{\frac{\mu}{1-\mu\eta^2}} = \frac{v\lambda}{\gamma}\sqrt{\frac{\mu}{v_s^2-\mu v^2}}$.

In the case if $\mu\eta^2$>1 $(v > v_s\sqrt{\frac{1}{\mu}})$, than

$$w_{steady}(\xi) = \frac{gL^2}{v^2}\{\frac{\gamma^2}{\mu\eta^2\lambda^2}[\frac{\sinh\bar{b}\xi - \sinh\bar{b} + \sinh\bar{b}(1-\xi)}{\sinh\bar{b}}] - \frac{1}{2}\xi(1-\xi)\}, \tag{59}$$

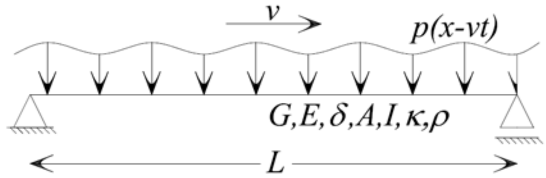


Fig. 6 Beam loaded by a randomly distributed load moving with a constant velocity.

$$\varphi_{steady}(\xi)=\frac{gL}{v^2}[-\frac{\cosh\bar{b}\xi-\cosh\bar{b}(1-\xi)}{\bar{b}\sinh\bar{b}}-\frac{1}{2}+\xi], \tag{60}$$

where $\bar{b}=\frac{v\lambda}{v_g}\sqrt{\frac{\mu}{\mu\eta^2-1}}=\frac{v\lambda}{\gamma}\sqrt{\frac{\mu}{\mu v^2-v_s^2}}$.

If the influence of the inertial part of the load (moving mass) is omitted than the solutions are given by Eqs. (24), (25) or (42), (43). Let notice that the closed solutions depends not only since the velocity parameter $\eta=v/v_s$ but also since proportion of moving mass m_1 to the beam mass $m=\rho A$.

5. Random vibrations of the beam

In many cases the moving loads obey laws that are of random character, particularly the effects of random motion of vehicles on irregularities of surface of bridge (Fryba 1976). Let the vibrations of the beam be caused by randomly distributed load $p(x-vt)$ moving with constant velocity v (Fig. 6).

In this case the Eq. (1) has form

$$-\frac{GA}{\kappa}\frac{\partial^2 w(x,t)}{\partial x^2}+\frac{GA}{\kappa}\frac{\partial\varphi(x,t)}{\partial x}+A\rho\frac{\partial^2 w(x,t)}{\partial t^2}=p(x-vt). \tag{1a}$$

Let us notice that the load process $p(x-vt)=p[L(\xi-T)]=p(L\,\tau)$, $(\xi-T=\tau)$, is a weak space-time stationary stochastic process and can be assumed as sum deterministic and random parts

$$p(x-vt)=\bar{p}+\tilde{p}(x-vt)=\bar{p}+\tilde{p}(L\tau), \tag{61}$$

where $\bar{p}=const., E[\tilde{p}(x-vt)]=0$ and a symbol $E[\ ]$ means expectation.

The solution for deterministic part of the load $\bar{p}=const.$ has been presented in the sections 2 and 3 for this reason we consider only vibration of the beam due to stochastic part.

Let us assume that covariance function of the moving load process

$$C_{pp}(\tau_1,\tau_2)=E[\tilde{p}(L\tau_1)\tilde{p}(L\tau_2)]=C_{pp}[L(\tau_1-\tau_2)], \tag{62}$$

is known.

For moving load modeled by space-time stationary stochastic process it is difficult to find solution using direct sine and cosine transformation as can be done in other stochastic excitations. For this reason we introduce the dynamic influence functions $H_w(\xi,T)$ and $H_\varphi(\xi,T)$ which are the responses of the Timoshenko beam loaded by a moving point force equal to one. These dynamic influence functions are solutions of the equations (see Śniady 2008).

$$-H''_w(\xi,T)+LH'_\varphi(\xi,T)+\eta^2\ddot{H}_w(\xi,T)=P_o\delta(\xi-T), \tag{63}$$

$$\frac{\lambda}{r}H_w(\xi,T)-\lambda^2H_\varphi(\xi,T)+\gamma^2H''_\varphi(\xi,T)-\eta^2\ddot{H}_\varphi(\xi,T)=0, \tag{64}$$

where $\delta(.)$ Dirac delta and $P_o=\dfrac{L\kappa}{AG}$.

The dynamic influence functions can be found in similar way, like it has been done in section 2 for the uniform distributed load and have forms (see Śniady 2008)

$$\begin{aligned}&H_w(\xi,T)=H_{wA}(\xi,T)+H_{wS}(\xi,T)=\\&=2P_o\sum_{n=1}^{\infty}\frac{[(n\pi)^2\lambda^2(\gamma^2-\eta^2)+1]\sin n\pi T\sin n\pi\xi}{(n\pi)^2[(n\pi)^2(1-\eta^2)\lambda^2(\gamma^2-\eta^2)-\eta^2]}+\sum_{n=1}^{\infty}[A_{nP}\sin r_nT+B_{nP}\sin s_nT]\sin n\pi\xi,\end{aligned} \tag{65}$$

and

$$\begin{aligned}&H_\varphi(\xi,T)=H_{\varphi A}(\xi,T)+H_{\varphi S}(\xi,T)=\\&=2P_0\sum_{n=1}^{\infty}\frac{\sin n\pi T\cos n\pi\xi}{(n\pi)L[(n\pi)^2(1-\eta^2)\lambda^2(\gamma^2-\eta^2)-\eta^2]}+\sum_{n=1}^{\infty}[C_{nP}\sin r_nT+D_{nP}\sin s_nT]\cos n\pi\xi,\end{aligned} \tag{66}$$

where

$$\begin{aligned}A_{nP}&=\frac{-2P_0\{[(n\pi)^2-\eta^2s_n^2][(n\pi)^2\lambda^2(\gamma^2-\eta^2)+1]-(n\pi)^2\}}{(n\pi)\eta^2r_n(r_n^2-s_n^2)[(n\pi)^2(1-\eta^2)\lambda^2(\gamma^2-\eta^2)-\eta^2]},\\B_{nP}&=\frac{2P_0\{[(n\pi)^2-\eta^2r_n^2][(n\pi)^2\lambda^2(\gamma^2-\eta^2)+1]-(n\pi)^2\}}{(n\pi)\eta^2s_n(r_n^2-s_n^2)[(n\pi)^2(1-\eta^2)\lambda^2(\gamma^2-\eta^2)-\eta^2]},\end{aligned} \tag{67}$$

and

$$C_{nP}=\frac{[(n\pi)^2-r_n^2\eta^2]}{L(n\pi)}A_{nP},$$
$$D_{nP}=\frac{[(n\pi)^2-s_n^2\eta^2]}{L(n\pi)}B_{nP}.$$

The closed solutions of the functions $H_{wA}(\xi,T)$ and $H_{\varphi A}(\xi,T)$ are given in the paper (Śniady 2008) and have, in the case $\eta<1$, forms

$$H_{wA}(\xi,T)=\frac{P_0}{(1-\eta^2)\eta^2}\frac{\sin\sigma(1-T)\sin\sigma\xi}{\sigma\sin\sigma}-\frac{P_0}{\eta^2}(1-T)\xi\ \text{ for }\ \xi\le T,$$
$$H_{wA}(\xi,T)=\frac{P_0}{(1-\eta^2)\eta^2}\frac{\sin\sigma T\sin\sigma(1-\xi)}{\sigma\sin\sigma}-\frac{P_0}{\eta^2}T(1-\xi)\ \text{ for }\ \xi\ge T, \tag{68}$$

and

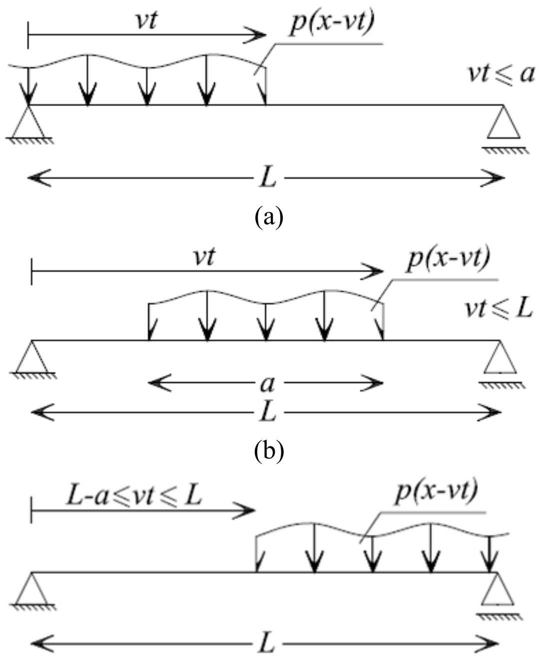


Fig.7 Phases of the moving random load on the beam

$$H_{\varphi A}(\xi,T)=-\frac{P_0}{2L\eta^2}+\frac{P_0\sin\sigma(1-T)\cos\sigma\xi}{L\eta^2\sin\sigma}\ \text{ for }\ \xi<T,$$

$$H_{\varphi A}(\xi,T)=\frac{P_0}{2L\eta^2}-\frac{P_0\sin\sigma T\cos\sigma(1-\xi)}{L\eta^2\sin\sigma}\ \text{ for }\ \xi>T, \tag{69}$$

where $\sigma^2=\dfrac{L^2\eta^2}{r^2(1-\eta^2)(\gamma^2-\eta^2)}$.

Let the vibrations of the beam be caused by the interval of the finite length $a<L$ randomly distributed load $p(x-vt)$ moving with constant velocity v. Using dynamic influence functions the response of the beam under moving load $\tilde{p}(L\tau)$ can be presented in the integral forms

(a) if $0\le T\le\dfrac{a}{L}$ (Fig. 7(a))

$$w(\xi,T)=\frac{L}{v}\int_0^T H_w(\xi,T-\tau)\tilde{p}(L\tau)d\tau, \tag{70}$$

$$\varphi(\xi,T)=\frac{L}{v}\int_0^T H_\varphi(\xi,T-\tau)\tilde{p}(L\tau)d\tau, \tag{71}$$

(b) if $\dfrac{a}{L}\le T\le 1$, (Fig. 7(b))

$$w(\xi,T)=\frac{L}{v}\int_{T-\frac{a}{L}}^{T}H_w(\xi,T-\tau)\tilde{p}(L\tau)d\tau, \tag{72}$$

$$\varphi(\xi,T)=\frac{L}{v}\int_{T-\frac{a}{L}}^{T}H_\varphi(\xi,T-\tau)\tilde{p}(L\tau)d\tau, \tag{73}$$

(c) if $1\le T\le 1+\dfrac{a}{L}$, (Fig. 7(c))

$$w(\xi,T)=\frac{L}{v}\int_{T-\frac{a}{L}}^{1}H_w(\xi,T-\tau)\tilde{p}(L\tau)d\tau, \tag{74}$$

$$\varphi(\xi,T)=\frac{L}{v}\int_{T-\frac{a}{L}}^{1}H_\varphi(\xi,T-\tau)\tilde{p}(L\tau)d\tau, \tag{75}$$

The Eqs. (70)-(75) can be used to obtain covariance function of the beam response. For example in the case b) the covariance functions have forms

Let us assume the moving excitation process to be stationary "white noise". The covariance function of load process has form $C_{pp}=\sigma_p^2\delta[L(\tau_1-\tau_2)]$ where σ_p^2 is variance of load. Since the Eqs. (76), (77) the variance of the Timoshenko beam are given by integral formulas

$$\sigma_w^2(\xi,T)=\frac{\sigma_p^2 L}{v^2}\int_{T-\frac{a}{L}}^{T}H_w^2(\xi,T-\tau)d\tau, \tag{78}$$

$$\sigma_\varphi^2(\xi,T)=\frac{\sigma_p^2 L}{v^2}\int_{T-\frac{a}{L}}^{T}H_\varphi^2(\xi,T-\tau)d\tau. \tag{79}$$

for $\dfrac{a}{L}\le T\le 1$.

The variance of the beam response can be obtained since the integral formulas (78), (79) using numerical procedure.

Remark

Let notice that for $p(x-vt)=p=const.$ after putting the (66) and (69) into (71) the part of the function $\varphi=(\xi,T)$ can be obtained in closed form. In this case instead the solutions (13), (15), (17) (20) and (21) we have

$$\varphi(\xi,T)=\varphi_1(\xi,T)+\varphi_2(\xi,T), \tag{80}$$

where

$$C_{ww}(\xi_1,\xi_2,T_1,T_2)=\left(\frac{L}{v}\right)^2\int_{T_1-\frac{a}{L}}^{T_1}\int_{T_2-\frac{a}{L}}^{T_2}H_w(\xi_1,T_1-\tau_1)H_w(\xi_2,T_2-\tau_2)C_{pp}[L(\tau_1-\tau_2)]d\tau_1 d\tau_2, \tag{76}$$

$$C_{\varphi\varphi}(\xi_1,\xi_2,T_1,T_2)=\left(\frac{L}{v}\right)^2\int_{T_1-\frac{a}{L}}^{T_1}\int_{T_2-\frac{a}{L}}^{T_2}H_{\varphi}(\xi_1,T_1-\tau_1)H_{\varphi}(\xi_2,T_2-\tau_2)C_{pp}[L(\tau_1-\tau_2)]d\tau_1 d\tau_2,\quad (77)$$

for $\frac{a}{L}\le T_i\le 1,\ i$=1,2.

$$\varphi_1(\xi,T)=\frac{qL}{v}\int_0^T H_{A\varphi}(\xi,\tau)d\tau=\begin{cases}\dfrac{qL\kappa}{v\eta^2 AG}[-\dfrac{T}{2}+\dfrac{\cos\sigma(1-T)-\cos\sigma}{\sigma\sin\sigma}\cos\sigma\xi] & \xi<T,\\ \dfrac{qL\kappa}{v\eta^2 AG}[\dfrac{T}{2}-\dfrac{1-\cos\sigma T}{\sigma\sin\sigma}\cos\sigma(1-\xi)] & \xi>T,\end{cases}\quad (81)$$

$$\varphi_2(\xi,T)=\frac{qL}{v}\int_0^T H_{P\varphi}(\xi,\tau)d\tau=\frac{qL}{v}\sum_{n=1}^{\infty}[C_{nP}\frac{1-\cos r_n T}{r_n}+D_{nP}\frac{1-\cos s_n T}{s_n}]\cos n\pi\xi.\quad (82)$$

6. Some numerical results.

The numerical calculations have been done using dimensionless parameter (Figs. 8-23): η=0,04; γ=1,9; ρ=0,5; λ=20. Also to analyse transverse displacement $w(x,t)$ and rotation of the cross-section of the beam $\varphi(x,t)$ depending on the velocity of the moving load the numerical calculations have been done (Figs. 14-21). Using ratio of moving mass to the beam mass μ=1 the vibrations of the beam under uniformly distributed moving mass m_1 have been presented (Figs. 22-23).

The bending moment $M(x,t)$, the shear force $Q(x,t)$, the transverse displacement $w(x,t)$ and the rotation of the cross-section of the beam $\varphi(x,t)$ increases together with T (Figs. 8-11).

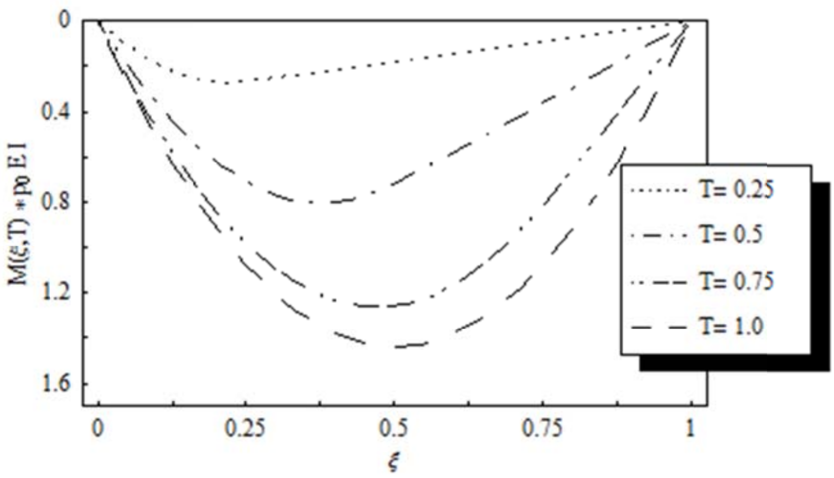


Fig. 8 The bending moments depending on the position of the load

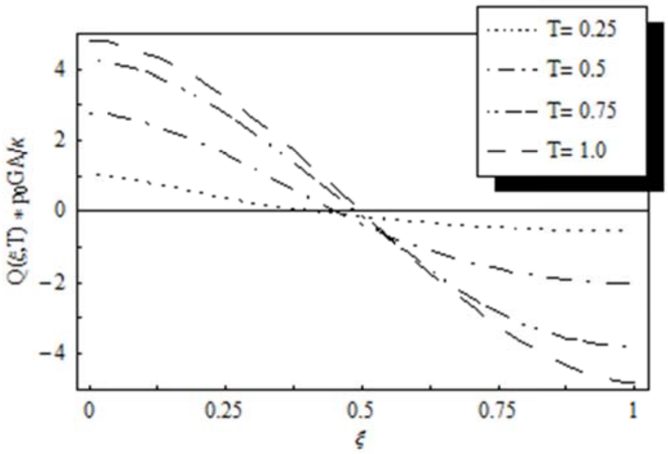


Fig. 9 The shear force depending on the position of the load

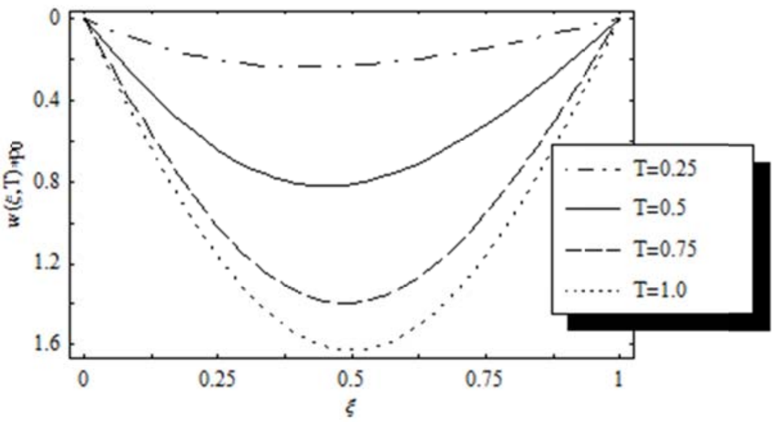


Fig. 10 The transverse displacement of the beam depending on the position of the load

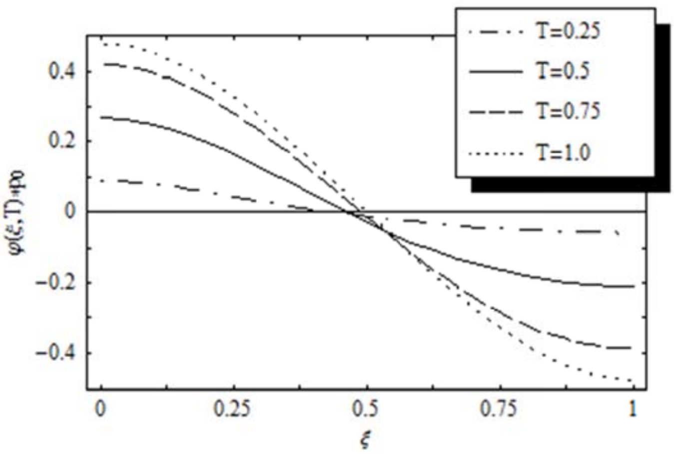


Fig. 11 The rotation of the cross-section of the beam depending on the position of the load.

The transverse displacement $w(x,t)$ is always the largest and the rotation of the cross-section of the beam $\varphi(x,t)$ is always the smallest in the middle of the beam (Figs.12-13).

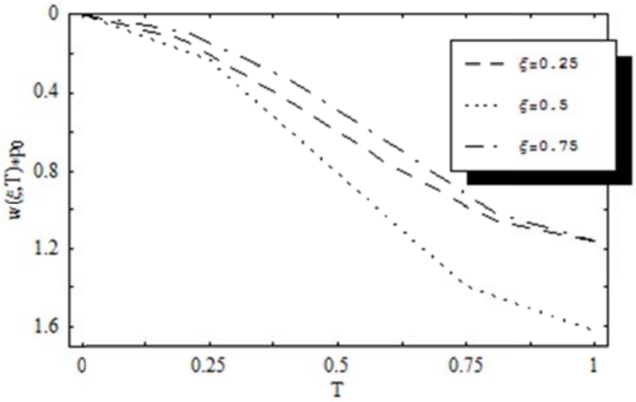


Fig. 12 The chart of the transverse displacement of the beam depending on the cross-section of the beam

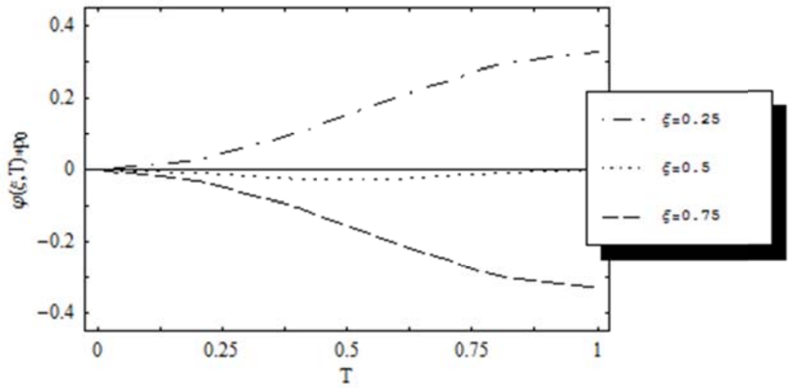


Fig. 13 The rotation of the cross-section of the beam depending on the cross-section of the beam

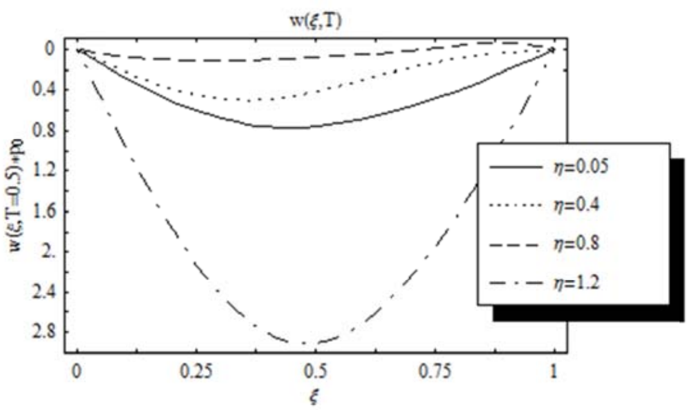


Fig. 14 The transverse displacement of the beam depending on the velocity of the load for T=0, 5

The transverse displacement $w(x,t)$ and the rotation of the cross-section of the beam $\varphi(x,t)$ decreases when the velocity of the load increases (Figs. 14-19) when $\eta<1$.

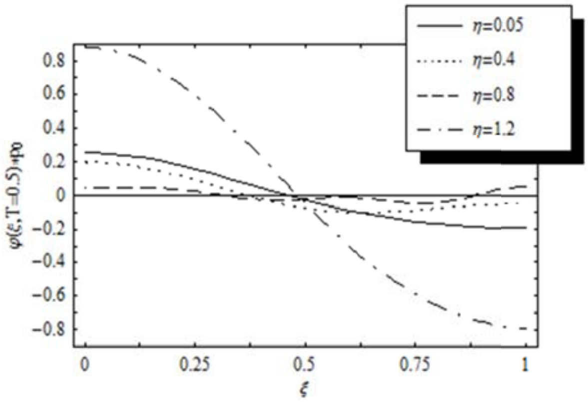


Fig. 15 The rotation of the cross-section of the beam depending on the velocity of the load for T=0, 5

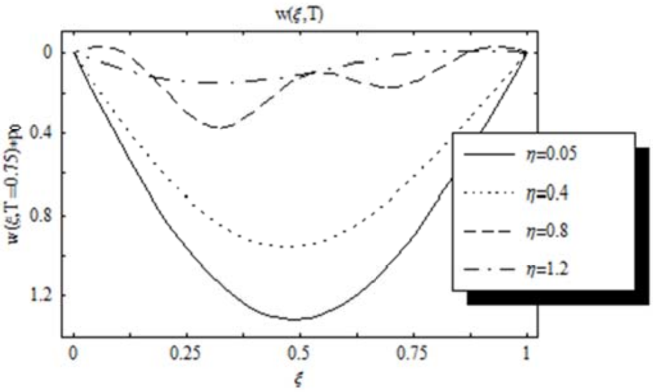


Fig. 16 The transverse displacement of the beam depending on the velocity of the load for T=0, 75

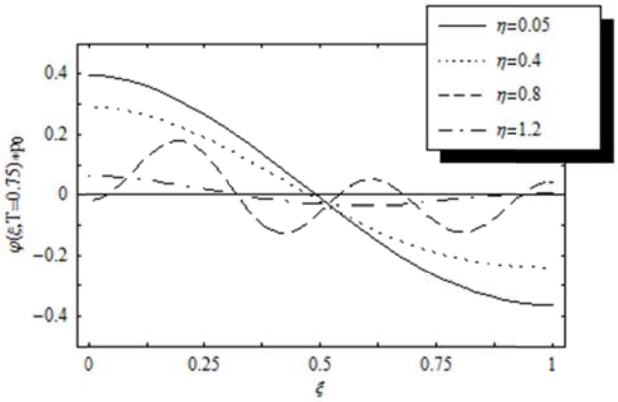


Fig. 17 The rotation of the cross-section of the beam depending on the velocity of the load for T=0, 75

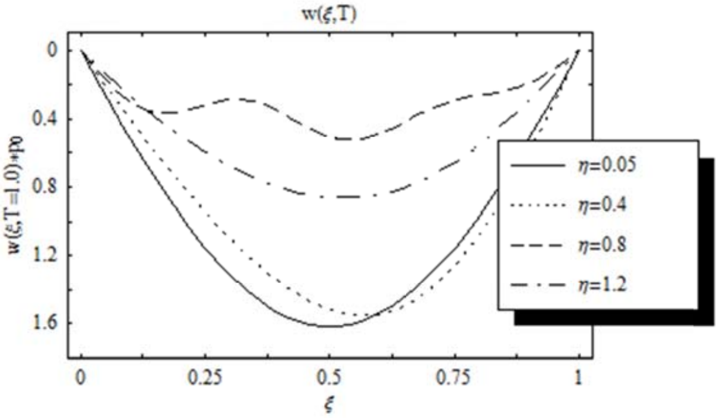


Fig. 18 The transverse displacement of the beam depending on the velocity of the load for T=1, 0

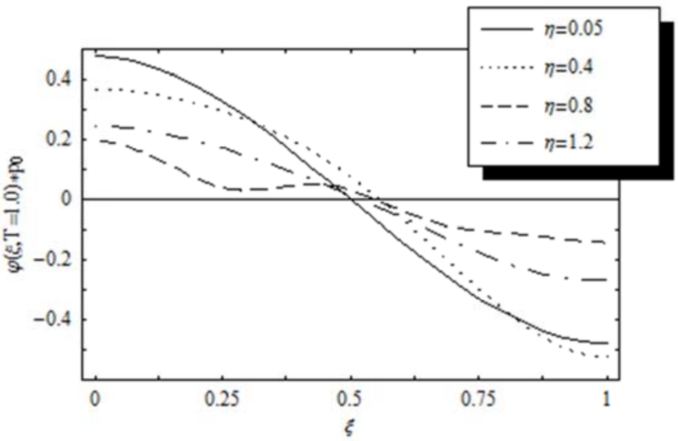


Fig. 19 The rotation of the cross-section of the beam depending on the velocity of the load for T=1, 0

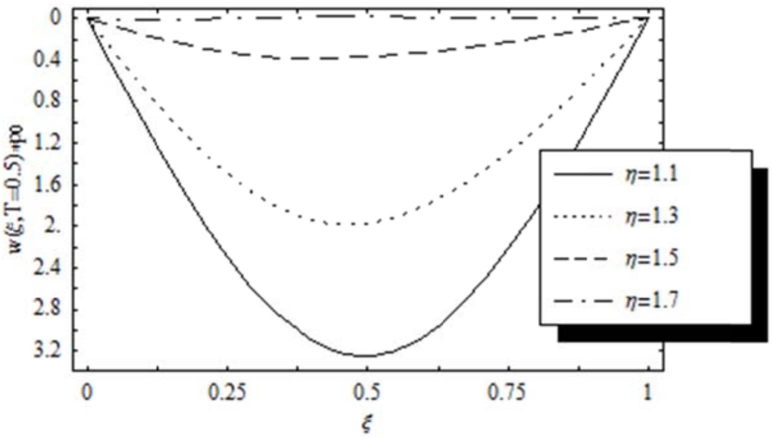


Fig. 20 The transverse displacement of the beam depending on the velocity of the load when partially distributed load is moving faster than the shear wave (η>1) for T=0.5

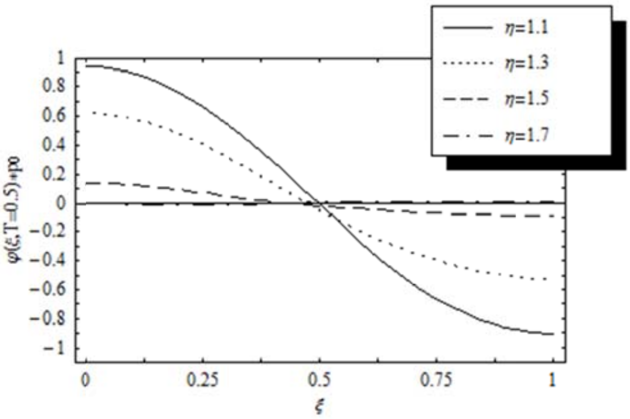


Fig. 21 The rotation of the cross-section of the beam depending on the velocity of the load when partially distributed load is moving faster than the shear wave (η>1) for T=0.5

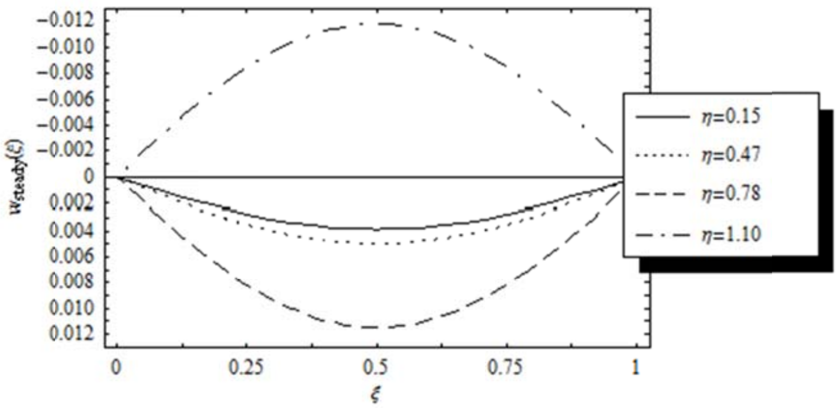


Fig. 22 The transverse displacement of the beam under uniformly distributed moving mass m_1 depending on the velocity of the load

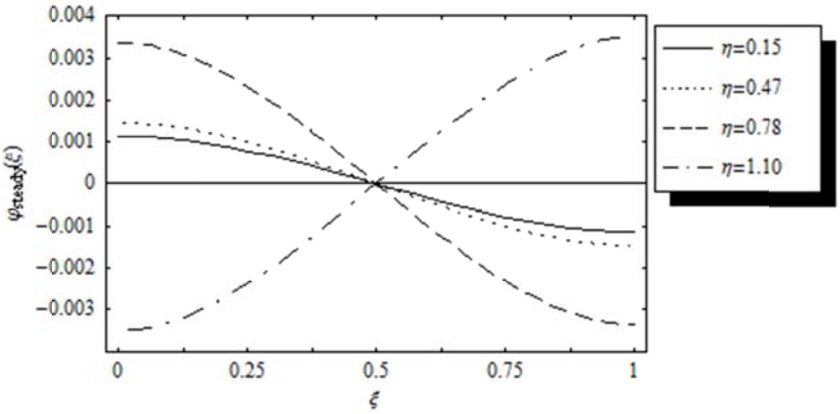


Fig. 23 The rotation of the cross-section of the beam under uniformly distributed moving mass m_1 depending on the velocity of the load

It is worth pointing out that when the velocity of the mass is bigger than the critical velocity, the beam displacement is opposite to the direction of the gravity force which is consistent with our intuition (Figs. 22, 23).

7. Conclusions

The dynamic response of a finite, simply supported Timoshenko beam loaded by a continuously distributed load moving with a constant velocity has been considered. Three problems have been considered. The dynamic response of the Timoshenko beam under a uniform distributed load moving with a constant velocity v has been considered as the first problem. Obtained solutions allow to find the response of the beam under the interval of the finite length a uniformly distributed moving load. Part of the solutions are presented in a closed form instead of an infinite series. As the second problem the steady-state vibrations of the beam under uniformly distributed mass m_1 moving with the constant velocity has been considered. The vibrations of the beam caused by the interval of the finite length randomly distributed load moving with constant velocity is considered as the last problem. It is assumed that load process is space-time stationary stochastic process. The last problem has been solved using dynamic influence function. The solutions are presented using dimensionless parameters making it easier to analyze the response of the beam.

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